QUANTUM INTEGERS AND CYCLOTOMY

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Abstract. A sequence of functions $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ satisfies the functional equation for multiplication of quantum integers if $f_{mn}(q) = f_m(q)f_n(q^m)$ for all positive integers m and n. This paper describes the structure of all sequences of rational functions with coefficients in $\mathbf{Q}$ that satisfy this functional equation.

1. The functional equation for multiplication of quantum integers

Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ denote the positive integers. For every $n \in \mathbf{N}$, we define the polynomial

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

This polynomial is called the *quantum integer* n. The sequence of polynomials $\{[n]_q\}_{n=1}^{\infty}$ satisfies the following functional equation:

$$f_{mn}(q) = f_m(q)f_n(q^m) \tag{1}$$

for all positive integers m and n. Nathanson [1] asked for a classification of all sequences $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ of polynomials and of rational functions that satisfy the functional equation (1).

The following statements are simple consequences of the functional equation. Proofs can be found in Nathanson [1].

Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be any sequence of functions that satisfies (1). Then $f_1(q) = f_1(q)^2 = 0$ or 1. If $f_1(q) = 0$, then $f_n(q) = f_1(q)f_n(q) = 0$ for all $n \in \mathbf{N}$, and $\mathcal{F}$ is a trivial solution of (1). In this paper we consider only nontrivial solutions of the functional equation, that is, sequences $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ with $f_1(q) = 1$.

Let P be a set of prime numbers, and let $S(P)$ be the multiplicative semigroup of $\mathbf{N}$ generated by P. Then $S(P)$ consists of all integers that can be represented as a product of powers of prime numbers belonging to P. Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a nontrivial solution of (1). We define the support

$$\mathrm{supp}(\mathcal{F}) = \{n \in \mathbf{N} : f_n(q) \neq 0\}.$$

There exists a unique set P of prime numbers such that $\mathrm{supp}(\mathcal{F}) = S(P)$. Moreover, the sequence $\mathcal{F}$ is completely determined by the set $\{f_p(q) : p \in P\}$. Conversely, if P is any set of prime numbers, and if $\{h_p(q) : p \in P\}$ is a set of functions such that

$$h_{p_1}(q)h_{p_2}(q^{p_1}) = h_{p_2}(q)h_{p_1}(q^{p_2}) \tag{2}$$

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for all $p_1, p_2 \in P$, then there exists a unique solution $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ of the functional equation (1) such that $\mathrm{supp}(\mathcal{F}) = S(P)$ and $f_p(q) = h_p(q)$ for all $p \in P$.

For example, for the set $P = \{2, 5, 7\}$, the reciprocal polynomials

$$\begin{aligned} h_2(q) &= 1 - q + q^2 \\ h_5(q) &= 1 - q + q^3 - q^4 + q^5 - q^7 + q^8 \\ h_7(q) &= 1 - q + q^3 - q^4 + q^6 - q^8 + q^9 - q^{11} + q^{12}. \end{aligned}$$

satisfy the commutativity condition (2). Since

$$h_p(q) = \frac{[p]_{q^3}}{[p]_q} \qquad \text{for } p \in P,$$

it follows that

$$f_n(q) = \frac{[n]_{q^3}}{[n]_q} \qquad \text{for all } n \in S(P). \tag{3}$$

Moreover, $f_n(q)$ is a polynomial of degree $2(n-1)$ for all $n \in S(P)$.

Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a solution of the functional equation (1) with $\mathrm{supp}(\mathcal{F}) = S(P)$ If $P = \emptyset$, then $\mathrm{supp}(\mathcal{F}) = \{1\}$. It follows that $f_1(q) = 1$ and $f_n(q) = 0$ for all $n \geq 2$. Also, for any prime p and any function $h(q)$, there is a unique solution of the functional equation (1) with $\mathrm{supp}(\mathcal{F}) = S(\{p\})$ and $f_p(q) = h(q)$. Thus, we only need to investigate solutions of (1) for $\mathrm{card}(P) \geq 2$.

If $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ and $\mathcal{G} = \{g_n(q)\}_{n=1}^{\infty}$ are solutions of (1) with $\mathrm{supp}(\mathcal{F}) = \mathrm{supp}(\mathcal{G})$, then, for any integers d, e, r, and s, the sequence of functions $\mathcal{H} = \{h_n(q)\}_{n=1}^{\infty}$, where

$$h_n(q) = f_n(q^r)^d g_n(q^s)^e,$$

is also a solution of the functional equation (1) with $\mathrm{supp}(\mathcal{H}) = \mathrm{supp}(\mathcal{F})$. In particular, if $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ is a solution of (1), then $\mathcal{H} = \{h_n(q)\}_{n=1}^{\infty}$ is another solution of (1), where

$$h_n(q) = \begin{cases} 1/f_n(q) & \text{if } n \in \mathrm{supp}(\mathcal{F}) \\ 0 & \text{if } n \notin \mathrm{supp}(\mathcal{F}). \end{cases}$$

The functional equation also implies that

$$f_m(q) f_n(q^m) = f_n(q) f_m(q^n) \tag{4}$$

for all positive integers m and n, and

$$f_{m^k}(q) = \prod_{i=0}^{k-1} f_m(q^{m^i}). \tag{5}$$

Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a solution in rational functions of the functional equation (1) with $\mathrm{supp}(\mathcal{F}) = S(P)$. Then there exist a completely multiplicative arithmetic function $\lambda(n)$ with support $S(P)$ and rational numbers t_0 and t_1 with $t_0(n-1) \in \mathbf{Z}$ and $t_1(n-1) \in \mathbf{Z}$ for all $n \in S(P)$ such that, for every $n \in S(P)$, we can write the rational function $f_n(q)$ uniquely in the form

$$f_n(q) = \lambda(n) q^{t_0(n-1)} \frac{u_n(q)}{v_n(q)}, \tag{6}$$

where $u_n(q)$ and $v_n(q)$ are monic polynomials with nonzero constant terms, and

$$\deg(u_n(q)) - \deg(v_n(q)) = t_1(n-1) \qquad \text{for all } n \in \mathrm{supp}(\mathcal{F}).$$

For example, let P be a set of prime numbers with $\mathrm{card}(P) \geq 2$. Let $\lambda(n)$ be a completely multiplicative arithmetic function with support $S(P)$, and let t_0 be a rational number such that $t_0(n-1) \in \mathbf{Z}$ for all $n \in S(P)$. Let R be a finite set of positive integers and $\{t_r\}_{r\in R}$ a set of integers. We construct a sequence $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ of rational functions as follows: For $n \in S(P)$, we define

$$f_n(q) = \lambda(n) q^{t_0(n-1)} \prod_{r\in R} [n]_{q^r}^{t_r}. \tag{7}$$

For $n \notin S(P)$ we set $f_n(q) = 0$. Then $\prod_{r\in R}[n]_{q^r}^{t_r}$ is a quotient of monic polynomials with coefficients in $\mathbf{Q}$ and nonzero constant terms. The sequence $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ satisfies the functional equation (1), and $\mathrm{supp}(\mathcal{F}) = S(P)$.

We shall prove that every solution of the functional equation (1) in rational functions with coefficients in $\mathbf{Q}$ is of the form (7). This provides an affirmative answer to Problem 6 in [1] in the case of the field $\mathbf{Q}$.

2. Roots of unity and solutions of the functional equation

Let K be an algebraically closed field, and let K^* denote the multiplicative group of nonzero elements of K. Let Γ denote the group of roots of unity in K^*, that is,

$$\Gamma = \{\zeta \in K^* : \zeta^n = 1 \text{ for some } n \in N\}.$$

Since Γ is the torsion subgroup of K^*, every element in $K^* \setminus \Gamma$ has infinite order. We define the *logarithm group*

$$L(K) = K^*/\Gamma,$$

and the map

$$L : K^* \to L(K)$$

by

$$L(a) = a\Gamma \text{ for all } x \in K^*.$$

We write the group operation in $L(K)$ additively:

$$L(a) + L(b) = a\Gamma + b\Gamma = ab\Gamma = L(ab).$$

Lemma 1. *Let K be an algebraically closed field, and $L(K)$ its logarithm group. Then $L(K)$ is a vector space over the field $\mathbf{Q}$ of rational numbers.*

Proof. Let $a \in K^*$ and $m/n \in \mathbf{Q}$. Since K is algebraically closed, there is an element $b \in K^*$ such that

$$b^n = a^m.$$

We define

$$\frac{m}{n} L(a) = L(b).$$

Suppose $m/n = r/s \in \mathbf{Q}$, and that

$$c^s = a^r$$

for some $c \in K^*$. Since $ms = nr$, it follows that

$$c^{ms} = a^{mr} = b^{nr} = b^{ms},$$

and so $c/b \in \Gamma$. Therefore,

$$\frac{m}{n} L(a) = L(b) = b\Gamma = c\Gamma = L(c) = \frac{r}{s} L(a),$$

and $(m/n)L(a)$ is well-defined. It is straightforward to check that $L(K)$ is a $\mathbf{Q}$-vector space. □

Lemma 2. *Let P be a set of primes, card$(P) \geq 2$, and let $S(P)$ be the multiplicative semigroup generated by P. For every integer $m \in S(P) \setminus \{1\}$ there is an integer $n \in S(P)$ such that $\log m$ and $\log n$ are linearly independent over $\mathbf{Q}$. Equivalently, for every integer $m \in S(P) \setminus \{1\}$ there is an integer $n \in S(P)$ such that there exist integers r and s with $m^r = n^s$ if and only if $r = s = 0$.*

Proof. If $m = p^k$ is a prime power, let n be any prime in $P \setminus \{p\}$. If m is divisible by more than one prime, let n be any prime in P. The result follows immediately from the Fundamental Theorem of Arithmetic. □

Let K be a field. A *function on* K is a map $f : K \to K \cup \{\infty\}$. For example, $f(q)$ could be a polynomial or a rational function with coefficients in K. We call $f^{-1}(0)$ the set of *zeros* of f and $f^{-1}(\infty)$ the set of *poles* of f.

Theorem 1. *Let K be an algebraically closed field. Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a sequence of functions on K that satisfies the functional equation (1). Let P be the set of primes such that supp$(\mathcal{F}) = S(P)$. If card$(P) \geq 2$ and if, for every $n \in$ supp$(\mathcal{F})$, the function $f_n(q)$ has only finitely many zeros and only finitely many poles, then every zero and pole of $f_n(q)$ is either 0 or a root of unity.*

Proof. The proof is by contradiction. Let Γ be the group of roots of unity in K. Suppose that

$$f_n(a) = 0 \text{ for some } n \in \operatorname{supp}(\mathcal{F}) \text{ and } a \in K^* \setminus \Gamma.$$

By Lemma 2, there is an integer $m \in S(P)$ such that $\log m$ and $\log n$ are linearly independent over $\mathbf{Q}$. Since a has infinite order in the multiplicative group K^* and $f_n^{-1}(0)$ is finite, there are positive integers k and $M = m^k$ such that a^M is not a zero of the function $f_n(q)$. By (4), we have

$$f_M(q)f_n(q^M) = f_n(q)f_M(q^n).$$

Therefore,

$$f_M(a)f_n(a^M) = f_n(a)f_M(a^n) = 0.$$

Since $f_n(a^M) \neq 0$, it follows from (5) that

$$0 = f_M(a) = f_{m^k}(a) = \prod_{i=0}^{k-1} f_m(a^{m^i}),$$

and so

$$f_m(a^{m^i}) = 0 \text{ for some } i \text{ such that } 0 \leq i \leq k-1.$$

Let

$$b = a^{m^i}.$$

Then

$$f_m(b) = 0,$$
$$b \in K^* \setminus \Gamma,$$

and

$$L(b) = m^i L(a) \tag{8}$$

Since $f_m^{-1}(0)$ is finite, there are positive integers ℓ and $N = n^\ell$ such that z^N is not a zero of $f_m(q)$ for every $z \in f_m^{-1}(0)$ with $z \in K^* \setminus \Gamma$. Since K is algebraically closed, we can choose $c \in K$ such that

$$c^N = b.$$

Then

$$f_m(c) \neq 0,$$
$$c \in K^* \setminus \Gamma,$$

and

$$NL(c) = L(b). \tag{9}$$

Again applying (4), we have

$$f_m(q)f_N(q^m) = f_N(q)f_m(q^N)$$

and so

$$f_m(c)f_N(c^m) = f_N(c)f_m(c^N) = f_N(c)f_m(b) = 0.$$

It follows that

$$0 = f_N(c^m) = f_{n^\ell}(c^m) = \prod_{j=0}^{\ell-1} f_n(c^{mn^j}),$$

and so

$$f_n(c^{mn^j}) = 0 \text{ for some } j \text{ such that } 0 \leq j \leq \ell - 1.$$

Let

$$a' = c^{mn^j}.$$

Then

$$f_n(a') = 0,$$
$$a' \in K^* \setminus \Gamma,$$

and

$$L(a') = mn^j L(c) \tag{10}$$

Combining (8), (9), and (10), we obtain

$$L(a') = \frac{mn^j}{N} L(b) = \frac{m^{i+1}}{n^{\ell-j}} L(a),$$

that is,

$$L(a') = \frac{m^{i'}}{n^{j'}} L(a), \text{ where } 1 \leq i' \leq k \text{ and } 1 \leq j' \leq \ell. \tag{11}$$

What we have accomplished is the following: Given an element $a \in f_n^{-1}(0)$ that is neither 0 nor a root of unity, we have constructed another element $a' \in f_n^{-1}(0)$ that is also neither 0 nor a root of unity, and that satisfies (11). Iterating this process, we obtain an infinite sequence of such elements. However, the number of zeros of $f_n(q)$ is finite, and so the elements in this sequence cannot be pairwise distinct. It follows that there is an element

$$a \in f_n^{-1}(0) \setminus (\Gamma \cup \{0\})$$

such that

$$L(a) = \frac{m^r}{n^s} L(a),$$

where r and s are positive integers. Then

$$a^{n^s}\Gamma = L\left(a^{n^s}\right) = n^s L(a) = m^r L(a) = L\left(a^{m^r}\right) = a^{m^r}\Gamma.$$

Since a is not a root of unity, it follows that

$$m^r = n^s,$$

which contradicts the linear independence of $\log m$ and $\log n$ over $\mathbf{Q}$. Therefore, the zeros of the functions $f_n(q)$ belong to $\Gamma \cup \{0\}$ for all $n \in \mathrm{supp}(\mathcal{F})$.

Replacing the sequence $\mathcal{F} = \{f_n(q)\}_{n\in\mathrm{supp}(\mathcal{F})}$ with $\mathcal{F}' = \{1/f_n(q)\}_{n\in\mathrm{supp}(\mathcal{F})}$, we conclude that the poles of the functions $f_n(q)$ also belong to $\Gamma \cup \{0\}$ for all $n \in \mathrm{supp}(\mathcal{F})$. This completes the proof. □

3. Rational solutions of the functional equation

In this section we shall completely classify sequences of rational functions with rational coefficients that satisfy the functional equation for quantum multiplication.

For $k \geq 1$, let $\Phi_k(q)$ denote the kth cyclotomic polynomial. Then

$$F_k(q) = q^k - 1 = \prod_{d|k} \Phi_d(q)$$

and

$$\Phi_k(q) = \prod_{d|k} F_d(q)^{\mu(k/d)}, \tag{12}$$

where $\mu(k)$ is the Möbius function. Let ζ be a primitive dth root of unity. Then $F_k(\zeta) = 0$ if and only if d is a divisor of k. We define

$$F_0(q) = \Phi_0(q) = 1.$$

Note that

$$F_k(q) = q^k - 1 = (q-1)(1+q+\cdots+q^{k-1}) = F_1(q)[k]_q \tag{13}$$

for all $k \geq 1$.

A *multiset* $U = (U_0, \delta)$ consists of a finite set U_0 of positive integers and a function $\delta : U_0 \to \mathbf{N}$. The positive integer $\delta(u)$ is called the *multiplicity* of u. Multisets $U = (U_0, \delta)$ and $U' = (U_0', \delta')$ are equal if $U_0 = U_0'$ and $\delta(u) = \delta'(u)$ for all $u \in U_0$. Similarly, $U \subseteq U'$ if $U_0 \subseteq U_0'$ and $\delta(u) \leq \delta'(u)$ for all $u \in U_0$. The multisets U and U' are *disjoint* if $U_0 \cap U_0' = \emptyset$. We define

$$\prod_{u\in U} f_u(q) = \prod_{u\in U_0} f_u(q)^{\delta(u)}$$

and

$$\max(U) = \max(U_0).$$

If $U_0 = \emptyset$, then we set $\max(U) = 0$ and $\prod_{u\in U} f_u(q) = 1$.

Lemma 3. *Let U and U' be multisets of positive integers. Then*

$$\prod_{u\in U} F_u(q) = \prod_{u'\in U'} F_{u'}(q), \tag{14}$$

if and only if $U = U'$.

Proof. Let $k = \max(U \cup U')$. Let ζ be a primitive kth root of unity. If $k \in U'$, then

$$\prod_{u \in U} F_u(\zeta) = \prod_{u' \in U'} F_{u'}(\zeta) = 0,$$

and so $k \in U$. Dividing (14) by $F_k(q)$, reducing the multiplicity of k in the multisets U and U' by 1, and continuing inductively, we obtain $U = U'$. □

Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a nontrivial solution of the functional equation (1), where $f_n(q)$ is a rational function with rational coefficients for all $n \in \text{supp}(\mathcal{F})$. Because of the standard representation (6), we can assume that

$$f_n(q) = \frac{u_n(q)}{v_n(q)},$$

where $u_n(q)$ and $v_n(q)$ are monic polynomials with nonzero constant terms. By Theorem 1, the zeros of the polynomials $u_n(q)$ and $v_n(q)$ are roots of unity, and so we can write

$$f_n(q) = \frac{\prod_{u \in U_n'} \Phi_u(q)}{\prod_{v \in V_n'} \Phi_v(q)},$$

where U_n' and V_n' are disjoint multisets of positive integers. Applying (12), we replace each cyclotomic polynomial in this expression with a quotient of polynomials of the form $F_k(q)$. Then

$$f_n(q) = \frac{\prod_{u \in U_n} F_u(q)}{\prod_{v \in V_n} F_u(q)}, \tag{15}$$

where U_n and V_n are disjoint multisets of positive integers. Let

$$f_n(q) = \frac{\prod_{u \in U_n} F_u(q)}{\prod_{v \in V_n} F_v(q)} = \frac{\prod_{u' \in U_n'} F_{u'}(q)}{\prod_{v' \in V_n'} F_{v'}(q)},$$

where U_n and V_n are disjoint multisets of positive integers and U_n' and V_n' are disjoint multisets of positive integers. Then

$$\prod_{u \in U_n \cup V_n'} F_u(q) = \prod_{v \in U_n' \cup V_n} F_v(q).$$

By Lemma 3, we have the multiset identity

$$U_n \cup V_n' = U_n' \cup V_n.$$

Since $U_n \cap V_n = \emptyset$, it follows that $U_n \subseteq U_n'$ and so $U_n = U_n'$. Similarly, $V_n = V_n'$. Thus, the representation (15) is unique.

We introduce the following notation for the *dilation* of a set: For any integer d and any set S of integers,

$$d * S = \{ds : s \in S\}.$$

Lemma 4. *Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a nontrivial solution of the functional equation (1) with supp$(\mathcal{F}) = S(P)$, where card$(P) \geq 2$. Let*

$$f_n(q) = \frac{\prod_{u \in U_n} F_u(q)}{\prod_{v \in V_n} F_v(q)}$$

and U_n and V_n are disjoint multisets of positive integers. For every prime $p \in P$, let

$$m_p = \max(U_p \cup V_p).$$

There exists an integer r such that $m_p = rp$ for every $p \in P$. Moreover, either $m_p \in U_p$ for all $p \in P$ or $m_p \in V_p$ for all $p \in P$.

Proof. Let p_1 and p_2 be prime numbers in P, and let

$$\frac{m_{p_1}}{p_1} \geq \frac{m_{p_2}}{p_2}.$$

Equivalently,

$$p_2 m_{p_1} \geq p_1 m_{p_2}.$$

Applying functional equation (4) with $m = p_1$ and $n = p_2$, we obtain

$$\frac{\prod_{u\in U_{p_1}} F_u(q)}{\prod_{v\in V_{p_1}} F_v(q)} \frac{\prod_{u\in U_{p_2}} F_u(q^{p_1})}{\prod_{v\in V_{p_2}} F_v(q^{p_1})} = \frac{\prod_{u\in U_{p_2}} F_u(q)}{\prod_{v\in V_{p_2}} F_v(q)} \frac{\prod_{u\in U_{p_1}} F_u(q^{p_2})}{\prod_{v\in V_{p_1}} F_v(q^{p_2})},$$

where

$$U_{p_1} \cap V_{p_1} = U_{p_2} \cap V_{p_2} = \emptyset.$$

The identity

$$F_n(q^m) = (q^m)^n - 1 = q^{mn} - 1 = F_{mn}(q),$$

implies that

$$\begin{aligned}
\frac{\prod_{u\in U_{p_1}\cup p_1 * U_{p_2}} F_u(q)}{\prod_{v\in V_{p_1}\cup p_1 * V_{p_2}} F_v(q)} &= \frac{\prod_{u\in U_{p_1}} F_u(q)}{\prod_{v\in V_{p_1}} F_v(q)} \frac{\prod_{u\in p_1 * U_{p_2}} F_u(q)}{\prod_{v\in p_1 * V_{p_2}} F_v(q)} \\
&= \frac{\prod_{u\in U_{p_2}} F_u(q)}{\prod_{v\in V_{p_2}} F_v(q)} \frac{\prod_{s\in p_2 * U_{p_1}} F_u(q)}{\prod_{t\in p_2 * V_{p_1}} F_v(q)} \\
&= \frac{\prod_{u\in U_{p_2}\cup p_2 * U_{p_1}} F_u(q)}{\prod_{v\in V_{p_2}\cup p_2 * V_{p_1}} F_v(q)}.
\end{aligned}$$

By the uniqueness of the representation (15), it follows that

$$U_{p_1} \cup (p_1 * U_{p_2}) \cup V_{p_2} \cup (p_2 * V_{p_1}) = U_{p_2} \cup (p_2 * U_{p_1}) \cup V_{p_1} \cup (p_1 * V_{p_2}).$$

Recall that

$$m_{p_1} = \max\left(U_{p_1} \cup V_{p_1}\right).$$

If

$$m_{p_1} \in U_{p_1},$$

then

$$p_2 m_{p_1} \in p_2 * U_{p_1}$$

and so

$$p_2 m_{p_1} \in U_{p_1} \cup (p_1 * U_{p_2}) \cup V_{p_2} \cup (p_2 * V_{p_1}).$$

However,

(i) $p_2 m_{p_1} \notin U_{p_1}$ since $p_2 m_{p_1} > m_{p_1} = \max\left(U_{p_1} \cup V_{p_1}\right)$,
(ii) $p_2 m_{p_1} \notin p_2 * V_{p_1}$ since $m_{p_1} \in U_{p_1}$ and $U_{p_1} \cap V_{p_1} = \emptyset$,
(iii) $p_2 m_{p_1} \notin V_{p_2}$ since $p_2 m_{p_1} \geq p_1 m_{p_2} > m_{p_2} = \max\left(U_{p_2} \cup V_{p_2}\right)$.

If $p_2 m_{p_1} > p_1 m_{p_2} = \max\left(p_1 * U_{p_2}\right)$, then $p_2 m_{p_1} \notin p_1 * U_{p_2}$. This is impossible, and so

$$p_2 m_{p_1} = p_1 m_{p_2} \in p_1 * U_{p_2},$$
$$m_{p_2} \in U_{p_2},$$

and

$$\frac{m_{p_1}}{p_1} = \frac{m_{p_2}}{p_2} = r \quad \text{for all } p_1, p_2 \in P.$$

Similarly, if $m_{p_1} \in V_{p_1}$ for some $p_1 \in P$, then $m_{p_2} \in V_{p_2}$ for all $p_2 \in P$. This completes the proof. $\square$

Theorem 2. *Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a sequence of rational functions with coefficients in $\mathbf{Q}$ that satisfies the functional equation (1). If supp$(\mathcal{F}) = S(P)$, where P is a set of prime numbers and card$(P) \geq 2$, then there are*

(i) *a completely multiplicative arithmetic function $\lambda(n)$ with support $S(P)$,*
(ii) *a rational number t_0 such that $t_0(n-1)$ is an integer for all $n \in S(P)$,*
(iii) *a finite set R of positive integers and a set $\{t_r\}_{r\in R}$ of integers*

such that

$$f_n(q) = \lambda(n) q^{t_0(n-1)} \prod_{r\in R} [n]_{q^r}^{t_r} \quad \text{for all } n \in \text{supp}(\mathcal{F}). \tag{16}$$

Proof. It suffices to prove (16) for all $p \in P$. Recalling the representation (6), we only need to investigate the case

$$f_p(q) = \frac{\prod_{u\in U_p} F_u(q)}{\prod_{v\in V_p} F_v(q)},$$

where U_p and V_p are disjoint multisets of positive integers. Let $m_p = \max(U_p \cup V_p)$. By Lemma 4, there is a nonnegative integer m such that $m_p = mp$ for all $p \in P$. We can assume that $m_p \in U_p$ for all $p \in P$.

The proof is by induction on m. If $m = 0$, then $U_p = V_p = \emptyset$ and $f_p(q) = 1$ for all $p \in P$, hence (16) holds with $R = \emptyset$.

Let $m = 1$, and suppose that $m_p = p \in U_p$ for all $p \in P$. Then

$$f_p(q) = \frac{\prod_{u\in U_p} F_u(q)}{\prod_{v\in V_p} F_v(q)} = \frac{(q^p - 1)\prod_{u\in U_p'} F_u(q)}{\prod_{v\in V_p} F_v(q)}.$$

Since $q^p - 1 = F_1(q)[p]_q$, we have

$$\begin{aligned} g_p(q) &= \frac{f_p(q)}{[p]_q} \\ &= \frac{(q^p - 1)\prod_{u\in U_p\setminus\{p\}} F_u(q)}{[p]_q \prod_{v\in V_p} F_v(q)} \\ &= \frac{F_1(q)\prod_{u\in U_p\setminus\{p\}} F_u(q)}{\prod_{v\in V_p} F_v(q)} \\ &= \frac{\prod_{u\in U_p'} F_u(q)}{\prod_{v\in V_p'} F_v(q)}, \end{aligned}$$

where $U_p' \cap V_p' = \emptyset$. The sequence of rational functions $\mathcal{G} = \{g_n(q)\}_{n=1}^{\infty}$ is also a solution of the functional equation (1), and either $\max(U_p' \cup V_p') = 0$ for all $p \in P$ or $\max(U_p' \cup V_p') = p$ for all $p \in P$.

If $\max(U_p' \cup V_p') = p$ for all $p \in P$, then we construct the sequence $\mathcal{H} = \{h_n(q)\}_{n=1}^{\infty}$ of rational functions

$$h_n(q) = \frac{g_n(q)}{[n]_q} = \frac{f_n(q)}{[n]_q^2}.$$

Continuing inductively, we obtain a positive integer t such that

$$f_n(q) = [n]_q^t \quad \text{for all } n \in \text{supp}(\mathcal{F}).$$

Thus, (16) holds in the case $m = 1$.

Let m be an integer such that the Theorem holds whenever $m_p < mp$ for all $p \in P$, and let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a solution of the functional equation (1) with $\mathrm{supp}(\mathcal{F}) = S(P)$ and $m_p = mp$ and $m_p \in U_p$ for all $p \in P$. The sequence $\mathcal{G} = \{g_n(q)\}_{n=1}^{\infty}$ with

$$g_n(q) = \frac{f_n(q)}{[n]_{q^r}}$$

is a solution of the functional equation (1). Since

$$F_{rp}(q) = q^{rp} - 1 = (q^r - 1)\left(1 + q^r + \cdots + q^{r(p-1)}\right) = F_r(q)[p]_{q^r},$$

it follows that

$$\begin{aligned} g_p(q) &= \frac{(q^{m_p} - 1)\prod_{u \in U_p \setminus \{m_p\}} F_u(q)}{[p]_{q^r} \prod_{v \in V_p} F_v(q)} \\ &= \frac{(q^{mp} - 1)\prod_{u \in U_p \setminus \{mp\}} F_u(q)}{[p]_{q^r} \prod_{v \in V_p} F_v(q)} \\ &= \frac{F_r(q)\prod_{u \in U_p \setminus \{mp\}} F_u(q)}{\prod_{v \in V_p} F_v(q)} \\ &= \frac{\prod_{u \in U_p'} F_u(q)}{\prod_{v \in V_p'} F_v(q)}, \end{aligned}$$

where $U_p' \cap V_p' = \emptyset$, and $\max(U_{p'} \cup V_{p'}) \leq mp$.

If $\max(U_{p'} \cup V_{p'}) = mp$, then $mp \in U_p'$. We repeat the construction with

$$h_n(q) = \frac{g_n(q)}{[n]_{q^r}} = \frac{f_n(q)}{[n]_{q^r}^2}.$$

Continuing this process, we eventually obtain a positive integer t_r such that the sequence of rational functions

$$\left\{ \frac{f_n(q)}{[n]_{q^r}^{t_r}} \right\}_{n=1}^{\infty}$$

satisfies the functional equation (1), and

$$\frac{f_p(q)}{[p]_{q^r}^{t_r}} = \frac{\prod_{u \in U_p'} F_u(q)}{\prod_{v \in V_p'} F_v(q)},$$

where $U_p' \cap V_p' = \emptyset$ and $\max(U_p' \cup V_p') < mp$. It follows from the induction hypothesis there is a finite set R of positive integers and a set $\{t_r\}_{r \in R}$ of integers such that

$$f_n(q) = \prod_{r \in R} [n]_{q^r}^{t_r} \quad \text{for all } n \in \mathrm{supp}(\mathcal{F}).$$

This completes the proof. □

There remain two related open problems. First, we would like to have a simple criterion to determine when a sequence of rational functions satisfying the functional equation (1) is actually a sequence of polynomials. It is sufficient that all of the integers t_r in the representation (16) be nonnegative, but the example in (3) shows that this condition is not necessary.

Second, we would like to have a structure theorem for rational function solutions and polynomial solutions to the functional equation (1) with coefficients in an arbitrary field, not just the field of rational numbers.

References

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