SELF-SIMILAR MEASURES ASSOCIATED TO IFS WITH NON-UNIFORM CONTRACTION RATIOS

SZE-MAN NGAI AND YANG WANG

Abstract. In this paper we study the absolute continuity of self-similar measures defined by iterated function systems (IFS) whose contraction ratios are not uniform. We introduce a transversality condition for a multi-parameter family of IFS and study the absolute continuity of the corresponding self-similar measures. Our study is a natural extension of the study of Bernoulli convolutions by Solomyak, Peres, et al.

1. Introduction

Let $\{S_i\}_{i=1}^q$ be an *iterated function system* (IFS) of contractive similitudes on $\mathbb{R}^d$ defined by

$$S_i(x) = \rho_i R_i x + b_i, \qquad 1 \le i \le q, \tag{1.1}$$

where $0 < \rho_i < 1$, $b_i \in \mathbb{R}^d$ and R_i is a $d \times d$ orthogonal matrix for each i. Let F be the corresponding *self-similar set* (i.e. the *attractor*) and let $\dim_{\mathrm{H}}(F)$ denote the Hausdorff dimension of F. For any set of probability weights $\{p_i\}$ such that $0 < p_i < 1$ and $\sum_{i=1}^q p_i = 1$, there corresponds a unique probability measure, known as a *self-similar measure*, which is supported on F and satisfies the following *self-similar identity*:

$$\mu = \sum_{i=1}^q p_i \mu \circ S_i^{-1} \tag{1.2}$$

(see e.g. Hutchinson [H], Falconer [F]). It is well known that μ is either singular or absolutely continuous with respect to the d-dimensional Lebesgue measure (see e.g. Dubins and Freedman [DF] or Peres et al. [PSS]). We are interested in conditions that determine this dichotomy.

Let us recall that for the IFS in (1.1) the *similarity dimension* of F is defined to be the unique real number α satisfying $\sum_{i=1}^q \rho_i^\alpha = 1$. It is easy to show (see [F, Sections 9.2 and 9.3]) that the similarity dimension of F is always greater than or equal to its Hausdorff dimension. Consequently, if

$$\sum_{i=1}^q \rho_i^d < 1, \tag{1.3}$$

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then the similarity dimension of F will be less than d and hence $\dim_{\mathrm{H}}(F) < d$. This forces μ to be singular. Therefore we only need to study the case $\sum_{i=1}^q \rho_i^d \geq 1$.

The absolute continuity of self-similar measures with all similitudes having the same contraction ratio, particularly in one dimension, has been investigated extensively, most notably in [PS1], [PS2], [So], [PSc]. In a seminal work Solomyak [So] proves that the infinite Bernoulli convolution, which is the self-similar measure given by

$$S_1(x) = \rho x, \qquad S_2(x) = \rho x + 1, \qquad p_1 = p_2 = \frac{1}{2}$$

is absolutely continuous for almost all $1/2 < \rho < 1$. The proof employs an important concept called the *transversality condition*, first introduced in Pollicott and Simon [PoS]. Subsequently more general results on absolute continuity of self-similar measures have been obtained in several studies (see [PS2], [PSc]), all of which employing the transversality condition. The transversality condition has been used to study other questions in fractal geometry (see e.g. [PSc] and [PSS]).

When the similitudes are allowed to have non-uniform contraction ratios little has been known on the absolute continuity of the corresponding self-similar measures, even for the simple IFS consisting of two maps

$$S_1(x) = \rho_1 x, \qquad S_2(x) = \rho_2 x + 1, \qquad p_1 = p_2 = \frac{1}{2}.$$

The main objective of this paper is to study the absolute continuity of self-similar measures associated with such iterated function systems. A basic theorem we establish is:

Theorem 1.1. *Let $\{S_i\}_{i=1}^q$ be an IFS on $\mathbb{R}^d$ given by $S_i(x) = \rho_i R_i x + b_i$, where $0 < \rho_i < 1$, $b_i \in \mathbb{R}^d$ and R_i is an orthogonal $d \times d$ matrix. Let μ be the corresponding self-similar measure with probability weights $p_1, \dots, p_q$.*

(i) *Suppose that $\prod_{i=1}^q p_i^{p_i} \rho_i^{-dp_i} > 1$. Then μ is singular.*

(ii) *Suppose that $\prod_{i=1}^q p_i^{p_i} \rho_i^{-dp_i} = 1$ but $p_i \neq \rho_i^d$ for some i. Then μ is singular.*

(iii) *Suppose that $p_i = \rho_i^d$ for all i. Then μ is absolutely continuous if and only if the IFS $\{S_i\}$ satisfies the open set condition (OSC). In this case $\mu = \alpha \mathcal{L}^d|_F$ where F is the attractor of the IFS, $\alpha = 1/\mathcal{L}^d(F)$ and $\mathcal{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$.*

We remark that part (i), the easier part of Theorem 1.1, is known. In fact Nicol et al. [NSB] showed that the condition $\prod_{i=1}^q p_i^{p_i} \rho_i^{-dp_i} > 1$ implies that the Hausdorff dimension of μ, denoted by $\dim_{\mathrm{H}}(\mu)$, is strictly less than d. This forces μ to be singular. Recall that

$$\dim_{\mathrm{H}}(\mu) := \inf \big\{ \dim_{\mathrm{H}}(E) : \mu(\mathbb{R}^d \setminus E) = 0 \big\}.$$

In §2 we give an elementary probabilistic proof of this result by using the Law of Large Numbers. Such a probabilistic approach also enables us to obtain sharper results, as in the proof of part (ii) in Theorem 1.1.

A well-known fact is that it is typically much harder to establish absolutely continuity than to prove singularity. In our case this fact is greatly exacerbated by the non-uniformity of the contraction ratios. The standard definition of the transversality condition needs to be modified into a more complex form, and is much harder to verify. Furthermore, we no longer have the powerful tool of convolution at our disposal. (We will discuss this in more detail later on.)

To establish absolute continuity we consider an IFS $\{S_i\}_{i=1}^q$ in the one dimensional case $d=1$, where each $S_i(x)=\rho_i x+b_i$. Fix $b_1,b_2,\dots,b_q$ and a set of probability weights $p_1,\dots,p_q>0$ with $\sum_{i=1}^q p_i=1$. For the corresponding self-similar measure to be absolutely continuous the contraction ratios $\rho_1,\dots,\rho_q$ must lie in the region $\prod_{i=1}^q p_i^{p_i}\rho_i^{-p_i}<1$. The transversality condition for a one-parameter family of similitudes can be generalized to a multi-parameter family of similitudes. We shall define it in details in §3. Using the transversality condition we establish the following generalization of a result in [PS2]:

Theorem 1.2. *Let $\{S_i(x)=\rho_i(\boldsymbol{\lambda})x+b_i(\boldsymbol{\lambda})\}_{i=1}^q$ be a C^2 family of IFS on $\mathbb{R}$ with parameters $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_m)$. Fix a set of probability weights $p_1,\dots,p_q>0$ with $\sum_{i=1}^q p_i=1$ and let $\mu_{\boldsymbol{\lambda}}$ be the corresponding self-similar measure. Suppose that Ω is an open subset of the region $\{\boldsymbol{\lambda}:\ \prod_{i=1}^q(p_i/\rho_i(\boldsymbol{\lambda}))^{p_i}<1\}$ such that for $\boldsymbol{\lambda}\in\Omega$ the IFS $\{S_i\}$ satisfies the transversality condition. Then μ is absolutely continuous for Lebesgue almost all $\boldsymbol{\lambda}\in\Omega$.*

The transversality condition for a one-parameter family of IFS is already difficult to verify, and for a multi-parameter family it is even harder. We consider the special case of $q=2$ with equal probability weights.

Theorem 1.3. *Let μ_{ρ_1,ρ_2} be the self-similar measure corresponding to the IFS $\{S_1,S_2\}$ on $\mathbb{R}$ with*

$$S_1(x)=\rho_1 x,\qquad S_2(x)=\rho_2 x+1,\qquad p_1=p_2=\frac{1}{2}.$$

Then μ_{ρ_1,ρ_2} is absolutely continuous for Lebesgue almost all (ρ_1,ρ_2) in the region $\rho_1\rho_2>1/4$ and $0<\rho_1,\rho_2<0.6491$.

The actual region of absolute continuity we have established is shown in Figure 1, and is substantially larger than the region stated in Theorem 1.3. Details about the absolute continuity of μ_{ρ_1,ρ_2} are in §4. We conjecture that for almost all (ρ_1,ρ_2) in the region $\rho_1\rho_2>1/4$ the measure μ_{ρ_1,ρ_2} is absolutely continuous. But the proof of this conjecture would perhaps require radically new ideas, in particular for the area where one of the parameters is very close to 1.

2. Proof of Theorem 1.1

We first introduce the following standard notation. Let $\Sigma_q=\{1,2,\dots,q\}$ be the set of letters, and let Σ_q^n be the set of words of length n in Σ_q. We use Σ_q^* and $\Sigma_q^{\mathbb{N}}$ to denote the set of all finite words in Σ_q and the set of all one-sided sequences in Σ_q, respectively.

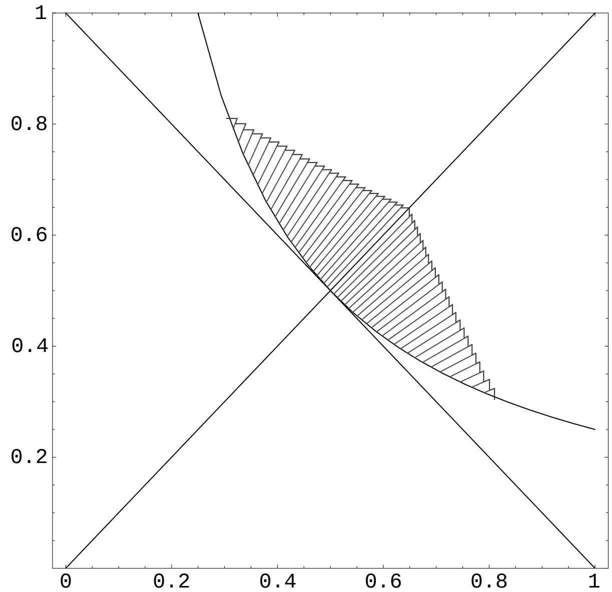

FIGURE 1. Regions of singularity and a.e. absolute continuity for the measure μ_{ρ_1,ρ_2} in Theorem 1.3. Condition (1.3) is satisfied on the left of the line $\rho_2 = 1 - \rho_1$. For (ρ_1, ρ_2) on the left of the hyperbola $\rho_2 = 1/(4\rho_1)$, μ_{ρ_1,ρ_2} is singular (Theorem 1.1 (i)). On the hyperbola, μ_{ρ_1,ρ_2} is singular except when $\rho_1 = \rho_2 = 1/2$ (Theorem 1.1 (ii) and (iii)). The shaded region bounded by the hyperbola and the "saw-tooth" curve is the known region of a.e. absolute continuity. The remaining region is unknown and μ_{ρ_1,ρ_2} is conjectured to be absolutely continuous for $\mathcal{L}^2$-a.e. (ρ_1, ρ_2) in that region.

For each $\mathbf{j} = (j_1, \ldots, j_n) \in \Sigma_q^*$ let $|\mathbf{j}|_i$ denote the number of letters i in $\mathbf{j}$, and let $|\mathbf{j}| = n$ be the length of $\mathbf{j}$. Furthermore, define

$$p_{\mathbf{j}} := p_{j_1} \cdots p_{j_n}, \quad \rho_{\mathbf{j}} := \rho_{j_1} \cdots \rho_{j_n}, \quad S_{\mathbf{j}} = S_{j_1} \circ \cdots \circ S_{j_n}.$$

Iterating equation (1.2) n times we get

$$\mu = \sum_{\mathbf{j} \in \Sigma_q^n} p_{\mathbf{j}} \mu \circ S_{\mathbf{j}}^{-1}. \tag{2.1}$$

Proof of Theorem 1.1 (i). This part is known (see [NSB]). We include a different proof here for self-containment. Let $\mathcal{L}^d$ denote the Lebesgue measure on $\mathbb{R}^d$. We will construct a sequence of Borel sets $\{F_n\}$ such that $\liminf_{n\to\infty} \mu(F_n) \geq C > 0$, where C is some constant independent of n and $\lim_{n\to\infty} \mathcal{L}^d(F_n) = 0$. This shows that μ is singular.

Let F be the attractor of the IFS $\{S_i\}_{i=1}^q$. F is the support of the self-similar measure μ. Since $\prod_{i=1}^q p_i^{p_i} \rho_i^{-dp_i} > 1$ there exists an $\varepsilon > 0$ sufficiently small such that

$$\prod_{i=1}^q p_i^{p_i+\varepsilon} \rho_i^{-d(p_i-\varepsilon)} = b > 1.$$

For each $n > 0$ define

$$\mathcal{K}_n := \{\mathbf{j} \in \Sigma_q^n : \ (p_i - \varepsilon)n \leq |\mathbf{j}|_i \leq (p_i + \varepsilon)n, \ 1 \leq i \leq n\}$$

and let $F_n := \bigcup_{\mathbf{j} \in \mathcal{K}_n} S_{\mathbf{j}}(F)$. Observe that $\mathcal{K}_n \neq \emptyset$ for all sufficiently large n. Moreover, for each $\mathbf{j} \in \mathcal{K}_n$ we have $S_{\mathbf{j}}^{-1}(F_n) \supseteq F$, and hence $\mu(S_{\mathbf{j}}^{-1}(F_n)) = 1$. By (2.1),

$$\mu(F_n) \geq \sum_{\mathbf{j} \in \mathcal{K}_n} p_{\mathbf{j}} \mu(S_{\mathbf{j}}^{-1}(F_n)) = \sum_{\mathbf{j} \in \mathcal{K}_n} p_{\mathbf{j}}.$$

The standard Law of Large Numbers (see e.g. Shiryaev [Sh], Ch. I, §5) yields $\mu(F_n) \to 1$ as $n \to \infty$. Now on the other hand,

$$\mathcal{L}^d(F_n) \leq \sum_{\mathbf{j} \in \mathcal{K}_n} \mathcal{L}^d(S_{\mathbf{j}}(F)) = \sum_{\mathbf{j} \in \mathcal{K}_n} \rho_{\mathbf{j}}^d \mathcal{L}^d(F).$$

But

$$\sum_{\mathbf{j} \in \mathcal{K}_n} \rho_{\mathbf{j}}^d = \sum_{\mathbf{j} \in \mathcal{K}_n} \rho_{\mathbf{j}}^d p_{\mathbf{j}}^{-1} p_{\mathbf{j}} \leq \sum_{\mathbf{j} \in \mathcal{K}_n} p_{\mathbf{j}} \Big(\prod_{i=1}^q \rho_i^{d(p_i - \varepsilon)n} p_i^{-(p_i+\varepsilon)n} \Big) = \sum_{\mathbf{j} \in \mathcal{K}_n} p_{\mathbf{j}} b^{-n} \leq b^{-n}.$$

Hence $\mathcal{L}^d(F_n) \to 0$ as $n \to \infty$. Thus, μ is singular. ∎

The proof of part (ii) of Theorem 1.1 is more involved. The Law of Large Numbers is no longer sufficient. We need more refined estimates. This is done through the use of the vector form Central Limit Theorem.

Proof of Theorem 1.1 (ii) and (iii). First we prove (iii). Suppose that $p_i = \rho_i^d$ for all i. Then the similarity dimension of the IFS $\{S_i\}$ is exactly d. Hence the support of the corresponding self-similar measure μ has positive Lebesgue measure if and only if $\{S_i\}$ satisfies the open set condition (OSC) (see Schief [Sc]). In particular $\{S_i\}$ must satisfy the OSC if μ is absolutely continuous. Conversely if $\{S_i\}$ satisfies the OSC then $\mu = \alpha \, \mathcal{L}^d|_F$, where F is the attractor of the IFS and $\alpha = 1/\mathcal{L}^d(F)$, obviously satisfies (1.2) and is absolutely continuous.

We now prove (ii). We again construct a sequence of sets F_n such that $\mu(F_n) > c > 0$ for all n where c is independent of n and $\mathcal{L}^d(F_n) \to 0$. This will prove that μ is singular.

For any nonempty $\mathcal{A}_n \subseteq \Sigma_q^n$ we define

$$\Omega_{\mathcal{A}_n} := \bigcup_{\mathbf{j} \in \mathcal{A}_n} S_{\mathbf{j}}(F).$$

Then

$$\mathcal{L}^d(\Omega_{\mathcal{A}_n}) \leq \sum_{\mathbf{j} \in \mathcal{A}_n} \mathcal{L}^d\big(S_{\mathbf{j}}(F)\big) = \mathcal{L}^d(F) \sum_{\mathbf{j} \in \mathcal{A}_n} \rho_{\mathbf{j}}^d. \tag{2.2}$$

Observe that $S_{\mathbf{j}}^{-1}(\Omega_{\mathcal{A}_n}) \supseteq F$, so $\mu(S_{\mathbf{j}}^{-1}(\Omega_{\mathcal{A}_n})) = 1$ for all $\mathbf{j} \in \mathcal{A}_n$. It follows from (2.1) that

$$\mu(\Omega_{\mathcal{A}_n}) = \sum_{\mathbf{j} \in \Sigma_q^n} p_{\mathbf{j}} \mu(S_{\mathbf{j}}^{-1}(\Omega_{\mathcal{A}_n})) \geq \sum_{\mathbf{j} \in \mathcal{A}_n} p_{\mathbf{j}}. \tag{2.3}$$

Our objective is to find a sequence $\{\mathcal{A}_n\}$ in Σ_q^n and some $\varepsilon_0 > 0$ such that

$$\lim_{n\to\infty} \sum_{\mathbf{j}\in\mathcal{A}_n} \rho_{\mathbf{j}}^d = 0 \qquad \text{and} \qquad \limsup_{n\to\infty} \sum_{\mathbf{j}\in\mathcal{A}_n} p_{\mathbf{j}} \geq \varepsilon_0 > 0. \tag{2.4}$$

Together with (2.2) and (2.3), this will force μ to be singular.

Now, write $\rho_i^d = p_i c_i$. Since $\prod_{i=1}^q c_i^{p_i} = 1$ and not all $p_i = \rho_i^d$, there must exist some $c_i < 1$ and some $c_i > 1$. Without loss of generality we assume that

$$c_1, \dots, c_r < 1, \quad c_{r+1}, \dots, c_q \geq 1.$$

For positive integers n sufficiently large define

$$\mathcal{A}_n := \Big\{\mathbf{j} \in \Sigma_q^n : \ |\mathbf{j}|_1 \geq p_1 n + \sqrt{n},\ |\mathbf{j}|_i \geq p_i n \text{ for } 2 \leq i \leq r,\ |\mathbf{j}|_i \leq p_i n \text{ for } r < i \leq q\Big\}.$$

Note that $\mathcal{A}_n \neq \emptyset$ for all n sufficiently large. Moreover,

$$\sum_{\mathbf{j}\in\mathcal{A}_n} \rho_{\mathbf{j}}^d = \sum_{\mathbf{j}\in\mathcal{A}_n} p_{\mathbf{j}} c_{\mathbf{j}} = \sum_{\mathbf{j}\in\mathcal{A}_n} p_{\mathbf{j}} c_1^{|\mathbf{j}|_1} \dots c_q^{|\mathbf{j}|_q}.$$

It follows from the definition of $\mathcal{A}_n$ that

$$\sum_{\mathbf{j}\in\mathcal{A}_n} \rho_{\mathbf{j}}^d \leq \sum_{\mathbf{j}\in\mathcal{A}_n} p_{\mathbf{j}} c_1^{p_1 n+\sqrt{n}} c_2^{p_2 n} \cdots c_q^{p_q n} = c_1^{\sqrt{n}} \sum_{\mathbf{j}\in\mathcal{A}_n} p_{\mathbf{j}} \Big(c_1^{p_1} c_2^{p_2} \dots c_q^{p_q}\Big)^n = c_1^{\sqrt{n}} \sum_{\mathbf{j}\in\mathcal{A}_n} p_{\mathbf{j}} \leq c_1^{\sqrt{n}}.$$

Hence $\sum_{\mathbf{j}\in\mathcal{A}_n} \rho_{\mathbf{j}}^d \to 0$ as $n \to \infty$. To complete the proof of (2.4) we only need to prove that $\limsup_{n\to\infty} \sum_{\mathbf{j}\in\mathcal{A}_n} p_{\mathbf{j}} \geq \varepsilon_0$ for some $\varepsilon_0 > 0$.

Lemma 2.1. *There exists an $\varepsilon_0 > 0$ such that for sufficiently large n,*

$$\sum_{\mathbf{j}\in\mathcal{A}_n} p_{\mathbf{j}} \geq \varepsilon_0.$$

Proof. We use the vector form Central Limit Theorem in probability theory to prove the lemma. Consider the random vector

$$\mathbf{X} = \mathbf{e}_i \quad \text{with probability } p_i,$$

where $\mathbf{e}_i$ is the standard coordinate vector in $\mathbb{R}^q$. Independently generate n such random vectors $\mathbf{X}_1, \dots, \mathbf{X}_n$. Let $\mathbf{Y}_n = \mathbf{X}_1 + \cdots + \mathbf{X}_n$. Then

$$\sum_{\mathbf{j}\in\mathcal{A}_n} p_{\mathbf{j}} = \mathrm{Prob}\Big\{Y_n^1 \geq p_1 n + \sqrt{n},\ Y_n^i \geq p_i n \text{ for } 2 \leq i \leq r,\, Y_n^i \leq p_i n \text{ for } r < i \leq q\Big\},$$

where $[Y_n^1, \dots, Y_n^q]^T = \mathbf{Y}_n$. Let

$$\mathbf{Z}_n = \frac{\mathbf{Y}_n - E(\mathbf{Y}_n)}{\sqrt{n}} = \frac{\mathbf{Y}_n - n[p_1, \dots, p_q]^T}{\sqrt{n}} =: [Z_n^1, \dots, Z_n^q]^T.$$

Let U be the subset of $\mathbb{R}^q$ defined by

$$U := \{(x_1, \dots, x_q) : \ x_1 \geq 1,\, x_2, \dots, x_r \geq 0,\, x_{r+1}, \dots, x_q \leq 0\}.$$

Then

$$\sum_{\mathbf{j}\in\mathcal{A}_n} p_{\mathbf{j}} = \mathrm{Prob}\Big\{\mathbf{Z}_n \in U\Big\}. \tag{2.5}$$

Observe that $\mathbf{Z}_n = [Z_n^1, \ldots, Z_n^q]^T$ lies on the hyperplane $x_1 + \cdots + x_q = 0$ in $\mathbb{R}^q$. So we consider the reduced random vector $\mathbf{Z}_n^* := [Z_n^1, \ldots, Z_n^{q-1}]^T$. Under this setting,

$$Z_n^q \le 0 \quad \Leftrightarrow \quad Z_n^1 + \cdots + Z_n^{q-1} \ge 0.$$

Define $U^* \subset \mathbb{R}^{q-1}$ by

$$U^* := \Big\{(x_1, \ldots, x_{q-1}) : \ x_1 \ge 1, \, x_2, \ldots, x_r \ge 0, \ x_{r+1}, \ldots, x_{q-1} \le 0, \, \sum_{k=1}^{q-1} x_k \ge 0\Big\}.$$

Hence we may rewrite (2.5) as

$$\sum_{\mathbf{j} \in \mathcal{A}_n} p_{\mathbf{j}} = \mathrm{Prob}\Big\{\mathbf{Z}_n^* \in U^*\Big\}.$$

Now, by the Central Limit Theorem (see Theorem 29.5 in Billingsley [B]),

$$\mathbf{Z}_n^* \quad \longrightarrow \quad \mathbf{Z} \qquad \text{weakly},$$

where $\mathbf{Z}$ has normal distribution centered at the origin whose covariance matrix $A = (\sigma_{ij})$ satisfies

$$\sigma_{ij} = E\Big((X^i - p_i)(X^j - p_j)\Big) = \begin{cases} p_i - p_i^2, & \text{if } i = j \\ -p_i p_j, & \text{if } i \ne j, \end{cases}$$

where $1 \le i, j \le q-1$ and $\mathbf{X} = [X^1, \ldots, X^q]^T$. Since $p_q > 0$, it is easy to see that A is diagonally dominant. So A is nonsingular. Hence (see p. 384 of [B]) $\mathbf{Z}$ has density

$$f_{\mathbf{Z}}(\mathbf{x}) = (2\pi)^{\frac{q-1}{2}} |\det A|^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}\mathbf{x}^T A^{-1} \mathbf{x}\Big), \quad \mathbf{x} \in \mathbb{R}^{q-1}.$$

This means that

$$\lim_{n\to\infty} \mathrm{Prob}\Big\{\mathbf{Z}_n^* \in U^*\Big\} = \int_{U^*} f_{\mathbf{Z}}(\mathbf{x})\, d\mathbf{x}.$$

The set U^* obviously has positive Lebesgue measure in $\mathbb{R}^{q-1}$. Therefore $\int_{U^*} f_{\mathbf{Z}}(\mathbf{x})\, d\mathbf{x} > 0$. Thus for sufficiently large n we have

$$\sum_{\mathbf{j} \in \mathcal{A}_n} p_{\mathbf{j}} = \mathrm{Prob}\Big\{\mathbf{Z}_n^* \in U^*\Big\} > \frac{1}{2}\int_{U^*} f_{\mathbf{Z}}(\mathbf{x})\, d\mathbf{x} =: \varepsilon_0 > 0,$$

proving the lemma. □

By proving this lemma we have now completed the proof of Theorem 1.1 (ii). ■

3. Transversality condition and a.e. absolute continuity

We consider an IFS $\{S_i\}_{i=1}^q$ on $\mathbb{R}$ where $S_i(x) = \rho_i x + b_i$, in which $\rho_i = \rho_i(\lambda_1, \ldots, \lambda_m)$ with $0 < \rho_i < 1$ and $b_i = b_i(\lambda_1, \ldots, \lambda_m)$ depend on the real parameters $\lambda_1, \ldots, \lambda_m$. We call the IFS $\{S_i\}$ an *m-parameter family* of IFS, and throughout the paper we will assume that all ρ_i and b_i are C^2 functions of $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$. A subset of $\mathbb{R}^m$ is called a *region* if it is the union of an open connected set with some, none, or all its boundary points.

Definition 3.1. *An m-parameter family $\{S_i\}_{i=1}^q$ of IFS satisfies the* transversality condition *for $\boldsymbol{\lambda} \in \Omega$, where Ω is a region in $\mathbb{R}^m$, if for any $\mathbf{i}, \mathbf{j} \in \Sigma_q^{\mathbb{N}}$ with $\mathbf{i}(1) \neq \mathbf{j}(1)$ we have*

$$\|\nabla(S_{\mathbf{i}}(0) - S_{\mathbf{j}}(0))\| > 0 \quad \textit{whenever} \quad S_{\mathbf{i}}(0) - S_{\mathbf{j}}(0) = 0, \tag{3.1}$$

where ∇ denotes the gradient with respect to the parameters $\boldsymbol{\lambda}$.

Observe that $S_{\mathbf{j}}(x) = S_{\mathbf{j}}(0)$ for any x and $\mathbf{j} \in \Sigma_q^{\mathbb{N}}$. Hence we shall adopt the notation $S_{\mathbf{j}} := S_{\mathbf{j}}[\boldsymbol{\lambda}] = S_{\mathbf{j}}[\lambda_1, \ldots, \lambda_m]$ for $S_{\mathbf{j}}(x) = S_{\mathbf{j}}(0)$. Furthermore, write $F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda}) := S_{\mathbf{i}}[\boldsymbol{\lambda}] - S_{\mathbf{j}}[\boldsymbol{\lambda}]$. Then (3.1) becomes

$$\|\nabla F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda})\| > 0 \quad \text{whenever} \quad F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda}) = 0.$$

Lemma 3.1. *Let $f(\boldsymbol{\lambda})$ be C^2 on a region $\Omega \subseteq \mathbb{R}^m$ and $\boldsymbol{\lambda}^* \in \Omega$ such that $B_r(\boldsymbol{\lambda}^*) \subseteq \Omega$. Let $f(\boldsymbol{\lambda}^*) = 0$ and $\|\nabla f(\boldsymbol{\lambda}^*)\| \geq \varepsilon > 0$. Suppose that all second partial derivatives of f satisfy $|\frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j}(\boldsymbol{\lambda})| \leq M$ on Ω. Then there exists a $\delta = \delta(\varepsilon, M, r) > 0$ and a $C_0 = C_0(\varepsilon, M, r) > 0$ such that for all $t > 0$,*

$$\mathcal{L}^m\Big(\{\boldsymbol{\lambda} \in \Omega \cap B_\delta(\boldsymbol{\lambda}^*) : \ |f(\boldsymbol{\lambda})| \leq t\}\Big) \leq C_0 t.$$

Proof. Since $\|\nabla f(\boldsymbol{\lambda}^*)\| \geq \varepsilon > 0$ we may without loss of generality assume that $|\frac{\partial f}{\partial \lambda_1}(\boldsymbol{\lambda}^*)| \geq \varepsilon/\sqrt{m}$. Write $\boldsymbol{\lambda}^* = [\lambda_1^*, \ldots, \lambda_m^*]^T$ and define $\mathbf{g} : \ \Omega \longrightarrow \mathbb{R}^m$ by

$$\mathbf{g}(\boldsymbol{\lambda}) := [f(\boldsymbol{\lambda}), \lambda_2 - \lambda_2^*, \ldots, \lambda_m - \lambda_m^*]^T.$$

The derivative $\mathbf{Dg}(\boldsymbol{\lambda}) := [\partial g_i/\partial \lambda_j]$ is upper triangular with $|\det \mathbf{Dg}(\boldsymbol{\lambda}^*)| \geq \varepsilon/\sqrt{m} > 0$. Now all second partial derivatives of f are bounded by M. The Mean Value Theorem applied to $\partial f/\partial \lambda_1$ restricted to the line segment from $\boldsymbol{\lambda}^*$ to $\boldsymbol{\lambda}$ implies that

$$\left|\frac{\partial f}{\partial \lambda_1}(\boldsymbol{\lambda}) - \frac{\partial f}{\partial \lambda_1}(\boldsymbol{\lambda}^*)\right| \leq M\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*\|$$

for $\boldsymbol{\lambda} \in B_r(\boldsymbol{\lambda}^*)$. Let $\delta_0 = \min\{r, \varepsilon/(2M\sqrt{m})\}$. Then $|\frac{\partial f}{\partial \lambda_1}(\boldsymbol{\lambda})| \geq \varepsilon/(2\sqrt{m})$ for all $\boldsymbol{\lambda} \in B_{\delta_0}(\boldsymbol{\lambda}^*)$. Hence $|\det \mathbf{Dg}(\boldsymbol{\lambda})| \geq \varepsilon/(2\sqrt{m})$ on $B_{\delta_0}(\boldsymbol{\lambda}^*)$.

Note that $\mathbf{Dg}(\boldsymbol{\lambda})$ satisfies the Lipschitz condition

$$\|\mathbf{Dg}(\boldsymbol{\lambda}_1) - \mathbf{Dg}(\boldsymbol{\lambda}_2)\| \leq C\|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|$$

for some $C = C(M)$ following again from the Mean Value Theorem. The norm on the left side of the inequality is the Euclidean norm on $\mathbb{R}^{m^2}$. Now $[\mathbf{Dg}]^{-1}(\boldsymbol{\lambda})$ is bounded on $B_{\delta_0}(\boldsymbol{\lambda}^*)$ because $\mathbf{Dg}$ is bounded and $|\det \mathbf{Dg}(\boldsymbol{\lambda})| \geq \varepsilon/(2\sqrt{m})$. It follows from the Inverse Function Theorem (cf. [HH]) that there exists a $\delta = \delta(\varepsilon, M, r) > 0$, $\delta \leq \delta_0$, such that $\mathbf{g} : \ B_\delta(\boldsymbol{\lambda}^*) \longrightarrow \mathbb{R}^m$ is invertible. Clearly

$$S := \mathbf{g}(B_\delta(\boldsymbol{\lambda}^*)) \subseteq \big\{\mathbf{y} \in \mathbb{R}^m : \ |y_i| < \delta \text{ for } 2 \leq i \leq m\big\} =: \widehat{S}.$$

Hence

$$\mathcal{L}^m\Big(\{\boldsymbol{\lambda} \in B_\delta(\boldsymbol{\lambda}^*) :\ |f(\boldsymbol{\lambda})| \le t\}\Big) = \int_{\{\mathbf{y}\in S:\ |y_1|\le t\}} |\det \mathbf{D}\mathbf{g}^{-1}(\mathbf{y})|\, d\mathbf{y}$$
$$\le \int_{\{\mathbf{y}\in \widehat{S}:\ |y_1|\le t\}} \frac{2\sqrt{m}}{\varepsilon}\, d\mathbf{y} \le C_0 t,$$

where $C_0 := 2(2\delta)^{m-1}$, using the fact that $\mathbf{D}\mathbf{g}^{-1}(g(\boldsymbol{\lambda})) = [\mathbf{D}\mathbf{g}]^{-1}(\boldsymbol{\lambda})$. Observe that all the constants in the proof ultimately depend only on ε, M and r. This completes the proof. ∎

Lemma 3.2. *Suppose an m-parameter family of IFS $\{S_i(x) = \rho_i(\boldsymbol{\lambda})x + b_i(\boldsymbol{\lambda})\}_{i=1}^q$ on $\mathbb{R}$ is C^2 for $\boldsymbol{\lambda}$ in an open set $U \subseteq \mathbb{R}^m$. Suppose further that the IFS satisfies the transversality condition on a compact region $\Omega \subset U$. Then there exists a constant $C > 0$ such that $F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda}) = S_{\mathbf{i}}[\boldsymbol{\lambda}] - S_{\mathbf{j}}[\boldsymbol{\lambda}]$ satisfies*

$$\mathcal{L}^m\Big(\{\boldsymbol{\lambda} \in \Omega :\ |F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda})| \le t\}\Big) \le Ct \tag{3.2}$$

for all $\mathbf{i}$, $\mathbf{j} \in \Sigma_q^{\mathbb{N}}$ with $\mathbf{i}(1) \ne \mathbf{j}(1)$ and all $t > 0$.

Proof. The class of all functions $F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda})$ with $\mathbf{i}(1) \ne \mathbf{j}(1)$ is compact in the space of continuous functions on Ω with the supremum norm. Hence there exists an $\varepsilon > 0$ such that $\|\nabla F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda})\| \ge \varepsilon$ whenever $F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda}) = 0$. The compactness of Ω implies that there exists an $r > 0$ such that

$$\Omega^r := \Big\{\boldsymbol{\lambda} \in \mathbb{R}^m :\ \operatorname{dist}(\boldsymbol{\lambda}, \Omega) \le r\Big\} \subset U.$$

Since Ω^r is compact, there exists a $\rho^* \in (0,1)$ such that $\rho_i(\boldsymbol{\lambda}) \le \rho^*$ for all i and $\boldsymbol{\lambda} \in \Omega^r$. Observe that all first and second partial derivatives of $b_i(\boldsymbol{\lambda})$ are bounded on Ω^r. Thus it follows from the geometric rate of convergence of all $S_{\mathbf{i}}[\boldsymbol{\lambda}]$ that there exists a constant $M > 0$ such that all second partial derivatives of $F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda})$ are bounded by M on Ω^r for all $\mathbf{i}$, $\mathbf{j} \in \Sigma_q^{\mathbb{N}}$.

For each $\boldsymbol{\lambda} \in \Omega$ we have $B_r(\boldsymbol{\lambda}) \subset \Omega^r$. Let $\delta = \delta(\varepsilon, M, r)$ be as in Lemma 3.1. Consider a partition of $\mathbb{R}^m$ into cubes of size $\delta/(2\sqrt{m})$. For each cube that intersects the set $\mathcal{Z}_{\mathbf{i},\mathbf{j}} := \{\boldsymbol{\lambda} \in \Omega :\ F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda}) = 0\}$ we pick a point in $\mathcal{Z}_{\mathbf{i},\mathbf{j}}$ within that cube. Let these points be $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_n$. Clearly $B_\delta(\boldsymbol{\lambda}_i)$ contains the entire cube in which $\boldsymbol{\lambda}_i$ lies. Hence $R_{\mathbf{i},\mathbf{j}} := \bigcup_{i=1}^n B_\delta(\boldsymbol{\lambda}_i)$ contains $\mathcal{Z}_{\mathbf{i},\mathbf{j}}$. Therefore $|F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda})| > 0$ on $\Omega \setminus R_{\mathbf{i},\mathbf{j}}$. The compactness of $\Omega \setminus R_{\mathbf{i},\mathbf{j}}$ and the compactness of the class of functions $F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda})$ with $\mathbf{i}(1) \ne \mathbf{j}(1)$ imply that there exists a $t_0 > 0$ such that $|F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda})| \ge t_0$ on $\Omega \setminus R_{\mathbf{i},\mathbf{j}}$ for all $\mathbf{i}$, $\mathbf{j} \in \Sigma_q^{\mathbb{N}}$ with $\mathbf{i}(1) \ne \mathbf{j}(1)$.

To prove (3.2) for all $t > 0$ it suffices to prove it for $0 < t < t_0$. Since $|F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda})| \le t$ implies $\boldsymbol{\lambda} \in R_{\mathbf{i},\mathbf{j}} = \bigcup_{i=1}^n B_\delta(\boldsymbol{\lambda}_i)$, we have

$$\{\boldsymbol{\lambda} \in \Omega :\ |F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda})| \le t\} = \bigcup_{i=1}^n \{\boldsymbol{\lambda} \in \Omega \cap B_\delta(\boldsymbol{\lambda}_i) :\ |F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda})| \le t\}.$$

It follows from Lemma 3.1 that

$$\mathcal{L}^m\Big(\{\boldsymbol{\lambda}\in\Omega:\ |F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda})|\le t\}\Big)\le nC_0t,$$

where $C_0=C_0(\varepsilon,M,r)$ is as in Lemma 3.1. Now the compactness of Ω means there can be at most $n_0:=\lfloor(\mathrm{diam}(\Omega)\cdot 2\sqrt{m}/\delta)^m\rfloor$ cubes intersecting $\mathcal{Z}_{\mathbf{i},\mathbf{j}}$, where $\lfloor x\rfloor$ denotes the smallest integer greater than or equal to x. Hence $n\le n_0$. This proves the lemma. ∎

Theorem 3.3. *Let $\{S_i(x)=\rho_i(\boldsymbol{\lambda})x+b_i(\boldsymbol{\lambda})\}_{i=1}^q$ be an m-parameter family of IFS on $\mathbb{R}$ satisfying the transversality condition on an open set Ω. For a fixed set of probability weights $p_1,\dots,p_q>0$ let $\mu_{\boldsymbol{\lambda}}$ be the corresponding self-similar measure. Suppose that for some $0<\alpha\le 1$ we have*

$$\sum_{i=1}^q p_i\left(\frac{p_i}{\rho_i(\boldsymbol{\lambda})}\right)^\alpha<1 \tag{3.3}$$

for all $\boldsymbol{\lambda}\in\Omega$. Then $\mu_{\boldsymbol{\lambda}}$ is absolutely continuous with its density function $d\mu_{\boldsymbol{\lambda}}/dx\in L^{1+\alpha}(\mathbb{R})$ for $\mathcal{L}^m$-a.e. $\boldsymbol{\lambda}\in\Omega$.

Proof. We prove $\mu_{\boldsymbol{\lambda}}$ is absolutely continuous for $\mathcal{L}^m$-a.e. $\boldsymbol{\lambda}\in\overline{B_\delta(\boldsymbol{\lambda}_0)}$ for any $\boldsymbol{\lambda}_0\in\Omega$ and sufficiently small $\delta>0$. This would imply the theorem.

Pick an arbitrary $\boldsymbol{\lambda}_0\in\Omega$ and a sufficiently small $\delta>0$ so that $R:=\overline{B_\delta(\boldsymbol{\lambda}_0)}\subset\Omega$. From (3.3) we have

$$\sum_{i=1}^q p_i\left(\frac{p_i}{\rho_i^*}\right)^\alpha<1 \tag{3.4}$$

where $\rho_i^*=\min_{\boldsymbol{\lambda}\in R}\rho_i(\boldsymbol{\lambda})$. We now prove that $\mu_{\boldsymbol{\lambda}}$ is absolutely continuous for $\mathcal{L}^m$-a.e. $\boldsymbol{\lambda}\in R$.

For any probability measure μ on $\mathbb{R}$ the *lower derivative* of μ at x is defined as

$$\underline{D}(\mu,x)=\varliminf_{r\to0^+}\frac{\mu\big(B_r(x)\big)}{r}.$$

It is known that μ is absolutely continuous if and only if $\underline{D}(\mu,x)<\infty$ for μ-a.e. $x\in\mathbb{R}$ (see Mattila [M], Ch. 2). In this case $\underline{D}(\mu,x)=d\mu/dx$ is the density of μ. Furthermore $d\mu/dx\in L^{1+\alpha}(\mathbb{R})$ if $\int_{\mathbb{R}}\underline{D}(\mu,x)^\alpha\,d\mu(x)<\infty$.

So to prove the theorem it suffices to prove

$$K_\alpha:=\int_R\int_{\mathbb{R}}\underline{D}(\mu_{\boldsymbol{\lambda}},x)^\alpha\,d\mu_{\boldsymbol{\lambda}}(x)\,d\boldsymbol{\lambda}<\infty. \tag{3.5}$$

It follows from Fatou's Lemma that

$$K_\alpha\le\varliminf_{r\to0^+}\frac{1}{r^\alpha}\int_R\int_{\mathbb{R}}\mu_{\boldsymbol{\lambda}}\big(B_r(x)\big)^\alpha\,d\mu_{\boldsymbol{\lambda}}(x)\,d\boldsymbol{\lambda}. \tag{3.6}$$

Now let ν be the product measure on $\Sigma_q^{\mathbb{N}}$ with weights $p_1, \ldots, p_q$. Let $\Pi_{\boldsymbol\lambda} : \Sigma_q^{\mathbb{N}} \longrightarrow \mathbb{R}$ be given by $\Pi_{\boldsymbol\lambda}(\mathbf{i}) = S_{\mathbf{i}}[\boldsymbol\lambda]$. Then $\mu_{\boldsymbol\lambda} = \nu \circ \Pi_{\boldsymbol\lambda}^{-1}$, and a change of variables in (3.6) yields

$$K_\alpha \le \varliminf_{r\to 0^+} \frac{1}{r^\alpha} \int_R \int_{\Sigma_q^{\mathbb{N}}} \mu_{\boldsymbol\lambda}\Big(B_r\big(\Pi_{\boldsymbol\lambda}(\mathbf{j})\big)\Big)^\alpha \, d\nu(\mathbf{j})\, d\boldsymbol\lambda. \tag{3.7}$$

Observe that $\mu_{\boldsymbol\lambda} = \nu \circ \Pi_{\boldsymbol\lambda}^{-1}$ implies

$$\begin{aligned} \mu_{\boldsymbol\lambda}\Big(B_r\big(\Pi_{\boldsymbol\lambda}(\mathbf{i})\big)\Big) &= \nu \circ \Pi_{\boldsymbol\lambda}^{-1}\Big(B_r\big(\Pi_{\boldsymbol\lambda}(\mathbf{i})\big)\Big) \\ &= \nu\Big(\{\mathbf{j} \in \Sigma_q^{\mathbb{N}} :\ |S_{\mathbf{i}}[\boldsymbol\lambda] - S_{\mathbf{j}}[\boldsymbol\lambda]| < r\}\Big) \\ &= \int_{\Sigma_q^{\mathbb{N}}} \chi_{\{\mathbf{j}\in\Sigma_q^{\mathbb{N}}:\ |F_{\mathbf{i},\mathbf{j}}(\boldsymbol\lambda)|<r\}} \, d\nu(\mathbf{j}), \end{aligned}$$

where the last two equalities follow from $\Pi_{\boldsymbol\lambda}(\mathbf{i}) = S_{\mathbf{i}}[\boldsymbol\lambda]$ and $F_{\mathbf{i},\mathbf{j}}(\boldsymbol\lambda) = S_{\mathbf{i}}[\boldsymbol\lambda] - S_{\mathbf{j}}[\boldsymbol\lambda]$. Substituting the above into (3.7) and changing the order of integration lead to

$$K_\alpha \le \varliminf_{r\to 0^+} \frac{1}{r^\alpha} \int_{\Sigma_q^{\mathbb{N}}} \int_R \bigg(\int_{\Sigma_q^{\mathbb{N}}} \chi_{\{\mathbf{j}\in\Sigma_q^{\mathbb{N}}:\ |F_{\mathbf{i},\mathbf{j}}(\boldsymbol\lambda)|<r\}} \, d\nu(\mathbf{j}) \bigg)^\alpha d\boldsymbol\lambda \, d\nu(\mathbf{i}).$$

Applying Jensen's inequality and again changing the order of integration, we get

$$\begin{aligned} K_\alpha &\le \varliminf_{r\to 0^+} \frac{C_1}{r^\alpha} \int_{\Sigma_q^{\mathbb{N}}} \bigg(\int_R \int_{\Sigma_q^{\mathbb{N}}} \chi_{\{\mathbf{j}\in\Sigma_q^{\mathbb{N}}:\ |F_{\mathbf{i},\mathbf{j}}(\boldsymbol\lambda)|<r\}} \, d\nu(\mathbf{j})\, d\boldsymbol\lambda \bigg)^\alpha d\nu(\mathbf{i}) \\ &= \varliminf_{r\to 0^+} \frac{C_1}{r^\alpha} \int_{\Sigma_q^{\mathbb{N}}} \bigg(\int_{\Sigma_q^{\mathbb{N}}} \mathcal{L}^m\{\boldsymbol\lambda \in R :\ |F_{\mathbf{i},\mathbf{j}}(\boldsymbol\lambda)| < r\} \, d\nu(\mathbf{j}) \bigg)^\alpha d\nu(\mathbf{i}) \end{aligned} \tag{3.8}$$

where $C_1 = \mathcal{L}^m(R)^{1-\alpha}$ is a constant.

Let σ denote the standard shift operator on $\Sigma_q^{\mathbb{N}}$. For any $\mathbf{i} \in \Sigma_q^{\mathbb{N}}$ let

$$\Lambda_{n,\mathbf{i}} := \Big\{ \mathbf{j} \in \Sigma_q^{\mathbb{N}} :\ \mathbf{i}(k) = \mathbf{j}(k) \text{ for all } 1 \le k \le n,\ \mathbf{i}(n+1) \ne \mathbf{j}(n+1) \Big\}.$$

For any $\mathbf{j} \in \Lambda_{n,\mathbf{i}}$ we have $F_{\mathbf{i},\mathbf{j}}(\boldsymbol\lambda) = \rho_{\mathbf{i}_n}(\boldsymbol\lambda) F_{\sigma^n\mathbf{i},\sigma^n\mathbf{j}}(\boldsymbol\lambda)$ where $\mathbf{i}_n = (i_1, i_2, \ldots, i_n)$ is the word consisting of the first n letters of $\mathbf{i}$. Hence by Lemma 3.2

$$\begin{aligned} \mathcal{L}^m\{\boldsymbol\lambda \in R :\ |F_{\mathbf{i},\mathbf{j}}(\boldsymbol\lambda)| < r\} &= \mathcal{L}^m\{\boldsymbol\lambda \in R :\ \rho_{\mathbf{i}_n}(\boldsymbol\lambda) |F_{\sigma^n\mathbf{i},\sigma^n\mathbf{j}}(\boldsymbol\lambda)| < r\} \\ &\le \mathcal{L}^m\{\boldsymbol\lambda \in R :\ |F_{\sigma^n\mathbf{i},\sigma^n\mathbf{j}}(\boldsymbol\lambda)| < (\rho^*_{\mathbf{i}_n})^{-1} r\} \le C(\rho^*_{\mathbf{i}_n})^{-1} r, \end{aligned}$$

for some constant C. Now (3.8) gives

$$\begin{aligned} K_\alpha &\leq \varliminf_{r\to 0^+} \frac{C_1}{r^\alpha}\int_{\Sigma_q^{\mathbb{N}}}\Big(\sum_{n=0}^\infty \int_{\Lambda_{n,\mathbf{i}}} \mathcal{L}^m\{\boldsymbol{\lambda}\in R:\ |F_{\mathbf{i},\mathbf{j}}(\boldsymbol{\lambda})|<r\}\,d\nu(\mathbf{j})\Big)^\alpha\,d\nu(\mathbf{i}) \\ &\leq \varliminf_{r\to 0^+} \frac{C_1}{r^\alpha}\int_{\Sigma_q^{\mathbb{N}}}\Big(\sum_{n=0}^\infty \int_{\Lambda_{n,\mathbf{i}}} C(\rho^*_{\mathbf{i}_n})^{-1} r\,d\nu(\mathbf{j})\Big)^\alpha\,d\nu(\mathbf{i}) \\ &= CC_1\int_{\Sigma_q^{\mathbb{N}}}\Big(\sum_{n=0}^\infty (\rho^*_{\mathbf{i}_n})^{-1}\nu(\Lambda_{n,\mathbf{i}})\Big)^\alpha\,d\nu(\mathbf{i}) \\ &\leq CC_1\int_{\Sigma_q^{\mathbb{N}}}\Big(\sum_{n=0}^\infty (\rho^*_{\mathbf{i}_n})^{-\alpha}\nu(\Lambda_{n,\mathbf{i}})^\alpha\Big)\,d\nu(\mathbf{i}), \end{aligned}$$

where the last inequality holds because $0<\alpha\leq 1$. It is standard to verify that

$$\int_{\Sigma_q^{\mathbb{N}}} |\rho^*_{\mathbf{i}_n}|^{-\alpha}\nu(\Lambda_{n,\mathbf{i}})^\alpha\,d\nu(\mathbf{i}) = \Big(\sum_{i=1}^q p_i\Big(\frac{p_i}{\rho_i^*}\Big)^\alpha\Big)^n.$$

Hence by (3.4)

$$K_\alpha \leq CC_1\sum_{k=0}^\infty\Big(\sum_{i=1}^q p_i\Big(\frac{p_i}{\rho_i^*}\Big)^\alpha\Big)^n<\infty.$$

This proves (3.5) and completes the proof of the theorem. ∎

Proof of Theorem 1.2: For each $0<\alpha\leq 1$ define

$$\Omega_\alpha = \Big\{\boldsymbol{\lambda}\in\Omega:\ \sum_{i=1}^q p_i\Big(\frac{p_i}{\rho_i(\boldsymbol{\lambda})}\Big)^\alpha<1\Big\}.$$

By Theorem 3.3 $\mu_{\boldsymbol{\lambda}}$ is absolutely continuous for $\mathcal{L}^m$-a.e. $\boldsymbol{\lambda}\in\Omega_\alpha$. We claim that any $\boldsymbol{\lambda}\in\Omega$ must also be in Ω_α for all sufficiently small $\alpha>0$. To prove this claim it suffices to prove that each $\boldsymbol{\lambda}\in\Omega$ has

$$\sum_{i=1}^q p_i\Big(\frac{p_i}{\rho_i(\boldsymbol{\lambda})}\Big)^\alpha<1\quad\text{for sufficiently small } \alpha>0.$$

The Taylor expansion yields $a^\alpha = 1+\ln(a)\alpha+o(\alpha)$. Hence

$$\begin{aligned} \sum_{i=1}^q p_i\Big(\frac{p_i}{\rho_i(\boldsymbol{\lambda})}\Big)^\alpha &= \sum_{i=1}^q p_i\Big(1+\alpha\ln\Big(\frac{p_i}{\rho_i(\boldsymbol{\lambda})}\Big)+o(\alpha)\Big) = 1+\alpha\sum_{i=1}^q p_i\ln\Big(\frac{p_i}{\rho_i(\boldsymbol{\lambda})}\Big)+o(\alpha) \\ &= 1+\alpha\sum_{i=1}^q \ln\Big(\frac{p_i}{\rho_i(\boldsymbol{\lambda})}\Big)^{p_i}+o(\alpha)<1 \end{aligned}$$

for sufficiently small $\alpha>0$, following from the assumption that $\sum_{i=1}^q \ln(p_i/\rho_i(\boldsymbol{\lambda}))^{p_i}<0$.

Now choose a sequence $\{\alpha_n\}$ such that $\alpha_n\to 0^+$. Then it follows that $\Omega=\bigcup_n\Omega_{\alpha_n}$. Hence $\mu_{\boldsymbol{\lambda}}$ is absolutely continuous for $\mathcal{L}^m$-a.e. $\boldsymbol{\lambda}\in\Omega$, proving the theorem. ∎

Remark. So far in this section we have considered the transversality condition only for IFS on $\mathbb{R}$. However, the transversality condition can also be defined for IFS on $\mathbb{R}^d$. Let $\{S_i(x)\}_{i=1}^q$ be a C^2 m-parameter family of IFS on $\mathbb{R}^d$ defined as

$$S_i(x) = \rho_i(\boldsymbol{\lambda})R_i(\boldsymbol{\lambda})x + b_i(\boldsymbol{\lambda}),$$

where $0 < \rho_i(\boldsymbol{\lambda}) < 1$, $R_i(\boldsymbol{\lambda})$ is an orthogonal matrix and $b_i(\boldsymbol{\lambda}) \in \mathbb{R}^d$. We say that $\{S_i\}$ satisfies the *transversality condition* for $\boldsymbol{\lambda} \in \Omega$, where $\Omega \subseteq \mathbb{R}^m$ is a region, if for any $\mathbf{i}, \mathbf{j} \in \Sigma_q^{\mathbb{N}}$ with $\mathbf{i}(1) \neq \mathbf{j}(1)$ the derivative matrix

$$\frac{\partial}{\partial \boldsymbol{\lambda}}(S_{\mathbf{i}}[\boldsymbol{\lambda}] - S_{\mathbf{j}}[\boldsymbol{\lambda}])$$

has full rank whenever $S_{\mathbf{i}}[\boldsymbol{\lambda}] = S_{\mathbf{j}}[\boldsymbol{\lambda}]$. All results in this section then generalize to $\mathbb{R}^d$.

4. IFS with two similitudes on $\mathbb{R}$

In this section we consider the IFS on $\mathbb{R}$ consisting of the two contractions

$$S_1(x) = \rho_1 x, \quad S_2(x) = \rho_2 x + 1, \tag{4.1}$$

where $0 < \rho_1, \rho_2 < 1$. We study the absolute continuity of the self-similar measure corresponding to probability weights $p_1 = p$, $p_2 = 1 - p$ and $0 < p < 1$. By Theorem 1.1 the measure is singular for $\rho_1^p \rho_2^{1-p} \leq p^p(1-p)^{1-p}$ with the exception of $\rho_1 = p$ and $\rho_2 = 1 - p$. So we consider the case $\rho_1^p \rho_2^{1-p} > p^p(1-p)^{1-p}$.

As in [So] and [PS1], we prove absolute continuity of the self-similar measure by establishing the transversality condition. For any $\mathbf{i} \in \{1,2\}^{\mathbb{N}}$ we have

$$S_{\mathbf{i}}(x) = S_{\mathbf{i}}(0) = \sum_{n=0}^{\infty} \epsilon_{\mathbf{i}(n+1)} \rho_{\mathbf{i}_n} = \sum_{n=0}^{\infty} \epsilon_{\mathbf{i}(n+1)} \rho_1^{|\mathbf{i}_n|_1} \rho_2^{|\mathbf{i}_n|_2}, \tag{4.2}$$

where $\rho_{\mathbf{i}_0} := 1$, $\mathbf{i}_n = (\mathbf{i}(1), \ldots, \mathbf{i}(n)) \in \{1,2\}^n$ denotes the word of the first n letters of $\mathbf{i}$, and $|\mathbf{j}|_1$ (respectively, $|\mathbf{j}|_2$) for a given $\mathbf{j} \in \{1,2\}^*$ denotes the number of letters '1' (respectively, '2') in $\mathbf{j}$, with $\epsilon_1 = 0$ and $\epsilon_2 = 1$. It seems rather hard to establish the transversality condition directly by using the parameters (ρ_1, ρ_2). Instead we perform the following change of variables

$$\rho_1 = a\lambda, \quad \rho_2 = \lambda,$$

where $0 < a, \lambda < 1$. In fact we shall simplify it further by fixing a and considering the transversality condition with respect to the single parameter λ.

With the new parameters (4.2) becomes

$$S_{\mathbf{i}}(0) = S_{\mathbf{i}}[\lambda, a] = \sum_{n=0}^{\infty} \epsilon_{\mathbf{i}(n+1)} a^{|\mathbf{i}_n|_1} \lambda^n. \tag{4.3}$$

As before let $F_{\mathbf{i},\mathbf{j}}(\lambda, a) = S_{\mathbf{i}}[\lambda, a] - S_{\mathbf{j}}[\lambda, a]$. For a fixed a denote

$$\mathcal{F}_a := \Big\{ f(\lambda) = F_{\mathbf{i},\mathbf{j}}(\lambda, a) : \ \mathbf{i}(1) = 2, \, \mathbf{j}(1) = 1 \Big\}.$$

We employ the technique of finding $(*)$-functions in [So] and [PS1]. However, as it turns out for a fixed a no single $(*)$-function leads to good results. Instead, we need to use more than one of them.

Definition 4.1. *Let $\mathcal{G} \subseteq \mathcal{F}_a$ for some $0 < a < 1$. Let $h(\lambda)$ be of the form*

$$h(\lambda) = 1 - \sum_{i=1}^{k} c_i \lambda^i + \sum_{i=k+1}^{\infty} c_i \lambda^i$$

where all $c_i \geq 0$ are bounded and $c_i \geq \varepsilon$ for some $\varepsilon > 0$ for sufficiently large i. Then $h(\lambda)$ is called a $(*)$-function *for $\mathcal{G}$ if for any given $g(\lambda) \in \mathcal{G}$ we have*

$$g(\lambda) - h(\lambda) = \sum_{i=1}^{n} b_i \lambda^i - \sum_{i=n+1}^{\infty} b_i \lambda^i$$

for some $n \geq 1$ and all $b_i \geq 0$.

Lemma 4.1. *Let $h(\lambda)$ be a $(*)$-function for some $\mathcal{G} \subseteq \mathcal{F}_a$ such that for some $\lambda_0 \in (0,1)$ we have*

$$h(\lambda_0) \geq \delta, \quad h'(\lambda_0) \leq -\delta \tag{4.4}$$

for some $\delta > 0$. Then for any $\varepsilon \in (0, \lambda_0)$ there exists some $\tilde{\delta} = \tilde{\delta}(\varepsilon) > 0$ such that for all $g \in \mathcal{G}$ and $\lambda \in [\varepsilon, \lambda_0]$,

$$g'(\lambda) \leq -\tilde{\delta} \quad \textit{whenever} \quad g(\lambda) \leq \tilde{\delta}.$$

Consequently all $g(\lambda) \in \mathcal{G}$ satisfy the transversality condition on $[0, \lambda_0]$.

Proof. This lemma is essentially proved in [PS1]; only minor modifications are needed since in our case the coefficient c_1 of $h(\lambda)$ can be 0, leading to $h'(0) = 0$. But this minor inconvenience can easily be fixed by considering the interval $[\varepsilon, \lambda_0]$ for some sufficiently small $\varepsilon > 0$. The proof in [PS1] then carries over. ∎

We next partition $\mathcal{F}_a$ so that good $(*)$-functions satisfying (4.4) can be found for each part of the partition. For $n \geq 1$ let

$$\mathcal{G}_{n,a} := \Big\{ F_{\mathbf{i},\mathbf{j}}(\lambda, a) \in \mathcal{F}_a : \ \mathbf{i}(k) = 2 \text{ for } 1 \leq k \leq n \Big\}, \qquad \mathcal{H}_{n,a} := \mathcal{F}_a \setminus \mathcal{G}_{n,a}.$$

Lemma 4.2. *Any $h_1(\lambda)$ of the form*

$$h_1(\lambda) = 1 - \sum_{i=1}^{k-1} a\lambda^i + c\lambda^k + \sum_{i=k+1}^{\infty} a\lambda^i \tag{4.5}$$

where $c \in \mathbb{R}$ and $k \geq n-1$ is a $(*)$-function for $\mathcal{H}_{n,b}$ for all b with $0 < b \leq a$.

Proof. Let $g(\lambda) = F_{\mathbf{i},\mathbf{j}}(\lambda, b) \in \mathcal{H}_{n,b}$. Then $\mathbf{i}(1) = 2$, $\mathbf{j}(1) = 1$ and $\mathbf{i}(l) = 1$ for some $l \leq n$. Hence

$$S_{\mathbf{i}}[\lambda, b] = \sum_{m=0}^{\infty} \epsilon_{\mathbf{i}(m+1)} b^{|\mathbf{i}_m|_1} \lambda^m =: 1 + \sum_{m=1}^{\infty} a_m \lambda^m$$

such that $a_m \le b \le a$ for all $m \ge n$, where $\epsilon_1 = 0$ and $\epsilon_2 = 1$. Furthermore, since $\mathbf{j}(1) = 1$ we know that $S_{\mathbf{j}}[\lambda, b]$ has the form

$$S_{\mathbf{j}}[\lambda, b] = \sum_{m=0}^{\infty} \epsilon_{\mathbf{j}(m+1)} b^{|\mathbf{j}_m|_1} \lambda^m =: \sum_{m=1}^{\infty} a'_m \lambda^m$$

such that $a'_m \le b \le a$ for all $m \ge 1$. Thus $g(\lambda) = F_{\mathbf{i},\mathbf{j}}(\lambda, b)$ satisfies

$$g(\lambda) - h_1(\lambda) = \sum_{m=1}^{p} b_m \lambda^m - \sum_{m=p+1}^{\infty} b_m \lambda^m$$

where $b_m \ge 0$ and $p = k - 1$ or $p = k$. ∎

Lemma 4.3. *Any $h_2(\lambda)$ of the form*

$$h_2(\lambda) = 1 - \sum_{i=n}^{n+k-1} \lambda^i + c\lambda^{n+k} + \sum_{i=n+k+1}^{\infty} \lambda^i \tag{4.6}$$

where $c \in \mathbb{R}$ and $k \ge 1$ is a $(*)$-function for $\mathcal{G}_{n,b}$ for all b with $0 < b < 1$.

Proof. Let $g(\lambda) = F_{\mathbf{i},\mathbf{j}}(\lambda, b) \in \mathcal{G}_{n,b}$. Then $\mathbf{i}(j) = 2$ for all $1 \le j \le n$ and $\mathbf{j}(1) = 1$. Hence

$$S_{\mathbf{i}}[\lambda, b] = 1 + \sum_{i=1}^{n-1} \lambda^i + \sum_{i=n}^{\infty} b_i \lambda^i$$

where $b_i \le 1$ for $i \ge n$. Since the coefficients of $S_{\mathbf{j}}[\lambda, b]$ cannot exceed 1, the first n coefficients of $g(\lambda)$ are nonnegative. It is now easy to see that

$$g(\lambda) - h_2(\lambda) = \sum_{i=1}^{p} c_i \lambda^i - \sum_{i=p+1}^{\infty} c_i \lambda^i$$

with all $c_i \ge 0$, where $p = n + k - 1$ or $p = n + k$. ∎

Let μ_{ρ_1,ρ_2} be the self-similar measure corresponding to the IFS (4.1) with probability weights $p_1 = p$, $p_2 = 1 - p$ and $0 < p < 1$. Define

$$\widehat{\Omega}_p := \Big\{(\rho_1, \rho_2) : \ \rho_1^p \rho_2^{1-p} > p^p (1-p)^{1-p}, \ 0 < \rho_1, \rho_2 < 1\Big\}.$$

For any pair of reals $\lambda_0, a \in (0, 1)$ let $K_{\lambda_0,a}$ be the two triangles

$$K_{\lambda_0,a} := \Big\{(\rho_1, \rho_2) : \ 0 < \rho_1 \le \lambda_0, \ \frac{\rho_2}{\rho_1} \le a\Big\} \cup \Big\{(\rho_1, \rho_2) : \ 0 < \rho_2 \le \lambda_0, \ \frac{\rho_1}{\rho_2} \le a\Big\}.$$

Theorem 4.4. *Let $n \ge 1$ and $0 < a \le 1$. Suppose that there exist $h_1(\lambda)$ and $h_2(\lambda)$ of the form (4.5) and (4.6), respectively, such that*

$$h_i(\lambda_0) \ge \delta, \quad h'_i(\lambda_0) \le -\delta, \quad i = 1, 2$$

for some $\lambda_0 \in (0, 1)$ and $\delta > 0$. Then μ_{ρ_1,ρ_2} is absolutely continuous for $\mathcal{L}^2$-a.e. $(\rho_1, \rho_2) \in K_{\lambda_0,a} \cap \widehat{\Omega}_p$.

Proof. Denote the two triangles of $K_{\lambda_0,a}$ by T_1 and T_2, respectively. Since $h_1(\lambda)$ is a $(*)$-function for $\mathcal{H}_{n,b}$ for all $0 < b \le a$, it follows from Lemma 4.1 that the transversality condition is satisfied for functions in $\mathcal{H}_{n,b}$ for $\lambda \in [0, \lambda_0]$. The existence of $h_2(\lambda)$ yields the transversality condition for functions in $\mathcal{G}_{n,b}$ for $\lambda \in [0, \lambda_0]$. Therefore for each fixed b with $0 < b \le a$ the IFS (4.1) parametrized by $\rho_1 = b\lambda$ and $\rho_2 = \lambda$ satisfies the transversality condition for $\lambda \in [0, \lambda_0]$. Hence μ_{ρ_1,ρ_2} as a one-parameter family with parameter λ is absolutely continuous for $\mathcal{L}^1$-a.e. $\lambda \in [0, \lambda_0]$, provided that $(b\lambda, \lambda) \in \widehat{\Omega}_p$. Therefore as a two-parameter family with parameters (λ, b) the measure μ_{ρ_1,ρ_2} is absolutely continuous for $\mathcal{L}^2$-a.e. $(\lambda, b) \in [0, \lambda_0] \times (0, a]$, provided that $(\rho_1, \rho_2) \in \widehat{\Omega}_p$. This region is $\widehat{\Omega}_p \cap T_2$.

To prove the a.e. absolute continuity of μ_{ρ_1,ρ_2} for $(\rho_1, \rho_2) \in \widehat{\Omega}_p \cap T_1$, we consider the self-similar measure $\tilde{\mu}_{\rho_1,\rho_2}$ associated to the IFS $\{\tilde{S}_1, \tilde{S}_2\}$ where

$$\tilde{S}_1(x) = \rho_1 x + 1, \quad \tilde{S}_2(x) = \rho_2 x$$

with the same probability weights $p_1 = p$ and $p_2 = 1 - p$. Note that

$$\tilde{S}_1 = \psi^{-1} \circ S_1 \circ \psi, \quad \tilde{S}_2 = \psi^{-1} \circ S_2 \circ \psi,$$

where $\psi(x) = \frac{\rho_1 - 1}{1 - \rho_2} x + \frac{1}{1 - \rho_2}$. It follows that $\mu_{\rho_1,\rho_2} = \tilde{\mu}_{\rho_1,\rho_2} \circ \psi^{-1}$. Hence μ_{ρ_1,ρ_2} is absolutely continuous if and only if $\tilde{\mu}_{\rho_1,\rho_2}$ is. But by symmetry the IFS $\{\tilde{S}_1, \tilde{S}_2\}$ satisfies the transversality condition on T_1. So $\tilde{\mu}_{\rho_1,\rho_2}$, and hence μ_{ρ_1,ρ_2}, is absolutely continuous for a.e. $(\rho_1, \rho_2) \in \widehat{\Omega}_p \cap T_1$. ∎

Theorem 4.4 allows us to establish a region of a.e. absolute continuity by looking for suitable $(*)$-functions. A quick search using Mathematica yields several pairs (λ_0, a) and their corresponding $(*)$-functions, given in Tables I and Table.

Corollary 4.5. *Let Λ be the set of pairs (λ_0, a) in Table I. Then the self-similar measure μ_{ρ_1,ρ_2} with probability weights $p_1 = p_2 = \frac{1}{2}$ is absolutely continuous for $\mathcal{L}^2$-a.e. (ρ_1, ρ_2) in the region*

$$\widehat{\Omega} \cap \bigcup_{(\lambda_0,a)\in\Lambda} K_{\lambda_0,a}, \quad \textit{where} \quad \widehat{\Omega} := \Big\{(\rho_1, \rho_2) : \ \rho_1\rho_2 > \frac{1}{4},\ 0 < \rho_1, \rho_2 < 1\Big\}.$$

The above region of a.e. absolute continuity is shown in the shaded region in Figure 1. It is clearly bigger than the region stated in Theorem 1.3. In the case of Bernoulli convolutions (see [So] and [PS1]), the random variable $S_{\mathbf{i}}(0) = S_{\mathbf{i}}[\lambda, a] = \sum_{n=0}^{\infty} \epsilon_{\mathbf{i}(n+1)} a^{|\mathbf{i}_n|_1} \lambda^n$ in (4.3) with $a = 1$ can be decomposed into a sum of two independent variables. The techniques of "thinning" and convolution can then be employed. In the case $a < 1$, a similar decomposition leads to a sum of two dependent random variables. It is thus not clear how such powerful techniques can be applied.

While Tables I and II are sufficient for the case $p = \frac{1}{2}$, it is not for all $0 < p < 1$. Our next two tables extend the values of a in Tables I and II to the interval $(0, 0.4)$. The corresponding $(*)$-functions are listed in Tables III and IV.

Note from the tables that the transversality condition holds on $[0, \lambda_0]$, where λ_0 is given by Tables I and III whenever $a > 0.100$, while it is given by Table IV whenever $a < 0.100$. We then have:

Theorem 4.6. *Let Λ be the set of pairs (λ_0, a) given by Tables I and III if $a > 0.100$, or given by Table IV if $a \leq 0.100$. Then the self-similar measure μ_{ρ_1,ρ_2} defined by the IFS (4.1) with probability weights $p_1 = p$ and $p_2 = 1-p$, $0 < p < 1$, is absolutely continuous for $\mathcal{L}^2$-a.e. (ρ_1, ρ_2) in the region*

$$\widehat{\Omega}_p \cap \bigcup_{(\lambda_0,a)\in\Lambda} K_{\lambda_0,a}.$$

Table I. $(*)$-functions $h_1(x)$

a	λ_0	$h_1(x)$
1.000	0.6491	$1-\sum_{i=1}^{3} x^i + 0.087530x^4 + x^5/(1-x)$
0.975	0.6542	$1-a\sum_{i=1}^{3} x^i - 0.053224x^4 + ax^5/(1-x)$
0.950	0.6594	$1-a\sum_{i=1}^{3} x^i - 0.189572x^4 + ax^5/(1-x)$
0.925	0.6646	$1-a\sum_{i=1}^{3} x^i - 0.321542x^4 + ax^5/(1-x)$
0.900	0.6698	$1-a\sum_{i=1}^{3} x^i - 0.449199x^4 + ax^5/(1-x)$
0.875	0.6750	$1-a\sum_{i=1}^{3} x^i - 0.572564x^4 + ax^5/(1-x)$
0.850	0.6802	$1-a\sum_{i=1}^{3} x^i - 0.691689x^4 + ax^5/(1-x)$
0.825	0.6854	$1-a\sum_{i=1}^{3} x^i - 0.806586x^4 + ax^5/(1-x)$
0.800	0.6918	$1-a\sum_{i=1}^{4} x^i + 0.630326x^5 + ax^6/(1-x)$
0.775	0.6983	$1-a\sum_{i=1}^{4} x^i + 0.418118x^5 + ax^6/(1-x)$
0.750	0.7049	$1-a\sum_{i=1}^{4} x^i + 0.214975x^5 + ax^6/(1-x)$
0.725	0.7114	$1-a\sum_{i=1}^{4} x^i + 0.020672x^5 + ax^6/(1-x)$
0.700	0.7178	$1-a\sum_{i=1}^{4} x^i - 0.165001x^5 + ax^6/(1-x)$
0.675	0.7243	$1-a\sum_{i=1}^{4} x^i - 0.342234x^5 + ax^6/(1-x)$
0.650	0.7307	$1-a\sum_{i=1}^{4} x^i - 0.511203x^5 + ax^6/(1-x)$
0.625	0.7372	$1-a\sum_{i=1}^{4} x^i - 0.672042x^5 + ax^6/(1-x)$
0.600	0.7451	$1-a\sum_{i=1}^{5} x^i + 0.297886x^6 + ax^7/(1-x)$
0.575	0.7527	$1-a\sum_{i=1}^{5} x^i + 0.049501x^6 + ax^7/(1-x)$
0.550	0.7602	$1-a\sum_{i=1}^{5} x^i - 0.184502x^6 + ax^7/(1-x)$
0.525	0.7677	$1-a\sum_{i=1}^{5} x^i - 0.404614x^6 + ax^7/(1-x)$
0.500	0.7750	$1-a\sum_{i=1}^{5} x^i - 0.611244x^6 + ax^7/(1-x)$
0.475	0.7824	$1-a\sum_{i=1}^{5} x^i - 0.804764x^6 + ax^7/(1-x)$
0.450	0.7896	$1-a\sum_{i=1}^{5} x^i - 0.985459x^6 + ax^7/(1-x)$
0.425	0.8006	$1-a\sum_{i=1}^{6} x^i - 0.487063x^7 + ax^8/(1-x)$
0.400	0.8101	$1-a\sum_{i=1}^{7} x^i - 0.004498x^8 + ax^9/(1-x)$

Table II. $(*)$-functions $h_2(x)$	
λ_0	$h_2(x)$
0.83	$1-\sum_{i=5}^{10} x^i - 0.5x^{11} + x^{12}/(1-x)$
0.85	$1-\sum_{i=6}^{12} x^i + 0.4x^{13} + x^{14}/(1-x)$
0.86	$1-\sum_{i=7}^{13} x^i - 0.98x^{14} + x^{15}/(1-x)$
0.87	$1-\sum_{i=8}^{15} x^i - 0.18x^{16} + x^{17}/(1-x)$
0.88	$1-\sum_{i=9}^{17} x^i + x^{18}/(1-x)$

Table III. $(*)$-functions $h_1(x)$		
a	λ_0	$h_1(x)$
0.375	0.8190	$1-a\sum_{i=1}^{7} x^i - 0.326319x^8 + ax^9/(1-x)$
0.350	0.8278	$1-a\sum_{i=1}^{7} x^i - 0.619668x^8 + ax^9/(1-x)$
0.325	0.8363	$1-a\sum_{i=1}^{7} x^i - 0.885920x^8 + ax^9/(1-x)$
0.300	0.8474	$1-a\sum_{i=1}^{8} x^i - 0.676359x^9 + ax^{10}/(1-x)$
0.275	0.8564	$1-a\sum_{i=1}^{8} x^i - 0.972115x^9 + ax^{10}/(1-x)$
0.250	0.8677	$1-a\sum_{i=1}^{9} x^i - 0.885398x^{10} + ax^{11}/(1-x)$
0.225	0.8801	$1-a\sum_{i=1}^{11} x^i - 0.511140x^{12} + ax^{13}/(1-x)$
0.200	0.8901	$1-a\sum_{i=1}^{11} x^i - 0.933492x^{12} + ax^{13}/(1-x)$
0.175	0.9026	$1-a\sum_{i=1}^{13} x^i - 0.808326x^{14} + ax^{15}/(1-x)$
0.150	0.9149	$1-a\sum_{i=1}^{15} x^i - 0.832987x^{16} + ax^{17}/(1-x)$
0.125	0.9278	$1-a\sum_{i=1}^{18} x^i - 0.819597x^{19} + ax^{20}/(1-x)$
0.100	0.9415	$1-a\sum_{i=1}^{23} x^i - 0.954917x^{24} + ax^{25}/(1-x)$
0.075	0.9558	$1-a\sum_{i=1}^{31} x^i - 0.912405x^{32} + ax^{33}/(1-x)$
0.050	0.9692	$1-a\sum_{i=1}^{43} x^i - 0.925572x^{44} + ax^{45}/(1-x)$
0.025	0.9842	$1-a\sum_{i=1}^{84} x^i - 0.981855x^{85} + ax^{86}/(1-x)$

Table IV. $(*)$-functions $h_2(x)$ for those $h_1(x)$ in Table III.		
a	λ_0	$h_2(x)$
—	0.89	$1-\sum_{i=10}^{18} x^i - 0.3x^{19} + x^{20}/(1-x)$
—	0.90	$1-\sum_{i=13}^{23} x^i + 0.6x^{24} + x^{25}/(1-x)$
—	0.91	$1-\sum_{i=15}^{26} x^i + 0.7x^{27} + x^{28}/(1-x)$
—	0.92	$1-\sum_{i=17}^{29} x^i + 0.8x^{30} + x^{31}/(1-x)$
—	0.929	$1-\sum_{i=20}^{33} x^i + 0.1x^{34} + x^{35}/(1-x)$
0.100	0.9393	$1-\sum_{i=25}^{40} x^i - 0.112691x^{41} + x^{42}/(1-x)$
0.075	0.9497	$1-\sum_{i=33}^{51} x^i - 0.398343x^{52} + x^{53}/(1-x)$
0.050	0.9595	$1-\sum_{i=45}^{68} x^i + 0.882207x^{69} + x^{70}/(1-x)$
0.025	0.9746	$1-\sum_{i=86}^{122} x^i + 0.580009x^{123} + x^{124}/(1-x)$

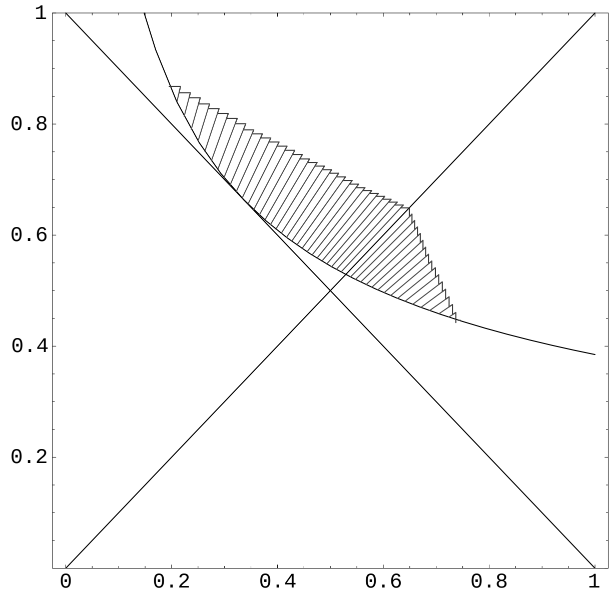

FIGURE 2. Regions of singularity and a.e. absolute continuity for the measure μ_{ρ_1,ρ_2} in Theorem 4.6 with $p_1 = 1/3$. On the left of the curve $\rho_2 = 2/(3\sqrt{3\rho_1})$, μ_{ρ_1,ρ_2} is singular. On the curve, μ_{ρ_1,ρ_2} is singular except when $(\rho_1, \rho_2) = (1/3, 2/3)$, the intersection of the curve with the line $\rho_2 = 1-\rho_1$. The two triangles making up $K_{\lambda_0,a}$ are symmetric with respect to the line $\rho_2 = \rho_1$. The shaded region is the known region of a.e. absolute continuity. The remaining region is unknown.

We illustrate Theorem 4.6 for $p_1 = 1/3$ in Figure 2.

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