GEOMETRY OF SELF–AFFINE TILES II

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Abstract. We continue the study in part I of geometric properties of self–similar and self–affine tiles. We give some experimental results from implementing the algorithm in part I for computing the dimension of the boundary of a self–similar tile, and we describe some conjectures that result. We prove that the dimension of the boundary may assume values arbitrarily close to the dimension of the tile. We give a formula for the area of the convex hull of a planar self–affine tile. We prove that the extreme points of the convex hull form a set of dimension zero, and we describe a natural gauge function for this set.

5. Introduction to Part II

This paper is a continuation of [SW], which we refer to as part I, and the sections are numbered accordingly. In Section 2 of part I we obtained an algorithm for computing the dimension (box and Hausdorff dimensions are equal) of the boundary in the case of a self–similar tile satisfying

$$AT = \bigcup_{d \in \mathcal{D}} (T + d) \tag{5.1}$$

for an expanding matrix A that is a similarity mapping the integer lattice $\mathcal{L}$ in $\mathbb{R}^n$ into itself, and the digit set $\mathcal{D}$ is a subset of $\mathcal{L}$ that it is a complete set of residues for $\mathcal{L}/A\mathcal{L}$. We wrote a program to implement the algorithm in the planar case ($n = 2$). In Section 6 we report some of the results obtained from running the program. These experimental results lead us to conjecture that for each fixed A, if we vary $\mathcal{D}$ over all allowable digit sets, there will be a minimum value for the dimension, and also that the limit of the dimensions will be n as $\mathcal{D}$ goes to infinity in the appropriate sense. However, we will see that there is no obvious candidate for the minimal digit set, and the convergence to the limit is not monotone in any obvious sense. In Section 7 we give a construction of self–similar tiles with a fixed A whose boundaries have dimension approaching n.

The remainder of the paper is devoted to the convex hull of the tile. In Section 8 we give a simple formula for the area of the convex hull in the planar case. In Section 9, also in the planar case, we complete the proof, begun in part I, of a formula for the perimeter of the convex hull, by showing that the set of extreme points E has dimension zero. We construct a natural dynamical system on E that is conjugate to a rotation on the circle, and yet is contractive except at a finite set of points. We construct a gauge function h such that the Hausdorff measure $\mathcal{H}_h$ of E is finite and positive. Modulo some elusive properties of a continued fraction expansion, we show that $h(t) = (\log(1/t))^{-1}$ for some values of t, but at other values of t it is larger.

The second author was supported by the National Science Foundation through the Research Experiences for Undergraduates (REU) program.

The third author is supported in part by the National Science Foundation, grant DMS–9623250.

The fourth author is supported in part by the National Science Foundation, grant DMS–9706793.

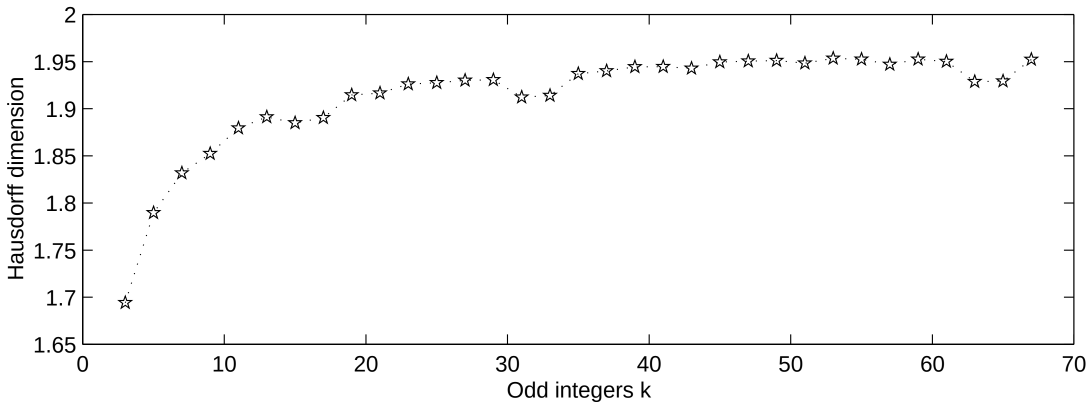

FIGURE 6.1. A scatter plot of the dimension of ∂T as a function of k for odd integers k, $3 \le k \le 67$, for $A = 2I$ and $\mathcal{D} = \{(0,0),(1,0),(0,1),(k,1)\}$.

Acknowledgments. Most of the research was carried out at Cornell University during the summer of 1997, with support from the Research Experiences for Undergraduates (REU) Program of the National Science Foundation, and from Cornell University (President's Discretionary Funding).

6. Computations of dimensions

We wrote a program to compute the dimension of the boundary of a self–similar tile. The program takes as input the similarity matrix A and the digit set $\mathcal{D}$. The program then finds $\mathcal{F}_2$ by the graph pruning algorithm of Theorem 2.3, and computes the matrix M. The spectral radius of M is computed by finding the roots of the characteristic polynomial and choosing the largest. The pruning step is useful in reducing the size of the matrix to make the last step feasible. A more sophisticated method of computing the spectral radius would allow one to handle much larger matrices.

We first looked at the case $A = 2I$. We chose the digit set to be $(0,0)$, $(1,0)$, $(0,1)$ and $(k,1)$ for k odd. A scatter plot of the data $\dim(\partial T)$ as a function of k is shown in Figure 6.1. The data shows the values of $\dim(\partial T)$ growing as k gets larger, but the increase is not monotonic in k. Significant dips are apparent at values $k = 2^m - 1$ and $k = 2^m + 1$. In fact, a similar phenomenon occurs in the one dimension [K]. In Figure 6.2 we give a similar scatter plot for the choice of digit set $(0,0)$, $(1,0)$, $(0,1)$, $(k,3)$ for k odd.

This leads us to conjecture that for fixed A, the dimension of ∂T approaches n as the digits go to infinity in an appropriate sense. However, it is not entirely trivial to make this precise, since if B is any matrix in $SL(n,\mathbb{Z})$ that commutes with A, then replacing $\mathcal{D}$ by $B\mathcal{D}$ transforms T to BT, and so leaves the dimension of the boundary unchanged, while the size of the digit set $B\mathcal{D}$ may become arbitrarily large.

One way to avoid this problem is to keep all the digits fixed except one.

Conjecture 6.1. *Fix A, and all digits of $\mathcal{D} = \{d_j\}$ except one, say d_1. Then* $\dim(\partial T) \to n$ *as* $|d_1| \to \infty$.

In the next section we will prove this conjecture under the additional hypothesis that $r > 2$, where r is the expansion ratio of A. It also holds for the case $A = 2I$ with the digits $(0,0)$, $(1,0)$ and $(0,1)$ held fixed, confirming the trend in our data. Incidentally, for $A = 2I$

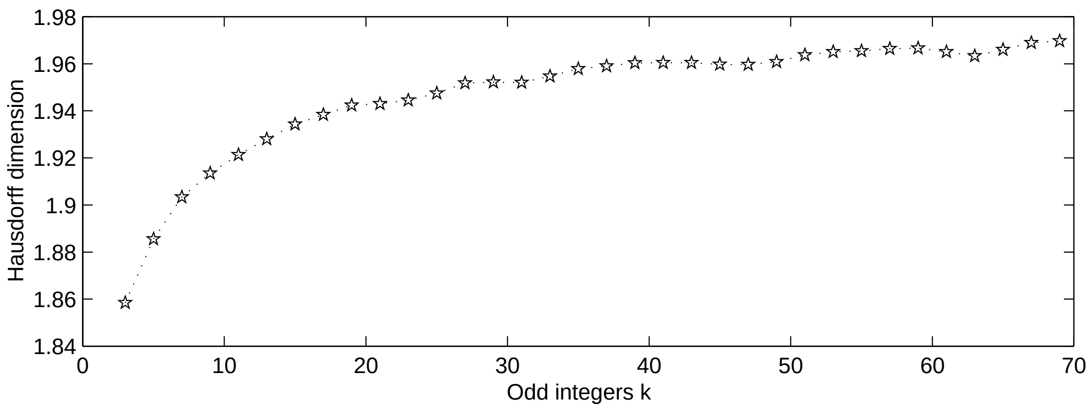

FIGURE 6.2. A scatter plot similar to Figure 6.1, for the digit set $\mathcal{D} = \{(0,0),(1,0),(0,1),(k,3)\}$.

the convergence of $\dim(\partial T)$ to 2 seems to occur at a faster rate if the last digit goes to infinity along a diagonal direction: we computed dimensions 1.9414812 and 1.9706678 for the last digit equal to $(9,9)$ and $(21,21)$, as compared to 1.9482966 for the last digit $(51,1)$.

At the other extreme, it appears that for each fixed A there is a minimum value for $\dim(\partial T)$, and it occurs when the digit set is very closely packed. However, it is not clear how to make this description precise. One naive idea is to take the digit set to be the lattice points lying in or on the square $A(S)$ where $S = [0,1)^2$. However, we found for $A = \left(\begin{smallmatrix} 3 & -1 \\ 1 & 3 \end{smallmatrix}\right)$ that there is a digit set that gives a lower dimension. Figure 6.3 shows the two digit sets and the corresponding dimensions; the reader can judge which of the two appears to be more closely packed. We also found for $A = \left(\begin{smallmatrix} 5 & -1 \\ 1 & 5 \end{smallmatrix}\right)$ that the apparent lowest dimension occurs for two choices of digit set that are not affinely equivalent. Both choices yield the same $\mathcal{F}_2$ set: $\{\pm(1,0), \pm(0,1), \pm(1,1)\}$, and the same M matrix:

$$\begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

The dimension of the boundary is 1.080509. The two digit sets are shown in Figure 6.4.

More generally, we can consider the matrix $A = \left(\begin{smallmatrix} a & -1 \\ 1 & a \end{smallmatrix}\right)$ for a an integer ($a \geq 3$). By choosing the digit set in $A(S)$ we obtain $\mathcal{F}_2 = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$, and by exploiting the $\pm$ symmetry we can reduce M to the 3×3 matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ a & a & 0 \\ a-2 & 1 & a-1 \end{pmatrix}$$

with characteristic polynomial

$$x^3 - 2ax^2 + (a^2+1)x - 2a. \tag{6.1}$$

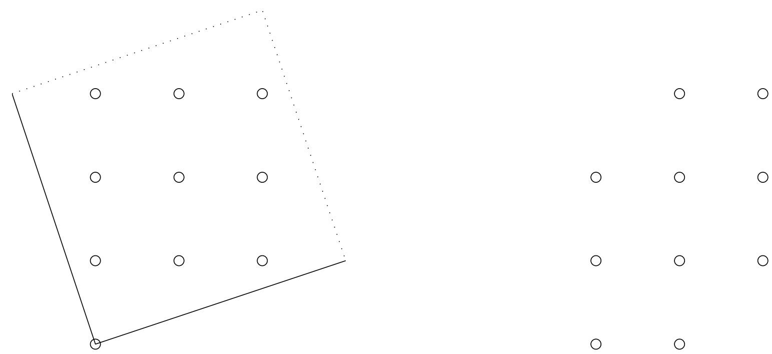

FIGURE 6.3. Two digit sets for $A = \left(\begin{smallmatrix}3 & -1\\ 1 & 3\end{smallmatrix}\right)$. The first is the inside of the square, with $\dim(\partial T) = 1.1525196$. The second substitutes the digit $(1,0)$ for $(0,3)$, and yields a lower dimension, 1.1353749. This appears to be the minimum for this matrix.

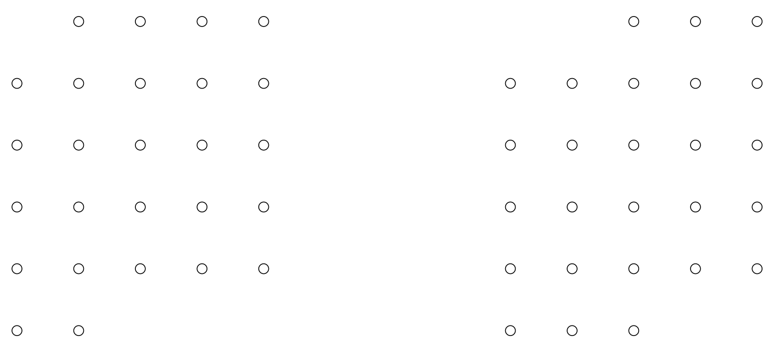

FIGURE 6.4. Two digit sets for the matrix $\left(\begin{smallmatrix}5 & -1\\ 1 & 5\end{smallmatrix}\right)$ that appear to yield the minimum dimension.

On the other hand, by modifying the digit set by replacing $(0, a)$ by $(1, 0)$, we obtain the same $\mathcal{F}_2$ and the reduced matrix is

$$\begin{pmatrix} 2 & 0 & 1 \\ a-1 & a & 0 \\ a-3 & 1 & a-1 \end{pmatrix}$$

with characteristic polynomial

$$x^3 - (2a+1)x^2 + (a^2+2a+1)x - (a^2+2a-1). \tag{6.2}$$

It is not difficult to show that the largest root of (6.2) is smaller than the largest root of (6.1). Suppose λ is the largest root of (6.1). It is not hard to verify that $a-1 < \lambda < a+1$ since (6.1) takes values ± 2 at $x = a \pm 1$ and the derivative is positive for $x > 0$. On the other hand the value of (6.2) for $x = \lambda$ is $-\lambda^2 + 2a\lambda - a^2 + 1 = 1 - (\lambda - a)^2 > 0$, so its largest root is smaller (the derivative of (6.2) is also positive for $x > 0$). It would be plausible to conjecture that this modified digit set produces the minimum for $\dim(\partial T)$ for this set of matrices.

A more modest conjecture that might be valid for all matrices A is that the minimum value for $\dim(\partial T)$ is always attained by a digit set belonging to a "small" finite collection characterized by some simple conditions. If such a conjecture were correct, then a reasonable algorithm for finding the minimum would be to search among this collection of digit sets. One reasonable condition on the digit set is that it be connected as a subset of $\mathcal{L}$, meaning that any two digits may be joined by a sequence of digits of unit distance apart. Modulo translation, there are only a finite number of connected digit sets.

Conjecture 6.2. *For fixed A, the minimum value for $\dim(\partial T)$ is attained for a connected digit set.*

Remark. Note that the converse of the conjecture is not true, since once we have one digit set attaining the minimum, we can find others simply by multiplying by a matrix B in $SL(n,\mathbb{Z})$ commuting with A, and these new digit sets may not be connected.

Bandt and Gelbrich [BG] study the related problem of when the tile is a topological disk.

7. Tiles with large boundaries

In order to show that the dimension of ∂T is large, we need a bound from below for the spectral radius of M. We will accomplish this by estimating the column sums of M. The first lemma shows that column sums are easy to compute.

Lemma 7.1. *Let $m(\beta) = \sum_{\alpha \in \mathcal{F}_2} M(\alpha,\beta)$ denote the column sums of M. Then*

$$m(\beta) = |\det A| - \mu(\beta)$$

where $\mu(\beta)$ denotes the multiplicity of β in $\mathcal{D}-\mathcal{D}$, i.e., the number of solutions of

$$\beta = d - d', \text{ for } d, d' \in \mathcal{D}.$$

Proof. Since $M_{\alpha\beta}$ is equal to the number of solutions of

$$\beta = A\alpha + d - d', \tag{7.1}$$

$m(\beta)$ is equal to the number of solutions of (7.1) as α varies over $\mathcal{F}_2$. Now given β and d', there is a unique α and d for which (7.1) holds. So (7.1) has $|\det A|$ solutions if we do not restrict α. But $0 \notin \mathcal{F}_2$, so this deletes $\mu(\beta)$ solutions. Also, it is easy to see that no other solutions are deleted, since if $\beta \in \mathcal{F}_2$ and $\alpha \neq 0$ satisfies (7.1), then α will never be removed in the pruning algorithm that generates $\mathcal{F}_2$, so $\alpha \in \mathcal{F}_2$. ∎

We obtain next a crude lower bound for the spectral radius $\rho(M)$ and the dimension of ∂T. We assume that A is a similarity, so $|\det A| = r^n$ where r is the expansion ratio for A.

Lemma 7.2. *Let μ denote the maximum value of $\mu(\beta)$ as β varies over $\mathcal{F}_2$. Equivalently, μ is the maximum cardinality of $\mathcal{D} \cap (\mathcal{D}+\beta)$ for $\beta \in \mathcal{F}_2$. Then*

$$\rho(M) \geq r^n - \mu \tag{7.2}$$

and

$$\dim(\partial T) \geq \log(r^n - \mu)/\log r. \tag{7.3}$$

Proof. Let e denote the row vector with all entries equal to 1. Then $eM \geq (r^n - \mu)e$ by Lemma 7.1. This implies (7.2), and then (7.3) follows by (2.8). ∎

To obtain better estimates we need to pass to iterates of the expansion identity (5.1). Thus we replace A by A^k and $\mathcal{D}$ by

$$\mathcal{D}_k = \mathcal{D} + A\mathcal{D} + A^2\mathcal{D} + \cdots + A^{k-1}\mathcal{D}.$$

Corollary 7.3. *Let μ_k denote the maximum cardinality of $\mathcal{D}_k \cap (\mathcal{D}_k + \beta)$ for β in $\mathcal{F}_2$. Then*

$$\dim(\partial T) \geq \log(r^{nk} - \mu_k)/\log r^k. \tag{7.4}$$

We note that $\mathcal{F}_2$ is defined in terms of the tile, not the expansion identity, so it does not vary with k. In applying Corollary 7.3 we will obtain estimates of the form

$$\mu_k \le \lambda r^{nk} \tag{7.5}$$

for a fixed $\lambda < 1$ and large k. Then (7.4) will give the estimate

$$\dim(\partial T) \ge n - \frac{\log(1-\lambda)^{-1}}{k \log r} \tag{7.6}$$

which makes $\dim(\partial T)$ close to n for large k.

Theorem 7.4. *Let T be a self–affine tile satisfying $A(T) = T + \mathcal{D}$ for some expanding similarity matrix $A \in M_n(\mathbb{Z})$ and digit set $\mathcal{D} \subset \mathcal{L}$ that is a residue system of $\mathcal{L}/A\mathcal{L}$. Suppose that the $\mathcal{L}$ translates of T tile $\mathbb{R}^n$ and the expansion ratio of A has $r > 2$. If $\mathcal{D} = \{d_j\}$ satisfies $\max_{j\ge 2} |d_j| \le b$ for some fixed constant $b > 0$, then $\dim(\partial T) \to n$ as $|d_1| \to \infty$.*

Proof. We may assume without loss of generality that 0 is in $\mathcal{D}$, say $d_2 = 0$. Write $B = |d_1|$. Since any β in $\mathcal{F}_2$ can be written

$$\beta = \sum_{k=1}^{\infty} A^{-k}(\delta_k - \delta'_k) \quad \text{for} \quad \delta_k, \delta'_k \in \mathcal{D},$$

we have the a priori bound

$$|\beta| \le (B+b)/(r-1) \quad \text{for} \quad \beta \in \mathcal{F}_2.$$

Since $r > 2$ this means

$$|\beta| \le \lambda_1 B \quad \text{for} \quad \beta \in \mathcal{F}_2 \tag{7.7}$$

for some fixed $\lambda_1 < 1$ if B is large enough.

We need to estimate the number of solutions of

$$\delta_0 + A\delta_1 + \cdots + A^{k-1}\delta_{k-1} = \beta + \delta'_0 + A\delta'_1 + \cdots + A^{k-1}\delta'_{k-1} \tag{7.8}$$

where δ_j, δ'_j are in $\mathcal{D}$ and β satisfies (7.7). We will choose k large, but not too large. Specifically, we choose λ_2 so that $\lambda_1 < \lambda_2 < 1$, and then require

$$2\Big(\frac{r^k - 1}{r-1}\Big) b \le (\lambda_2 - \lambda_1) B. \tag{7.9}$$

This still allows $k \to \infty$ as $B \to \infty$. We claim that (7.8) can hold only if

$$\delta_j = d_1 \Leftrightarrow \delta'_j = d_1 \text{ for all } j = 0, \dots, k-1. \tag{7.10}$$

We prove the claim by considering the largest value of j for which (7.10) fails to hold, say $j = m$. Then

$$\begin{aligned} A^m d_1 = \pm\beta &\pm (\delta_0 - \delta'_0) \pm A(\delta_1 - \delta'_1) \pm \cdots \pm A^{m-1}(\delta_{m-1} - \delta'_{m-1}) \\ &\pm A^m \delta_m \pm A^{m+1}(\delta_{m+1} - \delta'_{m+1}) \pm \cdots \pm A^{k-1}(\delta_{k-1} - \delta'_{k-1}) \end{aligned}$$

where $|\delta_j - \delta'_j| \le 2b$ for $j \ge m+1$, $|\delta_m| \le b$, and $|\delta_j - \delta'_j| \le B + b$ for $j \le m-1$. This yields

$$\begin{aligned} |A^m d_1| = r^m B &\le \lambda_1 B + (1 + r + \cdots + r^{m-1})(B+b) \\ &\quad + r^m b + 2(r^{m+1} + \cdots + r^{k-1}) b \\ &\le \Big(\lambda + \frac{r^m - 1}{r-1}\Big) B + \Big(\frac{r^k - 1}{r-1}\Big) 2b \\ &\le \Big(\lambda_2 + \frac{r^m - 1}{r-1}\Big) B \end{aligned}$$

by (7.9). This is a contradiction since

$$\lambda_2 + \frac{r^m - 1}{r - 1} < 1 + \frac{r^m - 1}{r - 1} \leq r^m.$$

Now we can establish the estimate (7.5) with $\lambda = 1 - r^{-n}$. Suppose first that β is not in $A\mathcal{L}$. Then (7.8) implies $\delta_0 - \delta_0'$ is not in $A\mathcal{L}$, hence $\delta_0 = \delta_0'$ is impossible. In particular $\delta_0 = d_1$ is not allowed. Since every choice of δ_0 determines a unique δ_0' that makes (7.8) valid mod $A\mathcal{L}$, we conclude that there are at most $r^n - 1$ choices for δ_0 and δ_0'. Again each choice of δ_k forces a unique choice of δ_k' for all k, so there are at most $(r^n - 1)r^{n(k-1)} = \lambda r^{nk}$ solutions to (7.8). Similarly, if β is in $A^m\mathcal{L}$ but not $A^{m+1}\mathcal{L}$ for some $m \leq k-1$ then we must have $\delta_j = \delta_j'$ for $j < m$, but $\delta_m \neq \delta_m'$ hence $\delta_m \neq d_1$, and so there are at most $r^n - 1$ choices for δ_m and δ_m'. Finally, if β is in $A^k\mathcal{L}$ then there are no solutions to (7.8) for $\beta \neq 0$.

Thus we have (7.6) holding under the assumption (7.9), so as $B \to \infty$ we have $k \to \infty$, hence $\dim(\partial T) \to n$. ∎

The theorem does not apply to the important example of $A = 2I$. However, in this case it is possible to show that the same conclusion holds if the fixed digits consist of the unit cube $(a_1, \dots, a_n)$ with each $a_j = 0$ or 1, with the vertex $(1, 1, \dots, 1)$ deleted. The argument (which we omit) is similar to the proof of the theorem, but a little more complicated. It does not appear likely that the same idea will work for matrices with $r < 2$. For example, if $\det A = 2$ then there are only two digits, so all tiles are similar.

8. Area of the convex hull

In this section and the next we return to the assumptions of Section 4: we assume that T satisfies (1.1) for some expanding matrix and some finite digit set $\mathcal{D}$, i.e., T is the attractor of the affine i.f.s. $\{S_j\}$ with $S_j x = A^{-1}(x + d_j)$, where all mappings have the same linear part. In addition, we assume $n = 2$. We also recall that $\mathcal{D}_0$ denotes the subset of $\mathcal{D}$ consisting of extreme points of P, where P denotes the convex hull of $\mathcal{D}$.

In this section we compute the area of K, the convex hull of T. We assume that the digits $d_1, \dots, d_N$ in $\mathcal{D}_0$ are arranged in counterclockwise order around P, and we set $d_0 = d_N$ to complete the circuit. In this notation, n_j is the outward normal to the segment joining d_{j-1} to d_j. If we choose u_j to be a positive multiple of $A^* n_j$, then $u_j \cdot A^{-1}d$ is maximized for $d = d_{j-1}$ and d_j. We define d_{jk} for $k \geq 2$ to be the digit that maximizes $u_j \cdot A^{-k}d$. This is well–defined under the generic assumption. We then define

$$\begin{cases} x_j' = A^{-1}d_j + \sum_{k=2}^{\infty} A^{-k} d_{jk}, & \text{and} \\ x_j'' = A^{-1}d_{j-1} + \sum_{k=2}^{\infty} A^{-k} d_{jk}. \end{cases} \tag{8.1}$$

Note that these points all lie on the boundary of K, hence determine a $2N$–polygon we will call P'. We have

$$x_j' - x_j'' = A^{-1}(d_j - d_{j-1}),$$

and so

$$\sum_{j=1}^{N} (x_j' - x_j'') = A^{-1}0 = 0.$$

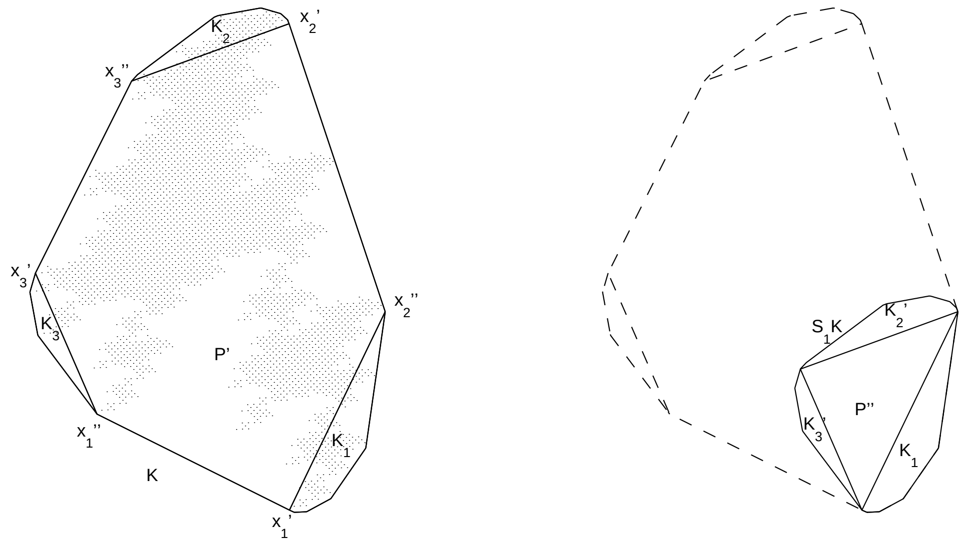

FIGURE 8.1. The scissors congruence of $K \setminus P' = K_1 \cup K_2 \cup K_3$ on the left with $S_1 K \setminus P''$ on the right by translating K_2 and K_3 to K_2' and K_3'. In this example $N = 3$.

It follows that

$$\sum_{j=0}^{N-1} (x''_{j+1} - x'_j) = 0 \quad (\text{here } x'_0 = x'_N).$$

That means there exists a convex N–polygon P'' whose sides are translates of $x''_{j+1} - x'_j$. In fact the vertices of P'' may be taken to be

$$\begin{cases} y_1 = x'_1 \\ y_2 = x''_2 \\ y_3 = x''_3 - (x'_2 - x''_2) \\ y_4 = x''_4 - (x'_2 - x''_2) - (x'_3 - x''_3) \\ \quad \cdots . \end{cases} \tag{8.2}$$

When $N = 2$, P'' degenerates to a line segment, and Area $P'' = 0$.

Theorem 8.1. *Let K be the convex hull of the nonempty compact self–affine set T satisfying $A(T) = T + \mathcal{D}$ for an expanding matrix $A \in M_2(\mathbb{R})$ and a finite digit set $\mathcal{D} \subset \mathbb{R}^2$. Let P', P'' be the convex polygons with vertices given by (8.1) and (8.2) respectively. Then under the generic assumption,*

$$\text{Area } K = \left(\frac{a}{a-1}\right)(\text{Area } P' - \text{Area } P'') \tag{8.3}$$

for $a = |\det A|$.

Proof. We will show the scissors congruence

$$K \setminus P' \cong S_1 K \setminus P'' \tag{8.4}$$

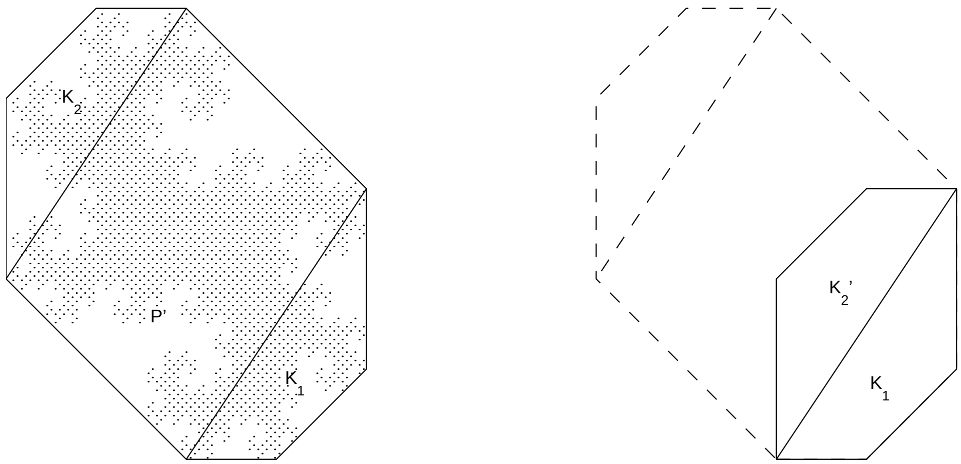

FIGURE 8.2. The twin dragon example, with $N = 2$. K is an octagon in this case, and P'' is a line segment. On the left $K \setminus P' = K_1 \cup K_2$, and on the right $S_1K = K_1 \cup K_2'$.

which immediately implies (8.3) since Area $S_1K = a^{-1}$ Area K. The proof of (8.4) is fairly evident from Figure 8.1, in which

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \qquad \mathcal{D} = \{(0,0),\ (0,1),\ (0,2)\ (1,1),\ (2,0)\}.$$

We break up $K \setminus P'$ into the N disjoint regions K_j defined by

$$K_j = (K \setminus P') \cap S_jK$$

which lie to the exterior of the line segment joining x_j' to x_{j+1}''. We then translate them so that their endpoints line up with y_j and y_{j+1}, so $K \setminus P'$ is scissors congruent to

$$K_1 \cup (K_2 - (x_2' - x_2'')) \cup (K_3 - (x_2' - x_2'') - (x_3' - x_3'')) \cup \cdots\ . \tag{8.5}$$

Note that the sets in (8.5) all lie in S_1K, and line up along the boundary of S_1K at the points $y_1, y_2, \ldots$. Thus the set (8.5) is equal to $S_1K \setminus P''$. ∎

If we drop the generic assumption, the digits d_{jk} are no longer well–defined since there may be ties. The same result will hold provided we resolve the ties in a consistent manner in (8.1) (in other words we pick a choice of d_{jk}, and use the same one to define x_j' and x_j''). This is illustrated in Figure 8.2 for the twin dragon. Of course if K is a polygon we can compute its area more directly in terms of its vertices.

9. Extreme points of the convex hull

In this section we give a description of the set E of extreme points of K, under the assumptions of Section 8. If K is a polygon then E is just a finite set, so we exclude this trivial case. If K is not a polygon, then we have shown in section 4 that its boundary contains a countable set of line segments. The endpoints of these segments belong to E, but it is convenient to exclude them from the discussion. The remaining points in E we denote by E', and it is easy to see that $E' \subseteq T$ and $E' = \{\widetilde{x} \in T :$ there exists u such that $x \cdot u$ has a unique maximum in T at $x = \widetilde{x}\}$.

Lemma 9.1. *For each $\widetilde{x} \in E'$ (with a finite number of exceptions) there exists a unique j such that $S_j\widetilde{x} \in E'$.*

Proof. We have $\widetilde{x} = \sum_{k=1}^{\infty} A^{-k} d_{j_k}$ and d_{j_k} is the unique maximizer for $((A^*)^{-k}u) \cdot d$ in $\mathcal{D}_0$. Then

$$S_j\widetilde{x} = A^{-1}d_j + \sum_{k=2}^{\infty} A^{-k} d_{j_{k-1}}.$$

For $\widetilde{u} = A^*u$ (normalized), we have $((A^*)^{-k}\widetilde{u}) \cdot d$ uniquely maximized for $d = d_{j_{k-1}}$ for $k \geq 2$. So $S_j\widetilde{x}$ uniquely maximizes $x \cdot \widetilde{u}$ if and only if d_j uniquely maximizes $d \cdot u$. There are only a finite number of vectors u for which there is more than one maximizer for $d \cdot u$. ∎

Write E'_j for the set of $\widetilde{x}$ in E' such that $S_j\widetilde{x} \in E'$. Then $E' = \cup E'_j$, disjoint except for point intersections, and moreover $E' = \cup S_j E'_j$ since the preimage of an extreme point is an extreme point. Since each S_j is a contraction, it follows immediately that E' has Hausdorff dimension zero, hence so does E. In particular, if A is a similarity, this proves that (4.1) is an equality.

Theorem 9.2. *Let T be a nonempty compact set satisfying $A(T) = T + \mathcal{D}$ for an expanding similarity matrix $A \in M_2(\mathbb{R})$ and a finite digit set $\mathcal{D} \subset \mathbb{R}^2$. Let K be the convex hull of T and P be the convex hull of $\mathcal{D}$. Then perimeter$(K) = \frac{1}{r-1}$ perimeter(P).*

Now assume A is a similarity. The condition that K is not a polygon means that the rotation angle $\widetilde{\theta}$ of A is an irrational multiple of 2π (we may assume A is orientation preserving, since if this is not the case we can always pass to A^2). We will give a more detailed description of the extreme set E'. For every unit vector $u_\theta = (\cos\theta, \sin\theta)$ not in a countable exceptional set, we associate a point x_θ in E', namely the unique maximizer of $x \cdot u_\theta$. If we denote by $\theta_1, \theta_2, \ldots, \theta_N$ the angles of the normals to the sides of P, in increasing order, and set $\theta_{N+1} = \theta_1 + 2\pi$, then the proof of Lemma 9.1 shows that $x_\theta \in E'_j$ for $\theta_j \leq \theta \leq \theta_{j+1}$, and x_{θ_j} is the unique point in $E'_{j-1} \cap E'_j$. We define a mapping $\widetilde{S}$ on E', except at the points x_{θ_j}, by

$$\widetilde{S}x = S_j x \quad \text{if} \quad x \in E'_j.$$

The proof of the Lemma also shows the intertwining property

$$\widetilde{S}x_\theta = x_{\theta - \widetilde{\theta}}.$$

In other words, the mapping $\theta \to x_\theta$ conjugates rotation through angle $-\widetilde{\theta}$ on the circle with the mapping $\widetilde{S}$ on E'. On each set E'_j, $\widetilde{S}$ is a contraction with ratio r^{-1}. Let μ denote the probability measure on E' that is the image of normalized Lebesgue measure on the circle under the mapping $\theta \to x_\theta$. This is an invariant measure with respect to the mapping $\widetilde{S}$, and the mapping is ergodic; both facts follow from the corresponding facts on the circle.

Although E is a set of dimension zero, since it is uncountable and compact, there must exist a gauge function h such that E has finite and positive Hausdorff h–measure $\mathcal{H}_h$ (see [R]). We will show that such a function must be close to $h(t) = (\log(1/t))^{-1}$. The measure μ will be used in the mass distribution principle for part of the explanation. Another point of view is to take the measure μ as the most natural object, and ask if there is a gauge function h such that $\mathcal{H}_h$ restricted to E is equivalent to μ.

Consider the continued fraction expansion of the number $\widetilde{\theta}/2\pi$, and let p_j/q_j denote the rational approximation (p_j and q_j relative prime) generated by the finite truncations of the expansion. We know

$$\frac{p_j}{q_j} \leq \frac{\widetilde{\theta}}{2\pi} < \frac{p_j}{q_j} + \frac{1}{q_j^2}, \tag{9.1}$$

which implies that if we divide up the circle into q_j intervals

$$\widetilde{J}_k = [2\pi k/q_j, 2\pi(k+1)/q_j),\ 0 \leq k \leq q_j - 1,$$

then the points $p\widetilde{\theta}$ for $0 \leq p \leq q_j - 1$ are distributed one to each interval: $p\widetilde{\theta} \in \widetilde{J}_k$ if and only if $p_j p \equiv k \bmod q_j$. This means that if we start at any point θ_* and generate the interval $\widetilde{I}_p = [\theta_* + p\widetilde{\theta}, \theta_* + p\widetilde{\theta} + 4\pi/q_j]$ for $0 \leq p \leq q_j - 1$, we have a covering of the circle with no point covered by more than 3 intervals. Denote by I_k the subset of E' corresponding to $\widetilde{I}_k$ under the map $\theta \to x_\theta$. We will continue to call these sets "intervals". Now we have

$$\mu(I_p) = 4\pi/q_j \text{ for all } p, \tag{9.2}$$

and

$$|I_{p+1}| = r^{-1}|I_p| \tag{9.3}$$

provided none of the points θ_j lies in $\widetilde{I}_p$, since $I_{p+1} = \widetilde{S}(I_p)$. (Here $|I_p|$ denotes the diameter of the interval I_p.) Note that we have a uniform upper bound of $3N$ for the number of intervals for which (9.3) fails to hold. As we will see, these exceptions do not materially affect the outcome. If θ_j belongs to $\widetilde{I}_p$ we can split I_p into two intervals $I_p = I'_p \cup I''_p$ by splitting $\widetilde{I}_p$ at θ_j and let $I'_{p+k} = \widetilde{S}^k(I'_p)$ and $I''_{p+k} = \widetilde{S}^k(I''_p)$ for $k \geq 1$, so $I_{p+k} = I'_{p+k} \cup I''_{p+k}$, and $|I'_{p+k}| \leq r^{-k}|I_p|$ and the same for I''_{p+k}. If we do this splitting every time we encounter θ_j in an interval, we end up with a covering of E' by at most $(3N+1)q_j$ intervals, all with diameter less than $|I_0|$.

By choosing the initial point θ_* carefully, we can make $|I_0|$ small. To see this we start out with an arbitrary choice, say $\theta_* = 0$. The idea is to repeatedly use (9.3), choosing a string of indices for which there are no obstacles. Since there are at most $3N$ obstacles among the q_j values of p, there must be a consecutive string of at least $(q_j/3N) - 1$ with no obstacles. Choosing p at the end of the string, we have

$$|I_p| \leq cr^{-q_j/3N} \tag{9.4}$$

with the constant c equal to r times the diameter of E'. Now redefine θ_* to make this interval be I_0.

Lemma 9.3. *Let $h(t)$ be any gauge function satisfying*

$$h(cr^{-q_j/3N}) \leq c_1 q_j^{-1} \text{ for all } j \tag{9.5}$$

(or just for an infinite sequence of values of j). Then $\mathcal{H}_h(E') < \infty$.

Proof. The coverings we have constructed contain at most $(3N+1)q_j$ intervals, each of diameter at most $cr^{-q_j/3N}$. Thus

$$\Sigma h(I) \leq (3N+1)q_j h(cr^{-q_j/3N}) \leq (3N+1)c_1,$$

and since $cr^{-q_j/3N} \to 0$ we obtain $\mathcal{H}_h(E') \leq (3N+1)c_1$. ∎

If we take equality in (9.5) this means $h(t) = c(\log(1/t))^{-1}$ for a sequence of t values going to 0, namely $t = cr^{-q_j/3N}$. However, the sequence of values q_j may be quite erratic, so it would not be wise to interpolate using the same formula. A better choice is

$$h(t) = \sup\{\mu(I) : |I| = t\}. \tag{9.6}$$

Theorem 9.4. *Let $h(t)$ be given by (9.6). Assume that there exists $\varepsilon > 0$ and an infinite sequence of indices j and $j' < j$ such that*

$$\varepsilon q_j \leq q_{j'} \leq q_j/4N. \tag{9.7}$$

Then $\mathcal{H}_h(E)$ is finite and positive, and the restriction of $\mathcal{H}_h$ to E' is equivalent to μ, in fact

$$\mu(A) \leq \mathcal{H}_h(A) \leq c_2\mu(A) \tag{9.8}$$

for any measureable set $A \subseteq E'$.

Proof. Since (9.6) implies

$$\mu(I) \leq h(|I|), \tag{9.9}$$

the mass distribution principle (see [Fa], p.24) gives the positivity of $\mathcal{H}_h(E')$ and the left inequality in (9.8). It is necessary to observe here that since E' is a subset of a C^1 curve, the boundary of K, all coverings used in computing the $\mathcal{H}_h$ measure may be taken to be intervals. Now if $A \subseteq \cup I_k$ with $\mathcal{H}_h(A)$ approximately equal to $\Sigma h(|I_k|)$, then (9.9) yields the left inequality in (9.8).

Next we will verify that (9.5) holds for j satisfying (9.7). We return to the covering constructed before the proof of Lemma 9.3 for the index j' associated to j in (9.7). This time we do not split intervals when we encounter obstacles, but simply observe that (9.3) can always be replaced by the inequality

$$|I_{p+1}| \geq r^{-1}|I_p|, \tag{9.10}$$

since the two maps $\widetilde{S}I'_p$ and $\widetilde{S}I''_p$ are moved apart after the contraction A^{-1}. Thus we have a covering of E' by intervals I_p satisfying

$$|I_p| \geq r^{-q_j}|I_0| \tag{9.11}$$

and

$$\mu(I_p) = 4\pi/q_{j'}. \tag{9.12}$$

Now we want to choose θ_* to make $|I_0|$ as large as possible. We can certainly arrange to have $|I_0| \geq c/q_{j'}$ for c independent of j' by choosing one of the $q_{j'}$ intervals I_p in the covering to be I_0, for if not then $|E'| \leq \Sigma|I_p|$ would be too small. By taking $q_{j'}$ large enough we can replace (9.11) by

$$|I_p| \geq r^{-(4/3)q_{j'}}, \tag{9.13}$$

and then using the right inequality in (9.7) we have

$$|I_p| \geq r^{-q_j/3N}. \tag{9.14}$$

On the other hand, using (9.12) and the left inequality in (9.7) we have

$$\mu(I_p) \leq 4\pi\varepsilon^{-1}/q_j. \tag{9.15}$$

Now suppose I is any interval satisfying $|I| = cr^{-q_j/3N}$. To prove (9.5) we need to show $\mu(I) \leq c_1 q_j^{-1}$ in view of (9.6). But this follows from (9.15), since I must be contained in a fixed number of I_p intervals (this again uses the fact that E' lies in a C^1 curve, hence diameters of adjacent small intervals are essentially additive). Since we have verified (9.5) for an infinite sequence of indices j, it follows from Lemma 9.3 that $\mathcal{H}_h(E')$ is finite.

To get the right inequality in (9.8) we observe first that it suffices to prove it for A an interval. Then we simply localize the argument in the proof of Lemma 9.3 to the interval A (select the intervals in the covering that meet A). ∎

There are two obvious concerns with this theorem. The first is that the formula (9.6) for the gauge function is not explicit. The second is that condition (9.7) depends on the continued fraction expansion of $\widetilde{\theta}/2\pi$, so there is at present no method of either proving it or disproving it for any given matrix A. Since the size of the ratio q_j/q_{j-1} is on the order of the corresponding continued fraction coefficient, condition (9.7) holds (with $j' = j-1$) whenever there are an infinite number of coefficients in the "moderate range". The upper inequality excludes very small values, and the lower inequality excludes very large values. (Of course (9.7) can also be satisfied with $j' < j-1$ if there are sequences of consecutive small values.) Since the control of ε in (9.7) allows the upper bound to be chosen at will, this condition appears to be generic, in that it should hold for a randomly chosen $\widetilde{\theta}$.

Part of the proof of the theorem shows that when the ratio q_j/q_{j-1} is not too large, we can interpolate the naive choice $h(t) = c(\log(1/t))^{-1}$ in between $cr^{-q_{j-1}/3N}$ and $cr^{-q_j/3N}$. We can also say something about the behavior of $h(t)$ in this range when q_j/q_{j-1} is large. In this case the excess of $\widetilde{\theta}/2\pi$ over p_{j-1}/q_{j-1} is on the order of $(q_j q_{j-1})^{-1}$, so the distribution of values $p\widetilde{\theta}$ for $0 \le p \le q_{j-1}-1$ is almost uniform, with $p\widetilde{\theta} \equiv 2\pi k/q_{j-1} + \varepsilon_p \mod 2\pi$ with $0 \le \varepsilon_p \le c/q_j$. This means that if we increase p beyond q_{j-1} the values of $p\theta$ will stay very close to $2\pi k/q_{j-1}$ also. More precisely, pick a small value of ε, and allow p to increase to εq_j. Then $p\widetilde{\theta}$ will exceed $2\pi k/q_{j-1}$ for the appropriate value of k by at most $\frac{\varepsilon q_j}{q_{j-1}} \cdot \frac{c}{q_j} = \frac{c\varepsilon}{q_{j-1}}$. If $c\varepsilon = 1/2$, then each interval $J_k = [2\pi k/q_{j-1}, 2\pi(k+1)/q_{j-1})$ will contain a subinterval of length at least π/q_{j-1} with no values of $p\widetilde{\theta} \mod 2\pi$ for $0 \le p \le \varepsilon q_j$. Now there are N obstacle values θ_i we are trying to avoid, so by choosing the starting value θ_* appropriately we can arrange to have all the intervals

$$I_p = [\theta_* + 2\pi p/q_{j-1}, \theta_* + 2\pi(p + 1/3N)/q_{j-1}]$$

for $0 \le p \le \varepsilon q_j$ avoid all N obstacles. Then since $I_p = \widetilde{S}^p(I_0)$ we have $|I_p| = r^{-p}|I_0|$ and in particular $|I_{\varepsilon q_j}| = cr^{-\varepsilon q_j}$, while $\mu(I_{\varepsilon q_j}) = 2\pi/3Nq_{j-1}$. This shows that

$$h(cr^{-\varepsilon q_j}) \ge 2\pi/3Nq_{j-1}. \tag{9.16}$$

This is considerably larger than the interpolated value $c/\varepsilon q_j$. In particular, if the set of continued fraction coefficients for $\widetilde{\theta}/2\pi$ is unbounded, then (9.16) shows that $h(t)$ is larger than $(\log(1/t))^{-1}$. Equivalently, if we used the gauge $(\log(1/t))^{-1}$ rather than (9.6), the Hausdorff measure of E' would be 0. Again, the assumption that the continued fraction coefficients are not bounded is true generically, but at present it cannot be decided in any particular instance.

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