Introduction

Recent work on sub–division schemes for surface generation [12]–[17], [6] and on wavelet surfaces [19] has led to the study of multi–dimensional *dilation equations*

$$f(\mathbf{x}) = \sum_{\boldsymbol{\eta}} c_{\boldsymbol{\eta}} f(2\mathbf{x} - \boldsymbol{\eta}) \tag{1}$$

Here $\boldsymbol{\eta} \in \mathbb{Z}^d$ is a multi–index, $\mathbf{x} \in \mathbb{R}^d$ and $\{c_{\boldsymbol{\eta}}\}$ are finitely supported *scaling coefficients.* In most applications the scaling coefficients satisfy the 2^d conditions

$$\sum_{\boldsymbol{\gamma}} c_{\boldsymbol{\eta}-2\boldsymbol{\gamma}} = 1, \quad \forall\, \boldsymbol{\eta} \tag{2}$$

Sub–division schemes proceed from a finitely supported set of control points $\mathbf{p}^0_{\boldsymbol{\eta}} \in \mathbb{R}^m$ defined on the integer lattice $\mathbb{Z}^d$. Points $\mathbf{p}^k_{\boldsymbol{\eta}}$ are defined on successively finer and finer grids $2^{-k}\mathbb{Z}^d$ through the recursion

$$p^k_{\boldsymbol{\eta}/2} = \sum_{\boldsymbol{\gamma}} c_{\boldsymbol{\eta}-2\boldsymbol{\gamma}} p^{k-1}_{\boldsymbol{\gamma}}$$

Observe that there are *two* spaces involved here: the *index space* $2^{-k}\mathbb{Z}^d$ which is sequentially being refined, over which the indices $\boldsymbol{\eta}$ of the points range; and the *physical space* $\mathbb{R}^m$ where the actual points $\{\mathbf{p}^k_{\boldsymbol{\eta}} : \boldsymbol{\eta} \in 2^{-k}\mathbb{Z}^d\}$ reside. Both of these spaces are compact, but they are not the same; i.e., the points $\mathbf{p}^k_{\boldsymbol{\eta}}$ need not be positioned over the index space in $2^{-k}\mathbb{Z}^d$ where $\boldsymbol{\eta}$ ranges.

If the denser and denser defined sets $\{\mathbf{p}^k_{\boldsymbol{\eta}}\}$ converge to form a continuous surface, then the scheme is said to be a *convergent sub–division scheme.* Algorithmically, since it is natural to work with integer and not rational indices, the grid is re–scaled each level, and the sub–division scheme is actually implemented on the re–scaled (fixed) grid $\mathbb{Z}^d$ as

$$\mathbf{p}^k_{\boldsymbol{\eta}} = \sum_{\boldsymbol{\gamma}} c_{\boldsymbol{\eta}-2\boldsymbol{\gamma}} \mathbf{p}^{k-1}_{\boldsymbol{\gamma}} \tag{3}$$

The scheme is convergent then if there exists a continuous function $g : \mathbb{R}^d \to \mathbb{R}^m$ such that

$$\sup_{\boldsymbol{\eta}} \left| \mathbf{p}^k_{\boldsymbol{\eta}} - g\left(\frac{\boldsymbol{\eta}}{2^k}\right) \right| \to 0 \quad \text{as } k \to \infty.$$

When the sets $\{\mathbf{p}_{\boldsymbol{\eta}}^k : \boldsymbol{\eta} \in \mathbb{Z}^d\}$ are nested, so that $\{\mathbf{p}_{\boldsymbol{\eta}}^0\} \subseteq \{\mathbf{p}_{\boldsymbol{\eta}}^1\} \subseteq \cdots$ then the scheme is *interpolatory.* The *normalized scaling function* f satisfying (1) and

$$\int_{\mathbb{R}^d} f = 1 \tag{4}$$

can be used to represent g as

$$g(\mathbf{x}) = \sum_{\boldsymbol{\eta}} f(x - \boldsymbol{\eta})\mathbf{p}_{\boldsymbol{\eta}}^0 \tag{5}$$

Examples of sub–division schemes abound. There is de Rham's construction [24], Chaikin's scheme [7], the schemes of Lane and Riesenfeld [21], Dyn et al.'s 4– and 6–point interpolatory schemes [13], [16], Dyn et al.'s surface schemes [14], [16], etc.

Just to illustrate the set–up described above, we present the scaling coefficients for Dyn et al.'s "butterfly scheme."

Table 1: Scaling coefficients $c_{\boldsymbol{\eta}}$ for Dyn et al.'s butterfly scheme [14].

Observe that the scaling coefficients have been partitioned into four groups, depending on the parity of $\boldsymbol{\eta}$ (even/even, even/odd, odd/even, odd/odd). Each set of coefficients sums to one, according to (2). The even/even set, marked by a circle, corresponds simply to the interpolatory condition

$$\mathbf{p}_{2i,2j}^k = \mathbf{p}_{i,j}^{k-1} \tag{6}$$

(or $\mathbf{p}^k_{i,j} = \mathbf{p}^{k-1}_{i,j}$ in terms of the non–rescaled grid $2^{-k}\mathbb{Z}^d$). The even/odd set, marked by triangles, corresponds to the refinement

$$\begin{aligned} \mathbf{p}^k_{2i,2j+1} = \frac{1}{2}\left(\mathbf{p}^{k-1}_{i,j} + \mathbf{p}^{k-1}_{i,j+1}\right) + 2w\left(\mathbf{p}^{k-1}_{i-1,j} + \mathbf{p}^{k-1}_{i+1,j+1}\right) \\ - w\left(\mathbf{p}^{k-1}_{i-1,j-1} + \mathbf{p}^{k-1}_{i-1,j+1} + \mathbf{p}^{k-1}_{i+1,j} + \mathbf{p}^{k-1}_{i+1,j+2}\right) \end{aligned} \tag{7}$$

In terms of the non–rescaled grid $2^{-k}\mathbb{Z}^d$, this refinement really defines the point $\mathbf{p}^k_{i,j+1/2}$, and is illustrated as follows:

Figure 2: The new point at $(i, j+1/2)$ is defined by (7) in terms of its neighbors. Here $i, j \in 2^{-k+1}\mathbb{Z}$.

Similarly the odd/even set, marked by squares, corresponds to the refinement

$$\begin{aligned} \mathbf{p}^k_{2i+1,j} = \frac{1}{2}\left(\mathbf{p}^{k-1}_{i,j} + \mathbf{p}^{k-1}_{i+1,j}\right) + 2w\left(\mathbf{p}^{k-1}_{i,j-1} + \mathbf{p}^{k-1}_{i+1,j+1}\right) \\ - w\left(\mathbf{p}^{k-1}_{i-1,j-1} + \mathbf{p}^{k-1}_{i+1,j-1} + \mathbf{p}^{k-1}_{i,j+1} + \mathbf{p}^{k-1}_{i+2,j+1}\right) \end{aligned} \tag{8}$$

and the odd/odd set, unmarked, corresponds to the refinement

$$\begin{aligned} \mathbf{p}^k_{2i+1,2j+1} = \frac{1}{2}\left(\mathbf{p}^{k-1}_{i,j} + \mathbf{p}^{k-1}_{i+1,j+1}\right) + 2w\left(\mathbf{p}^{k-1}_{i,j+1} + \mathbf{p}^{k-1}_{i+1,j}\right) \\ - w\left(\mathbf{p}^{k-1}_{i-1,j} + \mathbf{p}^{k-1}_{i,j-1} + \mathbf{p}^{k-1}_{i+1,j+2} + \mathbf{p}^{k-1}_{i+2,j+1}\right) \end{aligned} \tag{9}$$

In terms of the non–rescaled grid, these refinements really define the points $\mathbf{p}^k_{i+1/2,j}$ and

$\mathbf{p}^k_{i+1/2,j+1/2}$, respectively, as illustrated below.

Figure 3: The new point at $(i+1/2, j)$ is defined by (8) in terms of its neighbors. Here $i, j \in 2^{-k+1}\mathbb{Z}$.

Figure 4: The new point at $(i+1/2, j+1/2)$ is defined by (9) in terms of its neighbors. Here $i, j \in 2^{-k+1}\mathbb{Z}$.

The one refinement equation (3), then, actually gives rise to the four equations (6)–(9). The shape of Figures 2–4 accounts for the name "butterfly scheme."

If we start with control points

$$\{\mathbf{p}^0_{i,j} : -2 \le i, j \le 3\}$$

then the evolution of the index grids (for the non–rescaled refinement) is depicted in Figure 5.

Figure 5: Evolution of the index grid as the sub–division algorithm proceeds.

Some of the original boundary gets lost, so to render closed surfaces for example, one needs to have sufficient "wrap–around." That is, the physical points $\mathbf{p}$ near the boundary of the index grid have to coincide. In the limit as $k \to \infty$ one expects a surface to be generated, as a function $g : [0,1]^2 \to \mathbb{R}^3$. Observe that for this "butterfly scheme," the dilation equation (1) takes the form

$$\varphi(x,y) = \varphi(2x,2y) + \frac{1}{2}\varphi^{(1,0)}(x,y) + 2w[\varphi^{(1,-1)}(x,y) + \varphi^{(1,2)}(x,y)] \tag{10}$$

$$-w[\varphi^{(1,-2)}(x,y) + \varphi^{(1,3)}(x,y) + \varphi^{(2,3)}(x,y)]$$

where

$$\varphi^{(i,j)}(x,y) = \sum_{(i',j') \in S_{i,j}} \varphi(2x - i', 2y - j')$$

and

$$S_{i,j} = \{(i,j),(j,i),(-i,-j),(-j,-i)\}.$$

There is, of course, a lot of symmetry here. Some examples of surfaces constructed from the "butterfly scheme" are given in Figure 6 (at the end of this article).

In Section 1 below we study existence, uniqueness and regularity of solutions to (1). We show that if $\sum_{\boldsymbol{\eta}} c_{\boldsymbol{\eta}} = 2^{d+k}$ then one typically has $\binom{k+d-1}{k}$ linearly independent solutions to (1) — the same as the number of k^{th} order partial derivatives a function of d variables has. (See Theorem II(c).)

In Section 2 we describe an affine *iterated function system* (IFS) algorithm for constructing the normalized solution of (1) through an IFS attractor. This relates to the matrix product expansion approach developed in [4], [10], [11], [22], [23]. An affine IFS [1] consists of affine transformations $T_i : \mathbb{R}^M \to \mathbb{R}^M$, $i = 1, \ldots, N$; and it generates a discrete–time dynamical system $(\mathbf{X}_n)$ in $\mathbb{R}^M$ according to

$$\mathbf{X}_n = T_{\omega_n} \mathbf{X}_{n-1}$$

where (ω_n) is an appropriately chosen sequence of indices $\omega \in \{1, \ldots, N\}$. This sequence (ω_n) is said to *drive* the dynamics. In [2], [3], [5] we developed an IFS algorithm for the 1–D dilation equation, and applied it to construct compactly supported wavelets [8] and to generate curves through sub–division schemes. Here we present the multi–dimensional version of this work, geared to construct multi–dimensional wavelets and to generate surfaces through sub–division schemes.

Haar–type multi–dimensional wavelets are considered by Gröchenig and Madych [19], and they too work with IFS and fractal sets. But their IFS is different than ours — it is used to describe the support of f, as in Lemma I below, rather than generate the function f itself. Thus for equation (1), the IFS producing *supp* f is

$$w_{\boldsymbol{\eta}} : \mathbf{x} \mapsto \frac{\mathbf{x}+\boldsymbol{\eta}}{2}$$

where $\boldsymbol{\eta} \in \mathbb{Z}^d$ is in the support for $(c_{\boldsymbol{\eta}})$. (That is, $\boldsymbol{\eta}$ is such that $c_{\boldsymbol{\eta}} \neq 0$.) In Gröchenig and Madych's case where all the scaling coefficients are positive (in fact they are all equal), one can conclude that *supp* f is precisely the attractor for the IFS $\{w_{\boldsymbol{\eta}}\}$ (see the remark following Lemma I), but in general one can only infer inclusion.

Gröchenig and Madych have a nice extension of (1) which they study,

$$f(\mathbf{x}) = \sum_{\boldsymbol{\eta}} c_{\boldsymbol{\eta}} f(A\mathbf{x} - \boldsymbol{\eta}) \tag{11}$$

where $A \in Lin(\mathbb{R}^d)$ is an "acceptable dilation." We do not pursue this generalization here, but the interested reader will find that most of our techniques and results carry over to this setting as well.

§1. **Existence, Uniqueness, Regularity, etc.**

Consider the d–dimensional *two–scale dilation equation*

$$f(\mathbf{x}) = \sum_{\boldsymbol{\eta}} c_{\boldsymbol{\eta}} f(\alpha\mathbf{x} - \boldsymbol{\beta}_{\boldsymbol{\eta}}) \tag{12}$$

where $\boldsymbol{\eta} \in \mathbb{Z}^d$ is a multi–index, $\mathbf{x} \in \mathbb{R}^d$ and the *scaling coefficients* $(c_{\boldsymbol{\eta}})$ are finitely supported. The $c_{\boldsymbol{\eta}}$'s can be complex numbers, but the $\boldsymbol{\beta}_{\boldsymbol{\eta}}$'s and α are assumed to be real, and moreover $\alpha > 1$. Correspondingly let P be the multi–dimensional trigonometric polynomial

$$P(\boldsymbol{\xi}) = \frac{1}{\alpha^d} \sum_{\boldsymbol{\eta}} c_{\boldsymbol{\eta}} e^{i\langle \boldsymbol{\beta}_{\boldsymbol{\eta}}, \boldsymbol{\xi}\rangle}, \quad \boldsymbol{\xi} \in \mathbb{C}^d \tag{13}$$

If $f \in L^1(\mathbb{R}^d)$ satisfies (12) then its Fourier transform $\widehat{f}$ satisfies

$$\widehat{f}(\boldsymbol{\xi}) = P(\boldsymbol{\xi}/\alpha)\widehat{f}(\boldsymbol{\xi}/\alpha) \tag{14}$$

and conversely if $\widehat{f}$ satisfies (14) then its inverse transform (which may be a distribution) satisfies (12). Set

$$\Delta = P(\mathbf{0}) = \frac{1}{\alpha^d}\sum_{\boldsymbol{\eta}} c_{\boldsymbol{\eta}}$$

If $f \in L^1(\mathbb{R}^d)$ satisfies (12) then $\int f = 0$ whenever $\Delta \neq 1$.

Suppose $\Delta = 1$. We can estimate

$$\left|e^{i\langle\boldsymbol{\beta},\boldsymbol{\xi}\rangle} - 1\right| = \left|i\langle\boldsymbol{\beta},\boldsymbol{\xi}\rangle\int_0^1 e^{i\langle\boldsymbol{\beta},\boldsymbol{\xi}\rangle t}dt\right| \leq |\langle\boldsymbol{\beta},\boldsymbol{\xi}\rangle|\int_0^1 |e^{i\langle\boldsymbol{\beta},\boldsymbol{\xi}\rangle t}|dt \leq |\langle\boldsymbol{\beta},\boldsymbol{\xi}\rangle|e^{B(-\mathrm{Im}\,\boldsymbol{\xi})} \quad (\boldsymbol{\beta} \in \mathbb{R}^d, \boldsymbol{\xi} \in \mathbb{C}^d)$$

where $B(\mathbf{x}) = \max(0, \langle\boldsymbol{\beta},\mathbf{x}\rangle)$; and also

$$|e^{i\langle\boldsymbol{\beta},\boldsymbol{\xi}\rangle} - 1| \leq 1 + e^{B(-\mathrm{Im}\,\boldsymbol{\xi})} \leq 2e^{B(-\mathrm{Im}\,\boldsymbol{\xi})}$$

These lead to

$$|P(\boldsymbol{\xi}) - 1| \leq A_1|\boldsymbol{\xi}|e^{B(-\mathrm{Im}\,\boldsymbol{\xi})} \tag{15}$$

$$|P(\boldsymbol{\xi}) - 1| \leq A_2 e^{B(-\mathrm{Im}\,\boldsymbol{\xi})} \tag{16}$$

where $A_1 = \sum_{\boldsymbol{\eta}} |c_{\boldsymbol{\eta}}\boldsymbol{\beta}_{\boldsymbol{\eta}}|$, $A_2 = 2\sum_{\boldsymbol{\eta}} |c_{\boldsymbol{\eta}}|$ and $B(\mathbf{y}) = \max_{\boldsymbol{\eta}}(0, \langle\boldsymbol{\beta}_{\boldsymbol{\eta}}, \mathbf{y}\rangle)$. The estimate (15) guarantees that $\prod_{m=1}^{\infty} P(\boldsymbol{\xi}/\alpha^m)$ converges uniformly on compact subsets of $\mathbb{C}^d$ to an entire function. If $f \in L^1(\mathbb{R}^d)$ satisfies (12) then it follows from (14) that

$$\widehat{f}(\boldsymbol{\xi}) = \widehat{f}(0)\prod_{m=1}^{\infty} P(\boldsymbol{\xi}/\alpha^m) \tag{17}$$

Let C be a bound for $\prod_{m=1}^{\infty} P(\boldsymbol{\xi}/\alpha^m)$ on the unit disk $|\boldsymbol{\xi}| \leq 1$, say from (15) $C = \exp\left(\dfrac{A_1 e^{\beta}}{\alpha - 1}\right)$, $\beta = \max|\boldsymbol{\beta}_{\boldsymbol{\eta}}|$. For $|\boldsymbol{\xi}| > 1$ let k be such that $\alpha^k < |\boldsymbol{\xi}| \leq \alpha^{k+1}$. Then using (16) we can estimate

$$\left|\prod_{m=1}^{\infty} P(\boldsymbol{\xi}/\alpha^m)\right| \leq C\prod_{m=1}^{k+1}\left[1 + A_2 e^{\frac{B(-\mathrm{Im}\,\boldsymbol{\xi})}{\alpha^m}}\right] \leq C(1+A_2)^{k+1}e^{\frac{B(-\mathrm{Im}\,\boldsymbol{\xi})}{\alpha-1}} \leq C'|\boldsymbol{\xi}|^M e^{\frac{B(-\mathrm{Im}\,\boldsymbol{\xi})}{\alpha-1}}$$

where $M = \log_\alpha(1 + A_2)$. Thus we have the global estimate

$$\left|\prod_{m=1}^{\infty} P(\boldsymbol{\xi}/\alpha^m)\right| \leq C'(1+|\boldsymbol{\xi}|)^M e^{\frac{B(-\mathrm{Im}\,\boldsymbol{\xi})}{\alpha-1}} \tag{18}$$

Recall the

Payley-Wiener Theorem. *Let K be a convex compact subset of $\mathbb{R}^d$ with supporting function*

$$H(\mathbf{y}) = \max_{\mathbf{x}\in K}\langle \mathbf{x}, \mathbf{y}\rangle$$

Every entire analytic function $u(\boldsymbol{\xi})$ in $\mathbb{C}^d$ satisfying an estimate

$$|u(\boldsymbol{\xi})| \leq C(1+|\boldsymbol{\xi}|)^M e^{H(-\mathrm{Im}\,\boldsymbol{\xi})}$$

is the Fourier transform of a distribution of order M supported in K.

It thus follows from (17), (18) that if $f \in L^1(\mathbb{R}^d)$ satisfies (12) then

$$\mathit{supp}\ f \subseteq \frac{K}{\alpha - 1}, \quad \text{where } K = \mathit{conv{-}hull}(\boldsymbol{\beta_\eta})$$

Indeed, shifting f by $\dfrac{\boldsymbol{\beta_\eta}}{\alpha-1}$ effectively shifts $\boldsymbol{\beta_\eta}$ to zero, in which case $B(\mathbf{y}) = \max_{\boldsymbol{\eta}}\langle \boldsymbol{\beta_\eta}, \mathbf{y}\rangle$ is the support function for K.

We can say more about the support of compactly supported solutions of (12). For $\boldsymbol{\eta} \in \mathbb{Z}^d$ let $w_{\boldsymbol{\eta}} : \mathbb{R}^d \to \mathbb{R}^d$ be the strictly contractive affine transformation

$$w_{\boldsymbol{\eta}} : \mathbf{x} \mapsto \frac{\mathbf{x} + \boldsymbol{\beta_\eta}}{\alpha}$$

The system $\{w_{\boldsymbol{\eta}}\}$ forms an *iterated function system* (IFS) in $\mathbb{R}^d$ [1]. Such a system of strictly contractive affine transformations always possesses an *attractor* $\mathcal{A} \subseteq \mathbb{R}^d$, which is the unique non–empty compact set satisfying

$$\mathcal{A} = \bigcup_{\boldsymbol{\eta}} w_{\boldsymbol{\eta}}(\mathcal{A})$$

This set is the minimal non–empty closed set which is invariant under each $w_{\boldsymbol{\eta}}$. It can be constructed as $\mathcal{A} = \lim \mathcal{A}_k$ in the Hausdorff metric, where $\mathcal{A}_{k+1} = \bigcup_{\boldsymbol{\eta}} w_{\boldsymbol{\eta}}(\mathcal{A}_k)$ and $\mathcal{A}_0 \subseteq \mathbb{R}^d$ is any arbitrary non–empty compact set.

Lemma I. *Let f be a measurable compactly supported solution of (12). Then supp $f \subseteq \mathcal{A}$, where $\mathcal{A}$ is the attractor for the IFS $\{w_{\boldsymbol{\eta}}\}$.*

Proof. Let $\Omega = \mathit{supp}\ f$. Observe that $\mathit{supp}\ f(\alpha\mathbf{x} - \boldsymbol{\beta_\eta}) = w_{\boldsymbol{\eta}}(\Omega)$. Thus (12) implies that $\Omega \subseteq \bigcup_{\boldsymbol{\eta}} w_{\boldsymbol{\eta}}(\Omega)$, from which it follows that $\Omega \subseteq \mathcal{A}$. □

Clearly $\dfrac{1}{\alpha-1}\,conv\!-\!hull(\boldsymbol{\beta_\eta})$ is invariant under each $w_{\boldsymbol{\eta}}$, since $\dfrac{\boldsymbol{\beta_\eta}}{\alpha-1}$ is the fixed point of $w_{\boldsymbol{\eta}}$. Thus $\mathcal{A} \subseteq \dfrac{K}{\alpha-1}$ where $K = conv\!-\!hull(\boldsymbol{\beta_\eta})$, and Lemma I sharpens the support estimate we picked up from the Payley-Wiener Theorem. In fact many times $\mathcal{A}$ is a fractal Cantor–like set. For example if $\alpha = d = 2$ and if the $\boldsymbol{\beta_\eta}$'s are $\binom{0}{0}, \binom{1}{0}, \binom{0}{1}$ then $\mathcal{A}$ is a totally disconnected Sierpinski triangle with vertices $\boldsymbol{\beta_\eta}$ (see [1]); whereas if α is reduced to $3/2$ then $\mathcal{A}$ is the solid triangle with vertices $\binom{0}{0}, \binom{2}{0}, \binom{0}{2}$. In either case the Payley-Wiener Theorem only gives a solid triangle as bounding $supp\ f$. Gröchenig and Madych [19] have some nice illustrations of $supp\ f$. When $d = 1$ we find that $supp\ f \subseteq \left[\frac{\beta_{\min}}{\alpha-1}, \frac{\beta_{\max}}{\alpha-1}\right]$. The IFS $\{w_{\boldsymbol{\eta}}\}$ is typically overlapping (see [1]). For example the IFS $\left\{\frac{x+n}{\alpha} : n = 0, \ldots, N\right\}$, which comes from the $1-D$ dilation equation

$$f(x) = \sum_{n=0}^{N} c_n f(\alpha x - n),$$

is overlapping whenever $N \geq \alpha$. When the coefficients $c_{\boldsymbol{\eta}}$ are all non–negative, $\boldsymbol{\eta} \in \mathbb{Z}^d$, and $\Delta = 1$, then the normalized solution to (12), $\int f = 1$, is the invariant pdf for the IFS with probabilities [1]

$$\mathbb{P}(w = w_{\boldsymbol{\eta}}) = \frac{c_{\boldsymbol{\eta}}}{\alpha^d}$$

We next study existence and uniqueness of compactly supported solutions $f \in L^1(\mathbb{R}^d)$ to (12). It is no longer assumed that $\Delta = 1$, but in any event $\frac{1}{\Delta}P(\boldsymbol{\xi})$ satisfies the estimates (15), (16), (18).

Theorem II. *(a) If $|\Delta| \leq 1$, $\Delta \neq 1$ then the only $L^1(\mathbb{R}^d)$ solution to (12) is $f \equiv 0$.*

(b) If $\Delta = 1$ and (12) has a non–trivial $L^1(\mathbb{R}^d)$ solution f, then f is unique up to scale and $\widehat{f}$ is given by (17).

(c) If $|\Delta| > 1$ then a necessary condition for (12) to have a non–trivial compactly supported $L^1(\mathbb{R}^d)$ solution is $\Delta = \alpha^k$, for some $k \in \mathbb{Z}_+$. In this case

$$\widehat{f}(\boldsymbol{\xi}) = h(\boldsymbol{\xi}) \prod_{m=1}^{\infty} \frac{P(\boldsymbol{\xi}/\alpha^m)}{\Delta} \tag{19}$$

where h is a homogeneous polynomial of degree k.

Proof. (a) If $f \in L^1(\mathbb{R}^d)$ satisfies (12) then it follows from (14) that

$$\widehat{f}(\boldsymbol{\xi}) = \lim_{\ell\to\infty} \left[\prod_{m=1}^{\ell} P(\boldsymbol{\xi}/\alpha^m)\right] \widehat{f}(\boldsymbol{\xi}/\alpha^\ell) \tag{20}$$

If $|\Delta| < 1$ then $\prod_{m=1}^{\ell} P(\boldsymbol{\xi}/\alpha^m) \to 0$, and so this limit is zero. If $|\Delta| = 1$, $\Delta \neq 1$, we write

$$\widehat{f}(\boldsymbol{\xi}) = \lim_{\ell\to\infty} \Delta^{\ell} \left[\prod_{m=1}^{\ell} \frac{P(\boldsymbol{\xi}/\alpha^m)}{\Delta} \right] \widehat{f}(\boldsymbol{\xi}/\alpha^{\ell})$$

Since $\Delta \neq 1$, we get that $\widehat{f}(\mathbf{0}) = 0$, and since $\widehat{f}$ is continuous we get that $\widehat{f}(\boldsymbol{\xi}/\alpha^{\ell}) \to 0$, and so this limit is zero, too.

(b) Iterating (14) leads to the formula for $\widehat{f}$.

(c) Suppose $\Delta = \alpha^k \delta$, $1 < |\delta| \leq \alpha$, $\delta \neq \alpha$. We use induction on $n = k + d$ to show that f must be identically zero. If $k < 0$ then we can use (a) above, and so this covers the case $n \leq 0$.

If $d = 1$ then $\int f = 0$ since $\Delta \neq 1$. If $d > 1$ then define

$$f_i(\widehat{\mathbf{x}}) = \int_{-\infty}^{\infty} f(\mathbf{x})dx_i$$

where $\widehat{\mathbf{x}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Observe that f_i is a compactly supported $L^1(\mathbb{R}^{d-1})$ solution to the $(d-1)$–dimensional dilation equation

$$f_i(\widehat{\mathbf{x}}) = \frac{1}{\alpha} \sum_{\boldsymbol{\eta}} c_{\boldsymbol{\eta}} f_i(\alpha\widehat{\mathbf{x}} - \widehat{\boldsymbol{\beta}}_{\boldsymbol{\eta}})$$

where $\widehat{\boldsymbol{\beta}}$ is defined analogously to $\widehat{\mathbf{x}}$. For this equation the Δ would equal $\dfrac{1}{\alpha^{d-1}} \dfrac{1}{\alpha} \sum_{\boldsymbol{\eta}} c_{\boldsymbol{\eta}}$, which is the same as the original $\Delta = \alpha^k \delta$. But the dimension drops from d to $d-1$, so that $n = k + d - 1$. Hence the induction hypothesis applies, and we get that $f_i \equiv 0$ for each $i = 1, \ldots, d$.

Next define

$$F(\mathbf{x}) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} f(\mathbf{y})d\mathbf{y}$$

Since $f_i \equiv 0$, $\forall\ i$, it follows that F has compact support. Moreover F is in $L^1(\mathbb{R}^d)$ since f is compactly supported, and it satisfies the dilation equation (12) with scaling coefficients $\dfrac{1}{\alpha^d} c_{\boldsymbol{\eta}}$. This brings the n down from $k + d$ to k, and so the induction hypothesis again applies to conclude that $F \equiv 0$. ¿From this follows that $f \equiv 0$.

Suppose next that $\Delta = \alpha^k$ for some $k \in \mathbb{Z}_+$. Define

$$h(\boldsymbol{\xi}) = \widehat{f}(\boldsymbol{\xi}) / \prod_{m=1}^{\infty} \frac{P(\boldsymbol{\xi}/\alpha^m)}{\Delta}$$

Observe that h satisfies

$$h(\alpha\boldsymbol{\xi}) = \alpha^k h(\boldsymbol{\xi}) \tag{21}$$

at all points $\boldsymbol{\xi}$ where $\prod_{m=1}^{\infty} \frac{P(\boldsymbol{\xi}/\alpha^m)}{\Delta}$ is non–zero; and that $\prod_{m=1}^{\infty} \frac{P(\boldsymbol{\xi}/\alpha^m)}{\Delta}$ is bounded away from zero near $\boldsymbol{\xi} = \mathbf{0}$. Let $S \subseteq \mathbb{C}^d$ denote the zero set $S = \{\boldsymbol{\xi} \in \mathbb{C}^d : \prod_{m=1}^{\infty} \frac{P(\boldsymbol{\xi}/\alpha^m)}{\Delta} = 0\}$. If $(\boldsymbol{\xi}_n)$ is a sequence in $\mathbb{C}^d \backslash S$ with $\lim \boldsymbol{\xi}_n = \boldsymbol{\xi}^* \in S$ then $\lim h(\boldsymbol{\xi}_n)$ exists. Indeed $\alpha^{-\ell}\boldsymbol{\xi}_n \to \alpha^{-\ell}\boldsymbol{\xi}^*$, and for ℓ large enough these points will all be in a neighborhood of $\boldsymbol{\xi} = \mathbf{0}$ where h is continuous. So by (21) $h(\boldsymbol{\xi}_n) \to \alpha^{k\ell} h(\alpha^{-\ell}\boldsymbol{\xi}^*)$. By the Riemann Removable Singularity Theorem we get that h is analytic. Then by matching coefficients in the power series expansions of both sides in (21), we find that h must be a homogeneous polynomial of degree k. $\square$

Remark: If we extend our considerations to non–compactly supported solutions $f \in L^1(\mathbb{R}^d)$ of (12), then the restriction $\Delta = \alpha^k$ in Theorem II(c) gets removed. The same argument as above can be used to show that if $f \in L^1(\mathbb{R}^d)$ satisfies (12), with $|\Delta| > 1$, then

$$\widehat{f}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{\log_\alpha \Delta} G(\boldsymbol{\xi}) \prod_{m=1}^{\infty} \frac{P(\boldsymbol{\xi}/\alpha^m)}{\Delta} \tag{22}$$

where G is continuous on $\mathbb{C}^d \backslash 0$ and satisfies $G(\alpha\boldsymbol{\xi}) = G(\boldsymbol{\xi})$. If $d = 1$ this means that

$$G(x) = g_{\operatorname{sgn} x}(\log_\alpha |x|), \quad x \in \mathbb{R}$$

where $g_\pm$ are continuous periodic functions of period one. This is consistent with [9, Thm. 2.1].

¿From Theorem II(c) we conclude that *the solution space in $L^1(\mathbb{R}^d)$ for (12) is at most $\binom{k+d-1}{k}$–dimensional when $\Delta = \alpha^k$*, since this is the number of different monomials of degree k in d variables. One case where this dimension is achieved occurs when the dilation equation is a *tensor product* – i.e., when the scaling coefficients factor as

$$c_{\boldsymbol{\eta}} = c^{(1)}_{\eta_1} \cdots c^{(d)}_{\eta_d}$$

and also

$$\boldsymbol{\beta}_{\boldsymbol{\eta}} = \left(\beta^{(1)}_{\eta_1}, \ldots, \beta^{(d)}_{\eta_d}\right).$$

In this case the scaling function itself also factors as

$$f(\mathbf{x}) = f^{(1)}(x_1) \cdots f^{(d)}(x_d)$$

where each $f^{(i)}$ is the univariate scaling function satisfying

$$f^{(i)}(x) = \sum_{\eta_j} c^{(i)}_{\eta_j} f^{(i)}(\alpha x - \beta^{(i)}_{\eta_j})$$

By choosing the scaling coefficients so that $\Delta = 1$ and so that each $f^{(i)}$ is k times differentiable (see [8], [25] for how to ensure this), the k^{th} order partial derivatives of f will each satisfy the dilation equation with coefficients $\alpha^k c_{\boldsymbol{\eta}}$. By choosing the $c^{(i)}$'s so that fewer than k derivatives of $f^{(i)}$ exist, one can contrive things so that the solution space for the dilation equation with coefficients $\alpha^k c_{\boldsymbol{\eta}}$ has fewer than $\binom{k+d-1}{k}$ independent solutions in $L^1(\mathbb{R}^d)$.

Suppose f satisfies (12). If $f \in C^k$ then each of its k^{th} order partial derivatives is a solution of the corresponding dilation equation with scaling coefficients $\alpha^k c_{\boldsymbol{\eta}}$. The Δ for this latter dilation equation is α^k times the Δ for the equation which f satisfies; and this is consistent with (19) since differentiation with respect to x_j corresponds to multiplication by $-i\xi_j$ in the spectral domain. The converse is more involved, however – at least in higher dimensions. *Suppose $f \in L^1(\mathbb{R}^d)$ is a compactly supported solution of (12) with $\Delta \geq \alpha^k$, and $\widehat{f}(\boldsymbol{\xi}) = h(\boldsymbol{\xi})u(\boldsymbol{\xi})$ where h is a homogeneous polynomial of degree k and u is entire analytic. Is u necessarily the Fourier transform of a compactly supported function $g \in L^1(\mathbb{R}^d)$ satisfying the corresponding dilation equation with scaling coefficients $\frac{c_{\boldsymbol{\eta}}}{\alpha^k}$?* If so then we would have, of course, $f = h(i\boldsymbol{\nabla})g$ in the sense of distribution.

If $h(\boldsymbol{\xi}) = \langle \boldsymbol{\lambda}, \boldsymbol{\xi} \rangle$ then such a g always exists; namely,

$$g(\mathbf{x}) = \int_{-\infty}^{0} f(\mathbf{x} + \boldsymbol{\lambda} t)\,dt$$

Since f is $L^1(\mathbb{R}^d)$ and compactly supported, it follows that $g \in L^1(\mathbb{R}^d)$. Moreover since $\widehat{f}(\boldsymbol{\xi}) = 0$ whenever $\langle \boldsymbol{\lambda}, \boldsymbol{\xi} \rangle = 0$, it follows that $\int_{-\infty}^{\infty} f(\mathbf{x} + \boldsymbol{\lambda} t)\,dt \equiv 0$, and so g is also compactly supported – since f is.

Suppose next that h is elliptic, so that $h(\mathbf{y}) = 0$, $\mathbf{y} \in \mathbb{R}^d \Rightarrow \mathbf{y} = \mathbf{0}$. Say $\deg h = 2m$. Then $|h(\boldsymbol{\xi})| \geq C|\boldsymbol{\xi}|^{2m}$ for some $C > 0$, and so

$$|u(\boldsymbol{\xi})| \leq C(1 + |\boldsymbol{\xi}|^2)^{-m} \tag{23}$$

If $2m \geq d$ then $u \in L^2(\mathbb{R}^d)$ and its inverse transform $g \in L^2(\mathbb{R}^d)$. It follows from the Payley-Wiener Theorem that g is compactly supported, and so in fact $g \in L^1(\mathbb{R}^d)$.

If $d > 2m$ then it follows from [20, Thm. 7.1.20] that $h(\boldsymbol{\nabla})$ has a fundamental solution E which is homogeneous of degree $2m - d$ and C^∞ in $\mathbb{R}^d \backslash \mathbf{0}$. In particular

$$|E(\mathbf{x})| \leq \frac{C}{|\mathbf{x}|^{d-2m}} \tag{24}$$

for some $C > 0$. Since g is compactly supported, and since $\lim_{|\mathbf{x}| \to \infty} E * f(\mathbf{x}) = 0$, it follows from the ellipticity of $h(\boldsymbol{\nabla})$ that

$$g = E * f \tag{25}$$

Moreover since E is locally integrable in $\mathbb{R}^d$ (by (24)), and $f \in L^1(\mathbb{R}^d)$, it follows from (25) that g is (locally) integrable.

These arguments have led us to the following

Theorem III. *Let $f \in L^1(\mathbb{R}^d)$ be compactly supported, with*

$$\widehat{f}(\boldsymbol{\xi}) = h(\boldsymbol{\xi}) u(\boldsymbol{\xi})$$

where h is a homogeneous polynomial and u is entire analytic. Then under either of the conditions below, it follows that $u = \widehat{g}$ where $g \in L^1(\mathbb{R}^d)$ and is compactly supported

(i) h is linear; i.e., $h(\boldsymbol{\xi}) = \langle \boldsymbol{\lambda}, \boldsymbol{\xi} \rangle$

(ii) h is elliptic.

Remark. Since every homogeneous polynomial in $d = 2$ variables factors into a product of linears and irreducible quadratics, the conclusion of Theorem III always holds. That is, for any homogeneous polynomial h in two variables, it is always true that $u = \widehat{g}$ where $g \in L^1(\mathbb{R}^2)$ and is compactly supported. In the elliptic case we can get more information about the integrability of g. Observe from (23) that $u \in L^p(\mathbb{R}^d)$ for any $p > \dfrac{d}{2m}$. In particular we have that

(i) $g \in L^2(\mathbb{R}^d)$, if $d < 4m$;

(ii) $g \in L^\infty(\mathbb{R}^d)$, if $d < 2m$;

(iii) $g \in L^q(\mathbb{R}^d)$, for any $q < \dfrac{d}{d-2m}$, if $2m \le d < 4m$;

Here we use Parseval's Formula to derive (i), and the Hausdorff–Young Inequality to derive (ii) and (iii).

The significance of Theorem III as concerns the dilation equation (12) is as follows. Suppose $\Delta = \alpha^k$ and that $f \in L^1(\mathbb{R}^d)$ is a compactly supported solution of (12). Then according to Theorem II(c), $\widehat{f}(\boldsymbol{\xi}) = h(\boldsymbol{\xi})u(\boldsymbol{\xi})$, where h is a homogeneous polynomial of degree k, and u is entire analytic. If h can be factored into linear and elliptic factors, then we are able to conclude that $f = h(\boldsymbol{\nabla})g$, where g satisfies the corresponding dilation equation with scaling coefficients $c_{\boldsymbol{\eta}}/\alpha^k$. This allows us to identify solutions of (12) with $\Delta = \alpha^k$ as coming from k^{th} order partial derivatives of solutions of the corresponding scaled equation with $\Delta = 1$.

§2. IFS Algorithm

Algorithmically, sub–division schemes are easy to implement using a recursive 2^d–ary tree traversal. For example Figure 7 illustrates the case for the "butterfly scheme" which was described above. The root of the tree is a 6×6 index grid, where i, j range from -2 to 3, on which control points $\mathbf{p}^0_{\boldsymbol{\eta}}$ are defined. (See Figure 5.) In general the root grid in $\mathbb{Z}^d$ can be constructed as follows: Let $\{M_1, \ldots, N_1\} \times \cdots \times \{M_d, \ldots, N_d\} \subseteq \mathbb{Z}^d$ be the smallest (lattice) box containing the support $\{\boldsymbol{\eta} \in \mathbb{Z}^d : c_{\boldsymbol{\eta}} \neq 0\}$. Then the root grid can be taken as $-S$, where

$$S = \{M_1, \ldots, N_1 - 1\} \times \cdots \times \{M_d, \ldots, N_d - 1\} \tag{26}$$

The first sub–division level extends the grid to 7×7, where i, j now range from -2 to 4. (Note that in Figure 5 the grid is not re–scaled, but in this Figure it is.) This is broken up into four "overlapping" transformations $T_{0,0}, T_{1,0}, T_{0,1}, T_{1,1}$ as shown. Transformation $T_{0,0}$ maps the parent

Figure 7: Tree for implementation of sub–division algorithm. At each level the 6×6 grid is replaced by four staggered grids.

6×6 points indexed by the root grid into the child 6×6 points in the lower left of the 7×7 sub–divided grid. Similarly $T_{1,0}$ maps the parent 6×6 points indexed by the root grid into the child 6×6 points in the lower right of the 7×7 sub–divided grid, etc. Each $T_{\boldsymbol{\omega}}$ is thus a linear transformation

$$T_{\boldsymbol{\omega}} \in \mathcal{L}(\mathrm{I\!R}^S); \qquad \boldsymbol{\omega} \in \{0,1\}^d$$

whose action is determined by

$$\mathbf{p}_{-\boldsymbol{\eta}}^{\text{child}} = \sum_{\boldsymbol{\gamma} \in S} (T_{\boldsymbol{\omega}})_{\boldsymbol{\eta},\boldsymbol{\gamma}} \mathbf{p}_{-\boldsymbol{\gamma}}^{\text{parent}}; \quad \boldsymbol{\eta} \in S \tag{27}$$

Here $S \subseteq \mathbb{Z}^d$ is the finite set above; $-S = \{-2, \ldots, 3\}^2$ for the "butterfly scheme." Observe in (27) the convention that $-\boldsymbol{\eta}$ and $-\boldsymbol{\gamma}$ are the respective child and parent subscripts. Under this convention the coordinate description of $T_{\boldsymbol{\omega}}$ becomes

$$(T_{\boldsymbol{\omega}})_{\boldsymbol{\eta},\boldsymbol{\gamma}} = c_{2\boldsymbol{\gamma}-\boldsymbol{\eta}+\boldsymbol{\omega}}; \quad \boldsymbol{\eta}, \boldsymbol{\gamma} \in S \tag{28}$$

where we have shifted each 6×6 grid here so that i, j consistently range from -2 to 3. This is important so that each child can in turn be treated like the root. The tree continues to evolve this way, with four edges sprouting out of each node. *At every stage of the sub–division scheme the relationship of the parent* 6×6 *points to the child* 6×6 *points along the* $(0,0)$ *edge is the same;* namely, $T_{0,0}$ — and similarly for the other three types of edges. Thus the surface can be generated by recursively traversing this quad–tree, until a "deep enough" level ℓ is reached – at which stage the points can be plotted. The plotting depth ℓ is determined by the desired resolution of the graphics output.

Under condition (2) the transformations $T_{\boldsymbol{\omega}}$ are row–stochastic; i.e., $\sum_{\boldsymbol{\gamma} \in S} (T_{\boldsymbol{\omega}})_{\boldsymbol{\eta},\boldsymbol{\gamma}} = 1$ for all $\boldsymbol{\eta} \in S$. When the sub–division algorithm converges, longer and longer paths of the tree produce limiting singletons. This manifests itself in that

$$\lim_{n \to \infty} T_{\boldsymbol{\omega}_n} \cdots T_{\boldsymbol{\omega}_1} = T_\infty \tag{29}$$

exists, for any sequence $(\boldsymbol{\omega}_n) \in \Omega^d$ where Ω is the sequence space (or "code space") $\Omega = \{0,1\}^\infty$; and moreover this limit is of rank one. Identify $\mathbf{x} = \sum \boldsymbol{\omega}_n 2^{-n} \in [0,1]^d$ with this sequence $(\boldsymbol{\omega}_n)$; i.e., through the binary expansions

$$x_1 = .\omega_{1,1}\omega_{2,1}\cdots, \quad \ldots, \quad x_d = .\omega_{d,1}\omega_{d,2}\cdots$$

and denote correspondingly T_∞ by $T_\infty(\mathbf{x})$. Applying $T_\infty(\mathbf{x})$ to the original 6×6 set of control points gives 36 *identical* points — namely, that point on the surface corresponding to $\mathbf{x}$; i.e., the point

$$\sum_{\boldsymbol{\eta} \in S} f(\mathbf{x} + \boldsymbol{\eta}) \mathbf{p}^0_{-\boldsymbol{\eta}}$$

where f is the normalized solution to (1), $\int f = 1$.

It can be shown as in [5] that, on account of the shift relationship between the various $T_{\boldsymbol{\omega}}$'s, $T_\infty(\mathbf{x})$ is well–defined. That is, if some component of $\mathbf{x}$ is dyadic, then its two binary expansions (terminating and non–terminating) give rise to the same limiting product. Equivalently, for the first index component

$$\lim_{n\to\infty} T_{0,\boldsymbol{\omega}'_n} \cdots T_{0,\boldsymbol{\omega}'_2} T_{1,\boldsymbol{\omega}'_1} = \lim_{n\to\infty} T_{1,\boldsymbol{\omega}'_n} \cdots T_{1,\boldsymbol{\omega}'_2} T_{0,\boldsymbol{\omega}'_1} \tag{30}$$

for any sequence $(\boldsymbol{\omega}'_n) \in \Omega^{d-1}$; and similarly for any other index component. From the fact that $T_\infty(\mathbf{x})$ exists and is well–defined, it follows from its definition that

$$T_\infty(\mathbf{x}) = T_\infty(\tau \mathbf{x}) T_{\boldsymbol{\omega}'_1} \tag{31}$$

where

$$\tau \mathbf{x} = 2\mathbf{x} (\text{mod } 1), \qquad \mathbf{x} \in [0,1]^d$$

and $\boldsymbol{\omega}'_1$ is the first vector of bits in the binary expansion of $\mathbf{x}$. ¿From this it follows as in [5] that the rows of $T_\infty(\mathbf{x})$ comprise a solution to (1). That is, let $\mathbf{v} \in \mathbb{R}^S$, $\sum_{\boldsymbol{\eta} \in S} v_{\boldsymbol{\eta}} = 1$. Then $\mathbf{v}^t T_\infty(\mathbf{x})$ is the row vector

$$(f(\mathbf{x} + \boldsymbol{\eta}) : \boldsymbol{\eta} \in S),$$

where f is the normalized solution to (1), $\int f = 1$.

According to the recipe in [2], [5] this then gives rise to the following *IFS algorithm for surface generation.*

IFS Algorithm

initialize $X = (x_{\boldsymbol{\eta}}) \in \mathrm{IR}^S$ to be the fixed point of $T_{\mathbf{0}}$, normalized so that $\sum_{\boldsymbol{\eta}\in S} x_{\boldsymbol{\eta}} = 1$

for $n = 1, L$

plot $\sum_{\boldsymbol{\eta}\in S} x_{\boldsymbol{\eta}} \mathbf{p}^0_{-\boldsymbol{\eta}}$

$\boldsymbol{\omega} \leftarrow \boldsymbol{\omega}_n$ (from the bit string)

$X \leftarrow T_{\boldsymbol{\omega}} X$

endfor

The choice of the bit string $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_L$ must be done as in [5]. Given a desired resolution ℓ, the string must have the property that

($\mathcal{P}$) *as we slide a window of length ℓ across the string, every possible ℓ–bit vector pattern in $(\{0,1\}^d)^\ell$ should appear.*

Since there are $2^{d\ell}$ such patterns, it is clear that $L \geq 2^{d\ell} + \ell - 1$ (the last term due to the window spill–over). The bound $L = 2^{d\ell} + \ell - 1$ can in fact be attained by constructing a de–Bruijn sequence for 2^d symbols. With $\ell = d = 2$, for example, we can use the string

$$a\ a\ b\ c\ d\ b\ a\ d\ d\ c\ b\ b\ d\ a\ c\ c\ a$$

with $L = 2^4 + 1$ terms, where the symbols a, b, c, d stand for

$$a = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad c = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Observe that as we slide a window of length 2 across this string, we encounter the pairs

$$aa,\ ab,\ bc,\ cd,\ \ldots, ca$$

and indeed each of the 16 possible pairs occurs here.

In general if the initial set of control points spans more than a 6×6 index grid, then several trees can be traversed in parallel, each starting with a different 6×6 block of the initial grid as root — so long as the entire grid gets processed. These root 6×6 grids need to be offset in order to fit the full initial grid. This is illustrated in Figure 8 for $d = 1$. *Thus there is never any need to work in a larger space than* $\mathbb{R}^{6\times 6}$; i.e., $\mathbb{R}^S$ for the S given by (26). For $d > 2$ the tree in Figure 7 generalizes to a 2^d–ary tree in the obvious way.

Recall from [11] the following setup. If Σ is a bounded set of linear transformations and $\|\cdot\|$ is an operator norm, set

$$\|\Sigma\| = \sup\{\|T\| : T \in \Sigma\}, \qquad \Sigma^n = \{T_1 \cdots T_n : T_i \in \Sigma, 1 \leq i \leq n\}$$

Define

$$\widehat{\rho}(\Sigma) = \limsup_{n\to\infty} \|\Sigma^n\|^{1/n}$$

It follows from [11] and [4, Lemma II(b)] that

$$\widehat{\rho}(\Sigma) = \inf_{\|\cdot\|} \|\Sigma\|,$$

where the inf is over *all* operator norms $\|\cdot\|$. It follows from [11, Sec. 4] that a necessary and sufficient condition for all products (29) to converge to a continuous limit is that simultaneously

$$T_{\boldsymbol{\omega}} \sim \begin{array}{cc} \begin{pmatrix} I & \mathbf{b}^t_{\boldsymbol{\omega}} \\ & \\ 0 & A_{\boldsymbol{\omega}} \end{pmatrix} & \begin{array}{c} \uparrow \\ r \\ \downarrow \\ \uparrow \\ |S| - r \\ \downarrow \end{array} \end{array}$$

with $\widehat{\rho}(A_{\boldsymbol{\omega}} : \boldsymbol{\omega} \in \{0,1\}^d) < 1$. Moreover if the scaling coefficients satisfy (2), then on account of the uniqueness of solutions to (1), it follows as in [5, Thm. IV(a)] that necessarily $r = 1$. This gives the rank one property of T_∞.

Figure 8: Two parallel trees, each generating a different part of the limiting curve.

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Multi–Dimensional Two–Scale Dilation Equations

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