# Substitution Delone Sets 

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#### Abstract

Substitution Delone set families are families of Delone sets $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ which satisfy the inflation functional equation $$
X_{i}=\bigvee_{j=1}^{m}\left(\mathrm{~A}\left(X_{j}\right)+\mathcal{D}_{i j}\right), \quad 1 \leq i \leq m
$$ in which $A$ is an expanding matrix, i.e. all of the eigenvalues of $A$ fall outside the unit circle. Here the $\mathcal{D}_{i j}$ are finite sets of vectors in $\mathbb{R}^{d}$ and $\bigvee$ denotes union that counts multiplicity.

This paper characterizes families $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ that satisfy an inflation functional equation, in which each $X_{i}$ is a multiset (set with multiplicity) whose underlying set is discrete. It then studies the subclass of Delone set solutions, and gives necessary conditions on the coefficients of the inflation functional equation for such solutions $\mathcal{X}$ to exist. It relates Delone set solutions to a narrower subclass, called self-replicating multi-tiling sets, which arise as tiling sets for self-replicating multi-tilings.


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## 1. Introduction

Aperiodic and self-similar structures in $\mathbb{R}^{d}$ have been extensively studied using tilings of $\mathbb{R}^{d}$ as models. Among these are the classes of self-similar tilings and, more generally, selfaffine tilings. Such tilings have been proposed as models for quasicrystalline structures ([2], [6], [7].[21], [23]), [31]); they also arise in constructions of compactly supported wavelets and multiwavelets ( [1],[4], [5]). An alternate method of modelling quasicrystalline structures uses discrete sets, more specifically Delone sets (defined below), see [13], [14], [15], [25], [26], [27], [32]. These sets model the atomic positions occupied in the structure. In terms of tilings, such discrete sets can be viewed either as tiling sets, representing the translations used in forming tilings by translation of a finite number of different prototile types, or as control points marking in some way the location of each tile.

Comparison of these two types of models, which appear rather different, motivates the question: Is there an appropriate notion of self-similarity appropriate to discrete sets and Delone sets? This paper develops such a notion, which is based on a system of functional equations dual to the functional equations associated to self-affine tilings and multi-tilings.

We first recall the functional equation associated to the construction of finite sets of tiles $\left\{T_{1}, \ldots, T_{n}\right\}$ which tile $\mathbb{R}^{d}$ with special kinds of self-affine tilings. The tiles are solutions to a finite system of set-valued functional equations which we call multi-tile equations, which encode a self-affine property. An inflation map $\phi: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ is an expanding linear map $\phi(\mathbf{x})=\mathrm{Ax}$ in which A is an expanding $n \times n$ real matrix, i.e. all its eigenvalues $|\lambda|>1$. Let $\left\{\mathcal{D}_{i j}: 1 \leq i, j \leq n\right\}$ be finite sets in $\mathbb{R}^{d}$ called digit sets.

Multi-Tile Functional Equation. The family of compact sets $\left(T_{1}, T_{2}, \ldots, T_{n}\right)$ satisfy the system of equations

$$
\begin{equation*}
\mathrm{A}\left(T_{i}\right)=\bigcup_{j=1}^{n}\left(T_{j}+\mathcal{D}_{j i}\right) \tag{1.1}
\end{equation*}
$$

for $1 \leq i \leq n$.
This functional equation is set-valued, i.e. points are counted without multiplicity in the set union on the right side of (1.1). The subdivision matrix associated to (1.1) is

$$
\begin{equation*}
\mathbf{S}=\left[\left|\mathcal{D}_{i j}\right|\right]_{1 \leq i, j \leq n} \tag{1.2}
\end{equation*}
$$

These functional equations have a nice solution theory when the substitution matrix satisfies the following extra condition.

Definition 1.1. A nonnegative real matrix $S$ is primitive if some power $S^{k}$ has strictly positive entries.

In the one-dimensional case the subdivision matrix associated to (1.1) is always primitive. It is known that when S is primitive the functional equation (1.1) has a unique solution

$$
\mathcal{T}:=\left(T_{1}, \ldots, T_{n}\right)
$$

in which all $T_{i}$ are nonempty compact sets (see [1, Theorem 2.3], [4]). In the imprimitive case it has a finite number of solutions $\mathcal{T}:=\left(T_{1}, \ldots, T_{n}\right)$ in which all $T_{i}$ are nonempty compact sets, see [4] for examples, but the theory becomes more complicated. In this paper we primarily consider functional equations where the subdivison matrix $S$ is primitive.

We are interested in the case where these sets $T_{i}$ have positive Lebesgue measure and can be used in tiling $\mathbb{R}^{n}$ by a self-affine tiling. The tiling sets for such tilings are special solutions to a second functional equation involving the same data, which is "adjoint" to the multi-tile functional equation. This functional equation counts multiplicities of sets, unlike (1.1), and we consider solutions to it that are multisets.

Inflation Functional Equation. The multiset family $\mathcal{X}:=\left(X_{1}, X_{2}, \ldots, X_{n}\right)$ satisfies the system of equations

$$
\begin{equation*}
X_{i}=\bigvee_{j=1}^{n}\left(\mathbf{A}\left(X_{j}\right)+\mathcal{D}_{i j}\right), \quad 1 \leq i \leq n \tag{1.3}
\end{equation*}
$$

where $\mathcal{D}_{i j}$ are finite sets of vectors in $\mathbb{R}^{d}$.
Here each $X_{i}$ is a multiset (as defined in $\S 2$ ), and $\bigvee_{j=1}^{n}$ denotes multiset union, as defined in $\S 2$. The multiset equation (1.3) can alternatively be written as a system of equations for the multiplicity functions:

$$
\begin{equation*}
m_{X_{i}}(\mathbf{x})=\sum_{j=1}^{n} m_{\mathbf{A}\left(X_{j}\right)+\mathcal{D}_{i j}}(\mathbf{x})=\sum_{j=1}^{n} \sum_{\mathbf{d} \in \mathcal{D}_{i j}} m_{X_{j}}\left(\mathbf{A}^{-1}(\mathbf{x}-\mathbf{d})\right), \quad \text { for all } \quad \mathbf{x} \in \mathbb{R}^{d} \tag{1.4}
\end{equation*}
$$

A self-replicating multi-tiling consists of a pair of solutions to the multi-tile functional equation and the inflation functional equation such that:
(1) The solution $\mathcal{T}:=\left(T_{1}, \ldots, T_{n}\right)$ to the multi-tile functional equation has all sets $T_{i}$ of positive Lebesgue measure.
(2) The solution $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ to the inflation functional equation are sets (multisets with all multiplicities equal to one) and $\bigcup_{i=1}^{n}\left(T_{i}+X_{i}\right)$ is a tiling of $\mathbb{R}^{d}$, using the $T_{i}^{\prime} s$ as prototiles.

We call the family $\mathcal{X}$ a self-replicating multi-tiling set. This notion extends the notion of self-replicating tilings studied in Kenyon [8], [9], which allow only one type of tile $(n=1)$; later studies ([10], [11]) allowed $n$ tile types but restricted the inflation matrix A to be a similarity.

It follows from this definition that each $X_{i}$ is a uniformly discrete set, and in fact is a Delone set. There are very strong restrictions on the data $\left(\mathrm{A}, \mathcal{D}_{i j}\right)$ on the functional equations for a self-replicating multi-tiling to exist, e.g. the Perron eigenvalue condition given below.

This paper studies solutions to the inflation functional equation that are discrete, including solutions that do not correspond to tilings. The inflation functional equation has properties that significantly differ from those of the multi-tile functional equation. For example, the multitile functional equation has a unique solution (in the case of primitive subdivision matrix) because the solution $\mathcal{T}$ is given by the unique fixed point of a contracting system of mappings. In contrast, the inflation functional equation involves an expanding system of mappings, and its solutions are not compact sets. It may have infinitely many different solutions, even very nice solutions in some cases. Our replacement for the "contracting" condition, is to restrict attention to solutions $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ having the property that all $X_{j}$ are discrete multisets. A set $X$ in $\mathbb{R}^{d}$ is discrete if each bounded set in $\mathbb{R}^{d}$ contains finitely many elements of $X$. A multiset $X$ is discrete if its underlying set $\underline{X}$ is discrete and each element in $X$ has a finite multiplicity.

In $\S 3$ we develop a structure theory for solutions to the inflation functional equation that are discrete multisets. We show they decompose uniquely into a finite number of irreducible discrete multisets, and show that each irreducible discrete multiset is characterized by a finite set of data. These results are proved for general inflation functional equations, with no primitivity restriction. We note however that the requirement that a discrete multiset solution exist puts very significant restrictions on the inflation functional equation data $\left(\mathrm{A}, \mathcal{D}_{i j}\right)$.

In the remainder of the paper we study solutions which correspond more closely to tilings. A multiset $X$ is weakly uniformly discrete if there is a positive radius $r$ and a finite constant $m \geq 1$ such that each ball of radius $r$ contains at most $m$ points of $X$, counting multiplicities; it is uniformly discrete if one can take $m=1$. A multiset $X$ is a weak Delone set $X$ if it is weakly uniformly discrete and relatively dense; it is a Delone set if it is also uniformly discrete. A solution $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ to the inflation functional equation is a weak substitution Delone set family if each multiset $X_{i}$ is a weak Delone set; it is a substitution Delone set family if in addition each multiset $X_{i}$ is a Delone set. In studying solutions which are weak Delone sets, we restrict attention to inflation functional equations that have a primitive subdivision matrix S . Then we can make use of Perron-Frobenius theory, which asserts that a primitive nonnegative real matrix M has a positive real eigenvalue $\lambda(\mathrm{M})$ such that:
(i) $\lambda(\mathrm{M})$ has multiplicity one.
(ii) $\lambda(\mathrm{M})>\left|\lambda^{\prime}\right|$ for all eigenvalues $\lambda^{\prime}$ of M with $\lambda^{\prime} \neq \lambda(\mathrm{M})$.
(iii) There are both right and left eigenvectors for $\lambda(M)$ with positive real entries.

We call $\lambda(\mathrm{M})$ the Perron eigenvalue of M ; it is equal to the spectral radius of M .

For inflation functional equations whose substitution matrix is primitive, we show that those multiset equations having a weak Delone set solution must satisfy the following:

Perron eigenvalue condition: The Perron eigenvalue $\lambda(\mathbf{S})$ of the subdivision matrix $\mathbf{S}$ has

$$
\lambda(\mathbf{S})=|\operatorname{det}(\mathbf{A})| .
$$

More generally, there are inequalities relating the Perron eigenvalue and properties of solutions of the multi-tile and inflation functional equations, as follows:
(1) A necessary condition for the multi-tile functional equation with primitive subdivision matrix to have $\mathcal{T}=\left(T_{1}, \ldots, T_{n}\right)$ with some (and hence all) $T_{i}$ of positive Lebesgue measure is that

$$
\lambda(\mathbf{S}) \geq|\operatorname{det}(\mathbf{A})| .
$$

(2) A necessary condition for the inflation functional equation with primitive subdivision matrix to have a solution $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ with some (and hence all) $X_{i}$ weakly uniformly discrete is that

$$
\lambda(\mathbf{S}) \leq|\operatorname{det}(\mathbf{A})| .
$$

Inequality (1) is established by taking the Lebesgue measure on both sides of the multitile equation (1.1). In the special case when A is a similarity Mauldin and Williams [24] showed the stronger result that if $\lambda(\mathbf{S})<|\operatorname{det}(\mathbf{A})|$ then the Hausdorff dimension of each $T_{i}$ must be less than $d$. Inequality (2) is shown in $\S 4$ as Theorem 4.3. We show that the existence of substitution Delone set solutions is closely associated with self-affine multi-tilings. In particular, the associated multi-tile functional equation has a solution $\mathcal{T}=\left(T_{1}, T_{2}, \ldots, T_{n}\right)$ with the $T_{i}$ all having positive Lebesgue measure, see Theorem 2.4 in $\S 2$.

In the following section we give precise definitions and statements of some of the main results in the paper. In the final section $\S 7$ we give various examples showing the limits of our results.

To conclude this introduction, given any Delone set $X$ one can associate a topological dynamical system $\left([[X]], \mathbb{R}^{d}\right)$ with an $\mathbb{R}^{d}$-action, in which $[[X]]$ is the orbit closure of $X$ with the translation action under an appropriate topology, see Solomyak [36]. For substitution Delone set families these dynamical systems can be viewed as a generalization of substitution dynamical systems (Queffelec [30]), in that every primitive substitution dynamical system is topologically conjugate to a suitable substitution Delone set dynamical system. Under sufficiently strong extra hypotheses substitution Delone set dynamical systems are minimal and uniquely ergodic. We hope to return to this question elsewhere.

## 2. Definitions and Main Results

We consider solutions to the inflation functional equation that are multisets.
Definition 2.1. (i) A multiset is a "set" in which an element may be counted for more than once. Given a multiset $X$ we shall use $\underline{X}$ to denote the underlying set of $X$, i.e. without counting the multiplicity, and either $m_{X}(\mathbf{x})$ or $m(X, \mathbf{x})$ to denote the multiplicity function, whcih can be viewed as a discrete measure. A multiset is called an ordinary set if every element in $X$ has multiplicity 1.
(ii) Given multisets $X_{1}$ and $X_{2}$ we say $X_{1} \subseteq X_{2}$ provided $\underline{X}_{1} \subseteq \underline{X}_{2}$ and $m_{X_{1}}(\mathbf{x}) \leq m_{X_{2}}(\mathbf{x})$ for all $\mathrm{x} \in X_{2}$. In particular, $\underline{X} \subseteq X$.

For example, $X=\{0,0,1,3,4,4,4\}$ is a multiset in which 0 is counted twice and 4 is counted 3 times. Thus $\underline{X}=\{0,1,3,4\}$ and

$$
m_{X}(0)=2, \quad m_{X}(1)=m_{X}(3)=1, \quad m_{X}(4)=3 .
$$

Definition 2.2. For any multisets $X$ and $Y$ the multiset union $X \vee Y$ is the multiset having the multiplicity function

$$
\begin{equation*}
m_{X \vee Y}=m_{X}+m_{Y} \tag{2.1}
\end{equation*}
$$

and the multiset intersection $X \wedge Y$ is the multiset having multiplicity function

$$
\begin{equation*}
m_{X \wedge Y}=\min \left\{m_{X}, m_{Y}\right\} . \tag{2.2}
\end{equation*}
$$

For a multiset $X$ and a set (or multiset) $\mathcal{D}$ the multiset sum $X+\mathcal{D}$ is

$$
\begin{equation*}
X+\mathcal{D}:=\bigvee_{\mathbf{d} \in \mathcal{D}}(X+\mathbf{d}) \tag{2.3}
\end{equation*}
$$

Definition 2.3. A multiset family $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ is a finite vector of multisets $X_{i}$. We call a multiset family $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ an $n$-multiset family. For $n$-multiset families $\mathcal{X}^{(1)}$ and $\mathcal{X}^{(2)}$ we define

$$
\mathcal{X}^{(1)} \vee \mathcal{X}^{(2)}=\left(X_{1}^{(1)} \vee X_{1}^{(2)}, \ldots, X_{n}^{(1)} \vee X_{n}^{(2)}\right)
$$

Definition 2.4. A multiset family $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ is discrete if for each $i$ the multiset $X_{i}$ is discrete, i.e. the underlying set $\underline{X}_{i}$ is discrete and elements in $X_{i}$ have finite multiplicity.

In $\S 3$ we develop a structure theory for discrete multiset families, which decomposes them into irreducible families. Let $\psi$ be the inflational operator associated to an inflational functional equation (defined in $\S 3$ ).

Definition 2.5. A multiset family $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ satisfying an inflation functional equation $\psi(\mathcal{X})=\mathcal{X}$ is irreducible if it cannot be partitioned as $\mathcal{X}=\mathcal{X}^{(1)} \vee \mathcal{X}^{(2)}$ with each $\mathcal{X}^{(i)}$ nonempty such that $\psi\left(\mathcal{X}^{(i)}\right)=\mathcal{X}^{(i)}$ for each $i=1,2$.

Theorem 2.1 (Decomposition Theorem) Let $\mathcal{X}$ be a multiset family $\mathcal{X}$ that is discrete and satisfies an inflation functional equation $\psi(\mathcal{X})=\mathcal{X}$. Then $\mathcal{X}$ uniquely partitions into a finite number of irreducible discrete multiset families that satisfy the same inflation functional equation.

An inflational functional equation $\psi(\mathcal{X})=\mathcal{X}$ may not have a discrete solution, see Example 7.4. In $\S 3$ we characterize irreducible discrete multiset families $\mathcal{X}$ satisfying inflation functional equations, as follows.

Theorem 2.2. Let $\mathcal{X}$ be a multiset family which satisfies an inflation functional equation $\psi(\mathcal{X})=\mathcal{X}$ and is discrete and irreducible. Then $\mathcal{X}$ is generated by a finite "seed" $\mathcal{S}^{(0)}=$ $\left(S_{1}, S_{2}, \ldots, S_{n}\right)$, which consists of a periodic cycle $Y=\left\{\left(\mathbf{x}_{j}, i_{j}\right): 1 \leq j \leq p\right\}$ in which $\mathbf{x}_{j} \in X_{i_{j}}$ and there is some $\mathbf{d}_{j} \in \mathcal{D}_{i_{j+1}, i_{j}}$ with $\mathbf{x}_{j+1}=\mathrm{A} \mathbf{x}_{j}+\mathbf{d}_{j}$, with $\left(\mathbf{x}_{p+1}, i_{p+1}\right)=\left(\mathbf{x}_{1}, i_{1}\right)$. That is,

$$
\mathcal{X}=\lim _{N \rightarrow \infty} \phi^{N}\left(\mathcal{S}^{(0)}\right)
$$

The periodic cycle $Y$ is the only periodic cycle in $\mathcal{X}$ and its elements have multiplicity one.

This result appears, stated in a slightly more precise form, as Theorem 3.3. For each $p$ there are only finitely many periodic cycles $Y$ and they can be effectively enumerated. However not all periodic cycles $Y$ generate irreducible discrete multiset families. We show that there is an algorithmic procedure, which when given any such "seed" as input, has one of three outcomes:
(1) If the generated multiset system $\mathcal{X}$ is discrete and irreducible, it eventually halts and says this holds.
(2) If the generated multiset system $\mathcal{X}$ is not irreducible, or if the limit does not exist it eventually halts and says this holds.
(3) If the generated multiset system $\mathcal{X}$ is irreducible and not discrete, the procedure may not halt.

We also prove in $\S 3$ a dichotomy concerning the multiplicities of elements appearing in the multisets $X_{i}$ in an irreducible discrete multiset family $\mathcal{X}$ satisfying an inflation functional equation having a primitive subdivision matrix : Either they are all bounded by the period $p$ of the generating cycle or else they all have unbounded multiplicities. (Theorem 3.4.) We do not know of an effective computational procedure to tell in general which of these alternatives occurs.

In $\S 4-\S 6$ we restrict to the case of inflation functional equations having primitive subdivision matrix. We study Delone set solutions and self-replicating multi-tilings.

Definition 2.6. A multiset $X$ is a weak Delone set if it has the following properties:
(i) Weakly Uniformly discrete. There is an $r>0$ and a finite $m \geq 1$ such that any open ball of radius $r$ contains at most $m$ points of $X$.
(ii) Relatively dense. There is an $R>0$ such that any closed ball of radius $R$ contains at least one point of $X$.

It is a Delone set if in addition it is a uniformly discrete set, i.e. one can take $m=1$ in (i); $X$ is an ordinary set in this case.

Note that a finite union of weak Delone sets is a weak Delone set.

Definition 2.7. A multiset family $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ is a weak substitution Delone set family if $\mathcal{X}$ satisfies the inflation functional equation (1.3) in $\mathbb{R}^{d}$ and each multiset $X_{i}$ is a weak Delone set. It is a substitution Delone set family if each $X_{i}$ is a Delone set (hence has all multiplicities equal to one.)

In $\S 4$ we prove the following result.

Theorem 2.3 (Perron Eigenvalue Condition) If the inflation functional equation $\psi(\mathcal{X})=$ $\mathcal{X}$ with primitive subdivision matrix $\mathrm{S}=\left[\left|\mathcal{D}_{i j}\right|\right]$ has a solution $\mathcal{X}$ that is a weak Delone set family, then the Perron eigenvalue $\lambda(\mathbf{S})$ of $\mathbf{S}$ satisfies

$$
\lambda(\mathbf{S})=|\operatorname{det}(\mathbf{A})| .
$$

The narrowest class of solutions to the inflation functional equation are those having the following tiling property.

Definition 2.8. A multiset family $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ is called a self-replicating multi-tiling for a given system $\left(\mathrm{A}, \mathcal{D}_{i j}\right)$ in (1.3) with primitive subdivision matrix S if
(i) $\mathcal{X}$ is a substitution Delone set family for $\left(\mathrm{A}, \mathcal{D}_{i j}\right)$.
(ii) The associated multitile equation (1.1) has a unique solution $\mathcal{T}:=\left(T_{1}, \ldots, T_{n}\right)$ with each $T_{i}$ of positive Lebesgue measure.
(iii) The sets $\left\{T_{i}+\mathbf{x}_{i}: 1 \leq i \leq n \quad\right.$ and $\left.\quad \mathbf{x}_{i} \in X_{i}\right\}$ tile $\mathbb{R}^{d}$.

Note that this definition requires that the family $\mathcal{X}$ give a 1 -tiling, rather than a $p$-tiling for some $p \geq 2$, or a $p$-packing for some $p \geq 1$ that is not a tiling.

In $\S 5$ we prove the following result, which shows that substitution Delone sets are related to the existence of self-replicating multi-tilings. In this result $\psi^{N}(\cdot)$ denotes the functional equation $\psi(\cdot)$ composed with itself $N$ times, whose associated inflation matrix is $\mathbf{A}^{N}$.

Theorem 2.4. Let $\psi(\mathcal{X})=\mathcal{X}$ be a inflation functional equation that has a primitive subdivision matrix. Then the following conditions are equivalent:
(i) For some $N>0$ there exists a substitution Delone set solution $\psi^{N}(\hat{\mathcal{X}})=\hat{\mathcal{X}}$.
(ii) For some $N>0$ there exists a self-replicating multi-tiling $\hat{\mathcal{X}}$ such that $\psi^{N}(\hat{\mathcal{X}})=\hat{\mathcal{X}}$.
(iii) $\psi$ satisfies the Perron eigenvalue condition $\lambda(\mathbf{S})=|\operatorname{det}(\mathrm{A})|$ and the unique compact solution $\left(T_{1}, \ldots, T_{n}\right)$ of the associated multi-tile functional equation consists of sets $T_{i}$ that have positive Lebesgue measure, $1 \leq i \leq n$.

In $\S 6$ we give a sufficient condition for a substitution Delone set family to be a self-replicating multi-tiling (Theorem 6.1). Only very special data $\psi=\left(\mathbf{A}, \mathcal{D}_{i j}\right)$ satisfy the conditions of Theorem 2.4. A number of different necessary (resp. sufficient) conditions on $\psi=\left(\mathbf{A}, \mathcal{D}_{i j}\right)$ are known, in order to satisfy Theorem 2.4(iii), for which see Kenyon [8]-[11], Lagarias and Wang [17], [18], [19], Praggastis [28], [29] and Solomyak [34]-[36]. In $\S 7$ we give examples and counterexamples showing the limits of our results.

## 3. Structure of Discrete Multiset Solutions

We consider multiset families $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ satisfing the inflation functional equation

$$
\begin{equation*}
X_{i}=\bigvee_{j=1}^{n}\left(\mathbf{A}\left(X_{j}\right)+\mathcal{D}_{i j}\right), \quad 1 \leq i \leq n \tag{3.1}
\end{equation*}
$$

Associated to this system is the inflation operator $\psi(\cdot)$ which takes $n$-multiset families to $n$ multiset families $\psi(\mathcal{X})=\mathcal{X}^{\prime}$, as follows. Given $\mathcal{X}$ (not necessarily satisfying (3.1)), we define $\mathcal{X}^{\prime}=\left(X_{1}^{\prime}, \ldots, X_{n}^{\prime}\right)$ by

$$
\begin{equation*}
X_{i}^{\prime}:=\bigvee_{j=1}^{n}\left(\mathrm{~A}\left(X_{j}\right)+\mathcal{D}_{i j}\right), \quad 1 \leq i \leq n \tag{3.2}
\end{equation*}
$$

We write $\mathcal{X}^{\prime}=\psi(\mathcal{X})$, and view $\psi(\cdot)$ as an operator on multiset families $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$. The functional equation (3.1) asserts that $\mathcal{X}=\psi(\mathcal{X})$, i.e. $\mathcal{X}$ is a fixed point of $\psi(\cdot)$.

In this section we determine the structure of multiset families $\mathcal{X}$ that are discrete and satisfy $\psi(\mathcal{X})=\mathcal{X}$, where we put no restriction on $\psi(\cdot)$, allowing imprimitive substitution matrices. We also describe the structure of such sets that are also irreducible (Theorems 3.3 and 3.4). We show that every multiset family $\mathcal{X}$ that is discrete and satisfies $\psi(\mathcal{X})=\mathcal{X}$ uniquely partitions into a finite number of irreducible sets (Theorem 2.1).

We consider solutions to the inflation functional equation (3.1) built up by an iterative process starting from a finite multiset which is a "seed." Let

$$
\begin{equation*}
\mathcal{S}^{(0)}:=\left(S_{1}^{(0)}, \ldots, S_{n}^{(0)}\right) \tag{3.3}
\end{equation*}
$$

be a system of finite multisets, and iteratively define the finite multiset family $\mathcal{S}^{(i)}$ by $\mathcal{S}^{(i)}=$ $\psi\left(\mathcal{S}^{(i-1)}\right)$. We say that $\mathcal{S}^{(0)}$ satisfies the inclusion property $\mathcal{S}^{(0)} \subseteq \mathcal{S}^{(1)}$ if

$$
\begin{equation*}
S_{i}^{(0)} \subseteq S_{i}^{(1)} \quad \text { for } \quad 1 \leq i \leq n \tag{3.4}
\end{equation*}
$$

in the sense of multisets.
Lemma 3.1. If a multiset family $\mathcal{S}^{(0)}$ is finite and satisfies the inclusion condition $\mathcal{S}^{(0)} \subseteq$ $\psi\left(\mathcal{S}^{(0)}\right)$ then

$$
\begin{equation*}
\mathcal{S}^{(k)} \subseteq \mathcal{S}^{(k+1)} \quad \text { for all } \quad k \geq 0 \tag{3.5}
\end{equation*}
$$

The following limit set is well-defined

$$
\begin{equation*}
\underline{X}_{i}:=\lim _{k \rightarrow \infty} \underline{S}_{i}^{(k)}, \quad 1 \leq i \leq n, \tag{3.6}
\end{equation*}
$$

where $\underline{X}_{i}$ is regarded as a countable point set. For each $\mathbf{x} \in \underline{X}_{i}$ the multiplicity function has a limit,

$$
\begin{equation*}
m_{X_{i}}(\mathbf{x}):=\lim _{k \rightarrow \infty} m\left(S_{i}^{(k)}, \mathbf{x}\right) \tag{3.7}
\end{equation*}
$$

where this multiplicity may take the value $+\infty$. If all multiplicities in (3.7) remain finite, then the multisets $X_{i}$ are well-defined and the multiset family $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ satisfies $\psi(\mathcal{X})=\mathcal{X}$.

Remark. If all the multiplicities remain finite, we write $\mathcal{X}:=\lim _{k \rightarrow \infty} \mathcal{S}^{(k)}$. The limit multiset family $\mathcal{X}$ need not be discrete, and the underlying sets could even be dense.

Proof. The inclusion (3.5) follows by induction on $k$, since $\mathcal{S}^{(k-1)} \subseteq \mathcal{S}^{(k)}$ yields

$$
\mathcal{S}^{(k)}=\psi\left(\mathcal{S}^{(k-1)}\right) \subseteq \psi\left(\mathcal{S}^{(k)}\right)=\mathcal{S}^{(k+1)}
$$

The other statements follow easily from (3.5).
To further analyze multiset families $\mathcal{X}$ that satisfy $\psi(\mathcal{X})=\mathcal{X}$, we put the structure of a colored directed graph on $\mathcal{X}$. The vertices of this graph are the points in the disjoint union of the underlying sets $\underline{X}_{i}$, with points in $\underline{X}_{i}$ being specified as ( $\mathbf{x}, i$ ), where $\mathbf{x} \in \mathbb{R}^{d}$ and $i$ is the color (or label). For each $\mathbf{x}_{j} \in X_{j}$ and each $\mathbf{d}_{j} \in \mathcal{D}_{i j}$, set

$$
\mathbf{x}_{i}^{\prime}:=\mathrm{A} \mathbf{x}_{j}+\mathbf{d}_{j} \in X_{i},
$$

and put a directed edge $\mathbf{x}_{j} \rightarrow \mathbf{x}_{i}^{\prime}$ in the graph. $\mathbf{x}_{i}^{\prime}$ is called an offspring of $\mathbf{x}_{j}$, and $\mathbf{x}_{j}$ a preimage of $\mathbf{x}_{i}^{\prime}$. We denote this (infinite) colored directed graph by $\mathcal{G}(\mathcal{X})$. We also assign to each vertex $\mathrm{x} \in \underline{X}_{i}$ a weight which is its multiplicity $m_{X_{i}}(\mathrm{x})$.

Lemma 3.2. Let $\mathcal{X}$ be a multiset family which is discrete and satisfies the inflation functional equation $\psi(\mathcal{X})=\mathcal{X}$.
(i) There is a finite multiset family $\mathcal{S}^{(0)} \subseteq \mathcal{X}$ with the inclusion property, such that

$$
\begin{equation*}
\mathcal{X}=\lim _{k \rightarrow \infty} \mathcal{S}^{(k)} \tag{3.8}
\end{equation*}
$$

(ii) The directed graph $\mathcal{G}(\mathcal{X})$ contains a directed cycle.

Proof. (i) We show that there exists a radius $R>0$, depending only on $\psi(\cdot)$, such that every vertex in $\mathcal{X}$ can be reached by a directed path in the graph $\mathcal{G}(\mathcal{X})$ from some point $\mathbf{x} \in \mathcal{X} \cap B_{R}(\mathbf{0})$, where $B_{R}(\mathbf{0}):=\left\{\mathbf{x} \in \mathbb{R}^{d}:\|\mathbf{x}\| \leq R\right\}$. To show this, we set

$$
\begin{equation*}
\lambda\left(A^{-1}\right)=\min \left\{|\lambda|: \lambda \text { is an eigenvalue of } A^{-1}\right\} \tag{3.9}
\end{equation*}
$$

and $\lambda\left(\mathbf{A}^{-1}\right)<1$ since $\mathbf{A}$ is expanding. Fix a $\rho>1$ such that $\lambda\left(\mathbf{A}^{-1}\right)<1 / \rho<1$. It is shown in Lind [22] that there exists a norm $\|\cdot\|_{\mathrm{A}}$ on $\mathbb{R}^{d}$ with the property that

$$
\begin{equation*}
\left\|A^{-1} \mathbf{x}\right\|_{\mathrm{A}} \leq \frac{1}{\rho}\|\mathbf{x}\|_{\mathrm{A}} \quad \text { for all } \quad \mathbf{x} \in \mathbb{R}^{d} \tag{3.10}
\end{equation*}
$$

The norm is defined by $\|\mathrm{x}\|_{\mathrm{A}}:=\sum_{k=1}^{\infty} \rho^{k}\left\|A^{-k} \mathbf{x}\right\|$.
We claim that there exists an $R>0$ such that for any vertex $\mathbf{x}_{i}^{\prime} \in \underline{X}_{i}$ with $\left\|\mathbf{x}_{i}^{\prime}\right\|_{\mathrm{A}} \geq R$, any preimage $\mathbf{x}_{j} \in \underline{X}_{j}$ of $\mathbf{x}_{i}^{\prime}$ must satisfy

$$
\left\|\mathbf{x}_{j}\right\|_{\mathrm{A}} \leq \frac{2}{1+\rho}\left\|\mathrm{x}_{i}^{\prime}\right\|_{\mathrm{A}} .
$$

To see this, let

$$
\begin{equation*}
C=\max \left\{\left\|\mathbf{d}_{j}\right\|_{\mathrm{A}}: \mathbf{d}_{j} \in \mathcal{D}_{i j}, 1 \leq i, j \leq n\right\} \text { and } R:=\frac{\rho+1}{\rho-1} C \tag{3.11}
\end{equation*}
$$

It follows from $\mathbf{x}_{j}=\mathrm{A}^{-1}\left(\mathbf{x}_{i}^{\prime}-\mathbf{d}_{j}\right)$ that

$$
\begin{equation*}
\left\|\mathbf{x}_{j}\right\|_{\mathrm{A}} \leq \frac{1}{\rho}\left(\left\|\mathbf{x}_{i}^{\prime}\right\|_{\mathrm{A}}+C\right) \leq \frac{2}{1+\rho}\left\|\mathbf{x}_{i}^{\prime}\right\|_{\mathrm{A}} . \tag{3.12}
\end{equation*}
$$

Thus if a vertex $\mathbf{x}$ has $\|\mathbf{x}\|_{\mathrm{A}} \geq R$ then every preimage of $\mathbf{x}$ in the graph $\mathcal{G}(\mathcal{X})$ is smaller in the norm $\|\cdot\|_{\mathrm{A}}$ by a multiplicative constant $\frac{2}{1+\rho}<1$. It follows that every vertex x in $\mathcal{G}(\mathcal{X})$ can be reached by a finite directed path in $\mathcal{G}(\mathcal{X})$ starting from some vertex $\mathbf{x}^{\prime}$ with $\left\|\mathbf{x}^{\prime}\right\|_{\mathrm{A}} \leq R$. We therefore take the multiset family $\mathcal{S}^{(0)}=\left(S_{1}^{(0)}, \ldots, S_{n}^{(0)}\right)$ such that $S_{i}$ consists of those elements $\mathbf{x}$ of $X_{i}$ with $\|\mathbf{x}\|_{\mathrm{A}} \leq R$ and counting multiplicities, i.e.

$$
m_{S_{i}^{(0)}}(\mathbf{x})=m_{X_{i}}(\mathbf{x}) \quad \text { if } \quad \mathbf{x} \in \underline{S}_{i}^{(0)}
$$

The multiset family $\mathcal{S}^{(0)}$ is a finite family since $\mathcal{X}$ is discrete. The inclusion property

$$
\begin{equation*}
\mathcal{S}^{(0)} \subseteq \mathcal{S}^{(1)}=\psi\left(\mathcal{S}^{(0)}\right) \tag{3.13}
\end{equation*}
$$

holds because all preimages of vertices of $\mathcal{G}(\mathcal{X})$ in $\left\{\mathrm{x} \in \mathbb{R}^{d}:\|\mathrm{x}\|_{\mathrm{A}} \leq R\right\}$ lie in this set, since $\left\|\mathbf{x}_{j}\right\|_{\mathrm{A}} \leq \frac{1}{\rho}(R+C) \leq R$. Since $\mathcal{S}^{(0)} \subseteq \mathcal{X}$ and $\psi(\mathcal{X})=\mathcal{X}$ we obtain $\mathcal{S}^{(k)} \subseteq \mathcal{X}$ for all $k$, viewed as multisets, so that

$$
\mathcal{X}^{\prime}=\lim _{k \rightarrow \infty} \mathcal{S}^{(k)}
$$

exists and is a multiset. Now $\mathcal{S}^{(0)}$ contains all points $\mathbf{x}$ of $\mathcal{X}$ with the correct multiplicities for $\|\mathrm{x}\|_{\mathrm{A}} \leq R$. By induction on $k$ one proves:
(1) For each k , and $1 \leq i \leq n$, (2) For all $\mathbf{x}$ with $\|\mathbf{x}\|_{\mathrm{A}} \leq\left(\frac{1+\rho}{2}\right)^{k} R$, one has for $1 \leq i \leq n$ that the multiplicity

$$
m\left(\mathcal{S}_{i}^{(k)}, \mathbf{x}\right)=m_{X_{i}}(\mathbf{x})
$$

The base case $k=0$ holds by construction, and the induction step follows because all preimages of a vertex in $\left\{\|\mathbf{x}\| \leq\left(\frac{1+\rho}{2}\right)^{k} R\right\}$ lie in $\left\{\|\mathbf{x}\| \leq\left(\frac{1+\rho}{2}\right)^{k-1} R\right\}$.) Thus $\mathcal{X}^{\prime}=\mathcal{X}$.
(ii) We choose $R$ as in (i). Pick a point $\mathbf{x}$ in $\mathcal{X}$ with

$$
\mathbf{x} \in B_{R}:=\left\{\mathbf{y} \in \mathbb{R}^{d}:\|\mathbf{y}\|_{\mathbf{A}} \leq R\right\}
$$

which must exist by the argument in (i). All preimages of $\mathbf{x}$ necessarily lie in $B_{R}$. Since $\mathcal{X}$ is discrete, if we follow a chain of successive preimages of a vertex x in $\mathcal{G}(\mathcal{X})$ we stay in the finite set $\mathcal{X} \cap B_{R}$, so some vertex must occur twice. The path from this vertex to itself forms a directed cycle in $\mathcal{G}(\mathcal{X})$.

Recall that a multiset family $\mathcal{X}$ satisfying an inflation functional equation is irreducible if it has no nontivial partition $\mathcal{X}=\mathcal{X}_{1} \vee \mathcal{X}_{2}$, with both $\mathcal{X}_{i}$ satisfying the same equation.

Theorem 3.3 (Irreducible Set Characterization) Suppose that the multiset family $\mathcal{X}$ is discrete and satisfies an inflation functional equation $\psi(\mathcal{X})=\mathcal{X}$. The following are equivalent:
(i) $\mathcal{X}$ is irreducible.
(ii) The graph $\mathcal{G}(\mathcal{X})$ contains exactly one directed cycle (which may be a loop), and the elements of this cycle have multiplicity one.

If condition (ii) holds, let $Y=\left\{\mathbf{x}_{1}, \ldots, \mathbf{x}_{p}\right\}$ be the cycle in $\mathcal{G}(\mathcal{X})$ with multiplicity one and define the multiset family $\mathcal{S}^{(0)}=\left(S_{1}^{(0)}, \ldots, S_{n}^{(0)}\right)$ by $S_{i}=\{\mathbf{x} \in Y: \mathbf{x}$ has color $i\}$. Then $\mathcal{S}^{(0)} \subseteq \psi\left(\mathcal{S}^{(0)}\right)$ and

$$
\begin{equation*}
\mathcal{X}=\lim _{k \rightarrow \infty} \mathcal{S}^{(k)} \tag{3.14}
\end{equation*}
$$

Remark. In view of (ii) and (3.14) we call $\mathcal{S}^{(0)}$ the generating cycle of the irreducible multiset system $\mathcal{X}$, and we call $p$ the period of this cycle.

Proof. (i) $\Rightarrow$ (ii). Lemma 3.2 shows that $\mathcal{X}$ is generated by the points $\mathbf{x}$ in $\mathcal{X}$ that lie in the compact region $\|\mathrm{x}\|_{\mathrm{A}} \leq R$, and that $\mathcal{G}(\mathcal{X})$ contains vertices forming a directed cycle $Y$ inside this region. The vertices of $Y$ define a finite multiset family $\mathcal{S}^{(0)}=\left(S_{1}^{(0)}, \ldots, S_{n}^{(0)}\right)$, with $S_{i}^{(0)}=$ $\underline{X}_{i} \cap Y$, and all elements of $S_{i}^{(0)}$ have multiplicity one. It is clear that $\mathcal{S}^{(0)} \subseteq \mathcal{S}^{(1)}=\psi\left(\mathcal{S}^{(0)}\right)$ since each element of $\mathcal{S}^{(0)}$ has a preimage in $\mathcal{S}^{(0)}$. Since each $\mathcal{S}^{(k)} \subseteq \mathcal{X}$ the limit

$$
\begin{equation*}
\mathcal{X}^{\prime}:=\lim _{k \rightarrow \infty} \mathcal{S}^{(k)} \tag{3.15}
\end{equation*}
$$

exists and is a multiset family with $\mathcal{X}^{\prime} \subseteq \mathcal{X}$.
We claim that $\mathcal{X}^{\prime}=\mathcal{X}$. If not, then we obtain a partition $\mathcal{X}=\mathcal{X}^{\prime} \vee \mathcal{X}^{\prime \prime}$ into multisets, and since $\psi\left(\mathcal{X}^{\prime}\right)=\mathcal{X}^{\prime}$ we have $\psi\left(\mathcal{X}^{\prime \prime}\right)=\mathcal{X}^{\prime \prime}$. Both $\mathcal{X}^{\prime}$ and $\mathcal{X}^{\prime \prime}$ are discrete since $\mathcal{X}$ is, and this contradicts the irreducibility of $\mathcal{X}$. This proves the claim, and (3.14) follows. Now (3.14) implies that every element of $\mathcal{X}$ is reachable by a directed path starting from any fixed vertex $\mathrm{x}_{0}$ in the cycle $Y$.

We next show that $\mathcal{X}$ has multiplicity one on the vertices of $Y$. Suppose not. The elements of $\mathcal{S}^{(0)}$ have multiplicity one, and at some later stage some vertex $(\mathbf{x}, i)$ of the cycle $Y$ has multiplicity exceeding one in some $\mathcal{S}^{(k)}$. Thus there exists a path to ( $\mathrm{x}, i$ ) of length $k$ arising from some vertex $(\mathbf{y}, j)$ in $\mathcal{S}^{(0)}$, which does not arise purely from moving on the cycle $Y$. Since the vertex $(\mathbf{y}, j)$ can be reached from ( $\mathbf{x}, i)$ by moving around the cycle $Y$, taking $l$ steps, say, we obtain a directed path from $(\mathbf{x}, i)$ to itself of length $k+l$ which does not stay in the cycle $Y$. Now we obtain two distinct directed paths of length $(k+l) p$ from ( $\mathbf{x}, i$ ) to itself, one by wrapping around the cycle $Y k+l$ times, the other by repeating the path of length $k+l$ $p$ times. Since we can concatenate these two distinct paths in any order, it follows that the multiplicity of $(\mathbf{x}, i)$ in $\mathcal{S}^{((k+l) p m)}$ is at least $2^{m}$. This says implies that ( $\left.\mathbf{x}, i\right)$ has unbounded multiplicity as $m \rightarrow \infty$, which contradicts the discreteness. Thus $\mathcal{X}$ has multiplicity one on the cycle $Y$.

Finally we show that $\mathcal{X}$ contains only one directed cycle. Suppose not, and that $\mathcal{G}(\mathcal{X})$ contains a directed cycle $Y^{\prime}$ different from $Y$. The same argument as above says that $\mathcal{X}$ is generated by the cycle $Y^{\prime}$ and that all elements of $Y^{\prime}$ in $\mathcal{X}$ have multiplicity one. By exchanging $Y$ and $Y^{\prime}$ if necessary we may suppose that $Y$ has a vertex not contained in $Y^{\prime}$. Now following a path from a vertex in $Y \backslash Y^{\prime}$ to $Y^{\prime}$ we see that there exists an $\mathrm{x}^{\prime} \in Y^{\prime}$ that has a preimage not in $Y^{\prime}$. But $\mathrm{x}^{\prime}$ also has a preimage in $Y^{\prime}$ because $Y^{\prime}$ is a cycle. This implies that the multiplicity of $\mathbf{x}^{\prime}$ is at least 2 , a contradiction. Thus $\mathcal{G}(\mathcal{X})$ contains exactly one directed cycle.
(ii) $\Rightarrow$ (i). Suppose that $\mathcal{G}(\mathcal{X})$ contains exactly one directed cycle $Y$, of multiplicity one. We argue by contradiction. Suppose $\mathcal{X}=\mathcal{X}^{\prime} \vee \mathcal{X}^{\prime \prime}$. Now Lemma 3.2 (ii) implies that both $\mathcal{G}\left(\mathcal{X}^{\prime}\right)$ and $\mathcal{G}\left(\mathcal{X}^{\prime \prime}\right)$ contain a directed cycle. Therefore $\mathcal{G}(\mathcal{X})$ either contains two directed cycles, or else contains a directed cycle of multiplicity at least two. This contradicts the hypothesis.

We can now prove the Decomposition Theorem.

Proof of Decomposition Theorem 2.1. By definition if $\mathcal{X}$ is not irreducible then $\mathcal{X}=$ $\mathcal{X}^{(1)} \vee \mathcal{X}{ }^{(2)}$ where each $\mathcal{X}^{(i)}$ satisfies $\mathcal{X}^{(i)}=\psi\left(\mathcal{X}^{(i)}\right)$. Lemma 3.2 showed that $\mathcal{X}$ and $\mathcal{X}^{(i)}$ are generated by elements in the bounded region $\|\mathrm{x}\|_{\mathrm{A}} \leq R$. So the total multiplicity of elements of each $\mathcal{X}^{(i)}$ in the region $\|\mathrm{x}\|_{\mathrm{A}} \leq R$ is strictly smaller than that of elements of $\mathcal{X}$ in the region $\|\mathbf{x}\|_{\mathrm{A}} \leq R$. If one of $\mathcal{X}^{(i)}$ is not irreducible then we can further partition it, and decrease the total multiplicity in $\|\mathrm{x}\|_{\mathrm{A}} \leq R$ again. Since $\mathcal{X}$ is discrete, this process will end in finitely many steps, yielding

$$
\mathcal{X}=\bigvee_{i=1}^{N} \mathcal{X}^{(i)}
$$

To see that the partition is unique, to each element $\mathcal{X}_{i}$ of the partition is associated a unique directed cycle in $\mathcal{G}(\mathcal{X})$ of multiplicity one. But each directed cycle in $\mathcal{G}(\mathcal{X})$, counted with multiplicity one, must appear in some element of each partition.

We return to the study of discrete multiset families $\mathcal{X}$ with $\psi(\mathcal{X})=\mathcal{X}$ that are irreducible. We can give some further information on the multiplicities that occur in such $\mathcal{X}$, under the extra hypothesis that the subdivision matrix is irreducible.

Theorem 3.4 (Multiplicity Dichotomy) Suppose that the multiset family $\mathcal{X}$ is discrete and satisfies the inflation functional equation $\psi(\mathcal{X})=\mathcal{X}$ with primitive subdivision matrix. If $\mathcal{X}=\left(X_{1}, X_{2}, \ldots, X_{n}\right)$ is irreducible, then exactly one of the following cases holds:
(i) Every element of each multiset $X_{i}$ has multiplicity at most $p$, the period of the generating cycle in the directed graph $\mathcal{G}(\mathcal{X})$.
(ii) For each multiset $X_{i}$ in $\mathcal{X}$, the multiplicities $m_{X_{i}}(\mathbf{x})$ of $\mathbf{x} \in X_{i}$ are unbounded.

Remarks. (1) In the general case of imprimitive $S$, one can prove a similar result giving a dichotomy that applies to those $X_{i}$ such that $i$ is in the strongly connected component of the underlying graph of $S$ which contains the generating cycle.
(2) There exist irreducible discrete $\mathcal{X}$ having a primitive subdivision matrix, such that all $X_{i}$ have bounded multiplicity but some of the elements have multiplicity greater than 1 , see Example 7.5.

Proof. The multiplicity of a vertex ( $\mathbf{x}, i)$ having color $i$ in $\mathcal{G}(\mathcal{X})$ is $m(\mathbf{x}, i):=m_{X_{i}}(\mathbf{x})$. Fix a point $\left(\mathrm{x}_{0}, i_{0}\right)$ in the generating cycle. We claim: For any point $(\mathbf{x}, i) \neq\left(\mathrm{x}_{0}, i_{0}\right)$ in $\mathcal{X}$ the multiplicity $m(\mathbf{x}, i)$ is equal to the number of cycle-free directed paths from ( $\mathbf{x}_{0}, i_{0}$ ) to $(\mathbf{x}, i)$. This is proved by induction on the length $k$ of the longest cycle-free directed path from $\mathbf{x}_{0}$ to $\mathbf{x}$, which is bounded by (3.12). The base case $k=1$ is immediate. We have

$$
m(\mathbf{x}, i)=\sum_{\left(\mathbf{x}^{\prime}, j\right) \in P(\mathbf{x}, i)} m\left(\mathbf{x}^{\prime}, j\right)
$$

where $P(\mathbf{x}, i)$ is the set of preimages of $(\mathbf{x}, i)$ in $\mathcal{G}(\mathcal{X})$, Assuming $(\mathbf{x}, i) \neq\left(\mathbf{x}_{0}, i_{0}\right)$ all preimages have shorter longest cycle-free path, and the induction step follows.

Now assume that some vertex $\left(\mathbf{x}_{1}, j\right)$ has multiplicity $m\left(\mathbf{x}_{1}, j\right)=q>p$. Without loss of generality we may assume that $\left(\mathrm{x}_{1}, j\right)$ and ( $\mathrm{x}_{0}, i_{0}$ ) have the same color $j=i_{0}$, for if not by the primitivity hypothesis we can always find a descendant $\left(\mathrm{x}_{1}^{\prime}, i_{0}\right)$ of $\left(\mathrm{x}_{1}, j\right)$, and clearly $\left.m\left(\mathbf{x}_{1}^{\prime}, i_{0}\right) \geq m\left(\mathbf{x}_{1}, j\right)\right)>p$. By the pigeonhole principle there exist two cycle-free directed paths from $\left(\mathbf{x}_{0}, i_{0}\right)$ to ( $\mathbf{x}_{1}, i_{0}$ ), whose lengths are $L$ and $L^{\prime}$ respectively with $L \equiv L^{\prime}(\bmod p)$. Say $L=L^{\prime}+s p$ for some $s \geq 0$. We can then create a second directed path of length $L$ by first going around the cycle $s$ times in the beginning and then following the path of length $L^{\prime}$. We show that these two different directed paths force the multiplicity of vertices to be unbounded.

Since each directed edge can be labeled by elements in $\mathcal{D}_{i j}, 1 \leq i, j \leq m$, we label the two directed paths by

$$
\mathcal{P}_{1}=\left(\mathbf{d}_{1}, \ldots, \mathbf{d}_{L}\right), \quad \mathcal{P}_{2}=\left(\mathbf{d}_{1}^{\prime}, \ldots, \mathbf{d}_{L}^{\prime}\right) .
$$

The fact that $\mathbf{x}_{0}$ and $\mathbf{x}_{1}$ have the same color $i$ implies that $\mathbf{d}_{1} \in \mathcal{D}_{j i}, \mathbf{d}_{L} \in \mathcal{D}_{i k}$ and $\mathbf{d}_{1}^{\prime} \in \mathcal{D}_{j^{\prime} i}$, $\mathbf{d}_{L}^{\prime} \in \mathcal{D}_{i k^{\prime}}$ for some $j, k, j^{\prime}, k^{\prime}$. Evaluating the two paths yields

$$
\begin{equation*}
\mathbf{x}_{1}=\mathrm{A}^{L} \mathbf{x}_{0}+\sum_{j=1}^{L} \mathrm{~A}^{j-1} \mathbf{d}_{j}=\mathrm{A}^{L} \mathbf{x}_{0}+\sum_{j=1}^{L} \mathrm{~A}^{j-1} \mathbf{d}_{j}^{\prime} \tag{3.16}
\end{equation*}
$$

So $\sum_{j=1}^{L} \mathrm{~A}^{j-1} \mathbf{d}_{j}=\sum_{j=1}^{L} \mathrm{~A}^{j-1} \mathbf{d}_{j}^{\prime}$. This means that following the two paths $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ from ( $\mathrm{x}_{1}, i_{0}$ ) will lead to the same vertex $\left(\mathrm{x}_{2}, i_{0}\right)$, which also has color $i_{0}$. This process can be continued to obtain vertices $\left(\mathbf{x}_{k}, i_{0}\right), k \geq 1$, all of which have the same color $i_{0}$. Note that there are at least $2^{k}$ distinct directed paths from $\left(\mathrm{x}_{0}, i_{0}\right)$ to ( $\mathrm{x}_{k}, i_{0}$ ) as we may concatenate $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ in any combination. These $2^{k}$ directed paths remain distinct after removing the cycles in the initial segment. Hence $m\left(\mathbf{x}_{k}, i_{0}\right) \geq 2^{k}$. So the multiplicity of $X_{i_{0}}$ is unbounded. Now primitivity of the subdivision matrix implies that any vertex ( $\mathbf{x}, i_{0}$ ) has a descendant in each $X_{i}$. It follows that the multiplicity function is unbounded on every $X_{i}$.

Lemma 3.5. Let $\psi(\cdot)$ be an inflation functional equation with data $\left(\mathbf{A}, \mathcal{D}_{i j}\right)$. Then for each $p \geq 1$ there are at most finitely many discrete multiset families (possibly none) $\mathcal{X}$ satisfying $\psi(\mathcal{X})=\mathcal{X}$ that are irreducible and have a periodic cycle in $\mathcal{G}(\mathcal{X})$ of minimal period $p$.

Proof. Any periodic cycle $\left\{\left(\mathbf{x}_{0}, i_{0}\right),\left(\mathbf{x}_{1}, i_{1}\right), \ldots,\left(\mathbf{x}_{p-1}, i_{p-1}\right)\right\}$ constructed in Theorem 3.3 has the form

$$
\mathbf{x}_{k}=\mathrm{Ax}_{k-1}+\mathbf{d}_{k}, \quad 1 \leq k \leq p
$$

with $\mathbf{d}_{k} \in \mathcal{D}_{i_{k}, i_{k-1}}$ and $\left(\mathbf{x}_{p}, i_{p}\right)=\left(\mathbf{x}_{0}, i_{0}\right)$. It follows that

$$
\mathbf{x}_{0}=\mathrm{A}^{p} \mathbf{x}_{0}+\sum_{k=1}^{p} \mathrm{~A}^{p-k} \mathbf{d}_{k} .
$$

Thus

$$
\begin{equation*}
\mathbf{x}_{0}=-\left(\mathbf{A}^{p}-\mathbf{I}\right)^{-1}\left(\sum_{k=1}^{p} \mathbf{A}^{p-k} \mathbf{d}_{k}\right) \tag{3.17}
\end{equation*}
$$

where the matrix $A^{p}-I$ is invertible since $A$ is expanding. There are only a finite set of choices for the digits $\left\{\mathbf{d}_{1}, \ldots, \mathbf{d}_{p}\right\}$, so the number of choices for $\mathbf{x}_{0}$ is finite.

As we have shown, all discrete irreducible multiset families $\mathcal{X}$ are generated iteratively from a cycle as "seed". However, the multiset family generated from a given cycle need not be discrete. The following lemma gives a criterion for discreteness.

Lemma 3.6. Let $\psi(\cdot)$ be an inflation functional equation with data $\left(\mathbf{A}, \mathcal{D}_{i j}\right)$, and let $\mathcal{S}^{(0)}$ be a finite multiset family with the inclusion property $\mathcal{S}^{(0)} \subseteq \mathcal{S}^{(1)}=\psi\left(\mathcal{S}^{(0)}\right)$. Set $R=R(\mathrm{~A}, \mathcal{D})$ equal to the constant in (3.11). If for some $k \geq 1$,

$$
\begin{equation*}
\mathcal{S}^{(k)} \cap\left\{\mathbf{x} \in \mathbb{R}^{d}:\|\mathbf{x}\|_{\mathbf{A}} \leq R\right\}=\mathcal{S}^{(k+1)} \cap\left\{\mathbf{x} \in \mathbb{R}^{d}:\|\mathbf{x}\|_{\mathrm{A}} \leq R\right\} \tag{3.18}
\end{equation*}
$$

counting multiplicities, then the limit

$$
\mathcal{X}:=\lim _{k \rightarrow \infty} \mathcal{S}^{(k)}
$$

is a multiset family that satisfies $\psi(\mathcal{X})=\mathcal{X}$, whose underlying sets $\left(\underline{X}_{1}, \ldots, \underline{X}_{n}\right)$ are discrete. Conversely, if $\mathcal{X}$ is discrete then (3.18) holds for all sufficiently large $k$.

Proof. The iterative scheme $\mathcal{S}^{(k+1)}=\psi\left(\mathcal{S}^{(k)}\right)$ is geometrically expanding outside the set $\left\{\mathbf{x} \in \mathbb{R}^{d}:\|\mathbf{x}\|_{\mathbf{A}} \leq R\right\}$, i.e. (3.12) gives

$$
\|\mathbf{A} \mathbf{x}+\mathbf{d}\| \geq \rho\|\mathbf{x}\|-C \geq \frac{\rho^{2}+1}{\rho+1}\|\mathbf{x}\|
$$

The condition (3.18) says that $\mathcal{S}^{(k)}$ stabilizes inside $\left\{\mathrm{x} \in \mathbb{R}^{d}:\|\mathrm{x}\|_{\mathrm{A}} \leq R\right\}$. By induction on $j \geq 0$ one obtains that $\mathcal{S}^{(k+j)}$ stabilizes inside the domain $\left\{\mathbf{x} \in \mathbb{R}^{d}:\|\mathrm{x}\|_{\mathrm{A}} \leq\left(\frac{\rho^{2}+1}{\rho+1}\right)^{j} R\right\}$. The limit family $\mathcal{X}$ is discrete inside each of these domains, hence is discrete.

Lemma 3.6 leads to an algorithm which recognizes irreducible discrete families. More precisely, consider the class of inflation functional equations $\psi(\cdot)$ whose data ( $\mathbf{A}, \mathcal{D}_{i j}$ ) is drawn from a computable subfield $K$ of $\mathbb{C}$, which we take to be one in which addition and multiplication are computable, and one can effectively test equality or inequality of field elements. Then all finite cycles $Y$ for such $\psi(\cdot)$ will have elements in $K$, one can give a procedure which tests whether a given seed $\mathcal{S}^{(0)}$ generates an irreducible discrete $X$, using the criterion of Lemma 3.6, and which will eventually halt for all such sets. (One does not need to compute $R$ or $\|\cdot\|_{\mathrm{A}}$ exactly to guarantee halting.) However it need not halt on sets which are irreducible but not discrete. This algorithm will detect non-irreducible sets eventually by finding a second cycle or a cycle with multiplicity.

There are substantial restrictions on $\left(\mathrm{A}, \mathcal{D}_{i j}\right)$ necessary for there to exist some irreducible discrete multiset family $\mathcal{X}$ with $\psi(\mathcal{X})=\mathcal{X}$. We give a simple case when all irreducible solutions to the inflation functional equation are discrete (if any exist).

Lemma 3.7. Suppose the inflation functional equation $\psi(\cdot)$ has data $\left(\mathbf{A}, \mathcal{D}_{i j}\right)$, in which $\mathbf{A}$ is an expanding integer matrix and all vectors in $\mathcal{D}_{i j}$ have rational entries. Then all irreducible multiset families generated by a periodic cycle are discrete.

Proof. All cycles $Y$ of period $p$ are generated by solutions of the form (3.17), and the hypothesis guarantee that such a solution $\mathbf{x}$ is a rational vector. Starting from ( $\mathbf{x}, i$ ) one generates the orbit $Y$ and a seed $\mathcal{S}^{(0)}$ which consists of rational vectors. All elements of $\mathcal{S}^{(k)}$ are rational vectors with denominators dividing $d D$, where $d$ is the denominator of $\mathbf{x}$ and $D$ is the greatest common denominator of all the rationals in $\mathcal{D}_{i j}$. Thus there are only a finite number of possible choices for rational vectors in the ball of radius $R=R(\mathbf{A}, \mathcal{D})$ in (3.11).

Thus the criterion of Lemma 3.6 applies to conclude that if $\mathcal{X}=\lim _{k \rightarrow \infty} \mathcal{S}^{(k)}$ exists, then it is necessarily discrete.

Lemma 3.7 does not assert existence of irreducible multiset families, only that they are discrete if they do exist. Indeed there are cases ( $\mathrm{A}, \mathcal{D}_{i j}$ ) satisfying the hypotheses of Lemma 3.7 where no discrete solutions exist, see Example 7.4. At the other extreme, there exist examples that have infinitely many different irreducible discrete multiset families, see Example 7.3.

We conclude this section by raising two related computational problems:
Computational Problem (i). Given a seed $\mathcal{S}_{0}$ which generates an irreducible discrete multiset family $\mathcal{X}$ satisfying a given inflation functional equation $\psi(\mathcal{X})=\mathcal{X}$ and an index $i$, determine whether the multiplicities of points in $X_{i}$ are bounded or not.

Computational Problem (ii). Given the same data as above, determine the maximum multiplicity of points in $X_{i}$.
We do not know of algorithms to resolve either of these questions in general. Note that if $\mathcal{X}$ is irreducible, the criterion of Lemma 3.6 will allow this to be certified, for some $k$.

## 4. Perron Eigenvalue Condition

In this section we consider inflation functional equations having a primitive subdivision matrix $\mathbf{S}$ so that the Perron eigenvalue $\lambda(\mathbf{S})$ is defined. Our main object in this section is to show that the weak Delone set family property constrains the Perron eigenvalue $S$ to satisfy the Perron eigenvalue condition

$$
\begin{equation*}
\lambda(\mathbf{S})=|\operatorname{det}(\mathbf{A})| \tag{4.1}
\end{equation*}
$$

as given in Theorem 2.3. To achieve this we prove several preliminary results. Some of them apply more generally to multiset families $\mathcal{X}$ which are weakly uniformly discrete in the following sense.

Definition 4.1. A multiset family $\mathcal{X}$ is weakly uniformly discrete if the union $\bigvee_{i=1}^{n} X_{i}$ is a uniformly discrete multiset.

Note that a finite union of weakly uniformly discrete multisets is itself a weakly uniformly discrete multiset. So a multiset family $\mathcal{X}$ is weakly uniformly discrete if and only if each $X_{i}$ is.

We first consider an irreducible multiset family $\mathcal{X}$ satisfying $\mathcal{X}=\psi(\mathcal{X})$. Let $\mathcal{S}^{(0)}$ be the generating cycle for $\mathcal{X}$ and set $\mathcal{S}^{(k+1)}=\psi\left(\mathcal{S}^{(k)}\right)$. Denote

$$
H_{i}^{(k)}=\sum_{\mathbf{x} \in \S_{( }^{(k)}} m_{S_{i}^{(k)}}(\mathbf{x}),
$$

where $\left(S_{1}^{(k)}, S_{2}^{(k)}, \ldots, S_{n}^{(k)}\right)=\mathcal{S}^{(k)}$ as usual. $H_{i}^{(k)}$ is the total multiplicity of the multiset $\mathcal{S}^{(k)_{i}}$.
Lemma 4.1. Let $\mathcal{X}$ be an irreducible discrete multiset family satisfying $\psi(\mathcal{X})=\mathcal{X}$ for the data $\left(\mathbf{A}, \mathcal{D}_{i j}\right)$, with subdivision matrix $\mathbf{S}$. Let $\mathbf{H}^{(k)}=\left[H_{1}^{(k)}, \ldots, H_{n}^{(k)}\right]^{T}$. Then $\mathbf{H}^{(k)}=\mathrm{S}^{k} \mathbf{H}^{(0)}$.

Proof. We prove that $\mathbf{H}^{(k+1)}=\mathbf{S} \mathbf{H}^{(k)}$. Observe that

$$
S_{i}^{(k+1)}=\bigvee_{j=1}^{n}\left(S_{j}^{(k)}+\mathcal{D}_{i j}\right)
$$

Hence

$$
H_{i}^{(k+1)}==\sum_{j=1}^{m}\left|\mathcal{D}_{i j}\right| H_{j}^{(k)},
$$

yielding $\mathbf{H}^{(k+1)}=\mathbf{S} \mathbf{H}^{(k)}$. Therefore $\mathbf{H}^{(k)}=\mathrm{S}^{k} \mathbf{H}^{(0)}$.
Theorem 4.2. Let $\mathcal{X}$ be an irreducible discrete multiset family satisfying $\psi(\mathcal{X})=\mathcal{X}$ for the data $\left(\mathbf{A}, \mathcal{D}_{i j}\right)$, with primitive subdivision matrix $\mathrm{S}=\left[\left|\mathcal{D}_{i j}\right|\right]$. Let $B_{1}(\mathbf{0})$ be the unit ball and set

$$
\begin{equation*}
M_{i}^{(k)}:=\sum_{\mathbf{x} \in \underline{X}_{i} \cap A^{k}{ }_{\left(B_{1}(\mathbf{0})\right)}} m_{X_{i}}(\mathbf{x}) . \tag{4.2}
\end{equation*}
$$

Then there exist constants $C_{1}, C_{2}>0$ such that

$$
\begin{equation*}
C_{1} \lambda(\mathrm{~S})^{k} \leq M_{i}^{(k)} \leq C_{2} \lambda(\mathrm{~S})^{k} \quad \text { for all } k . \tag{4.3}
\end{equation*}
$$

Proof. We first establish the lower bound. Observe that for each element $\mathbf{x} \in \bigvee_{i=1}^{n} S_{i}^{(k)}$ there exist digits $\mathbf{d}_{1}, \ldots, \mathbf{d}_{k}$ in the collection of digit sets $\left\{\mathcal{D}_{i j}\right\}$ such that $\mathbf{x}=\tau_{\mathbf{d}_{1}} \circ \cdots \circ \tau_{\mathbf{d}_{k}}\left(\mathbf{x}_{0}\right)$, where $\mathbf{x}_{0}$ is in the generating cycle and $\tau_{\mathbf{d}}(\mathbf{x}):=\mathbf{A x}+\mathbf{d}$. So

$$
\mathbf{x}=\mathrm{A}^{k} \mathbf{x}_{0}+\sum_{j=1}^{k} \mathrm{~A}^{j-1} \mathbf{d}_{j}=\mathrm{A}^{k}\left(\mathbf{x}_{0}+\sum_{j=1}^{k} \mathrm{~A}^{j-1-k} \mathbf{d}_{j}\right) .
$$

Note that $\mathbf{A}$ is expanding. Hence there exists a constant $C$ depending only on $\left\{\mathcal{D}_{i j}\right\}, \mathrm{A}$ and $\mathcal{S}^{(0)}$ such that

$$
\left\|\mathrm{x}_{0}+\sum_{j=1}^{k} \mathrm{~A}^{j-1-k} \mathbf{d}_{j}\right\| \leq C
$$

Fix an integer $k_{1}>0$ so that $B_{C}(\mathbf{0}) \subseteq \mathrm{A}^{k_{1}}\left(B_{1}\right)$. Then $\mathbf{x} \in \mathrm{A}^{k+k_{1}}\left(B_{1}\right)$. It follows that for any $\mathrm{x} \in \underline{X}_{i} \cap \mathrm{~A}^{k+k_{1}}\left(B_{1}\right)$ we always have

$$
m_{X_{i}}(\mathbf{x})=m_{S_{i}^{(k)}}(\mathbf{x})
$$

Therefore $M_{i}^{\left(k+k_{1}\right)} \geq H^{(k)}$ for all $k$, where by Lemma 4.1 and the primitivity of S we also have $H^{(k)} \geq C_{1}^{\prime} \lambda(\mathrm{S})^{k}$ for some constant $C_{1}^{\prime}$. The lower bound follows by taking $C_{1}=C_{1}^{\prime} \lambda(\mathrm{S})^{-k_{1}}$.

To establish the upper bound we first claim that there exists a $k_{2}$ such that for any $k, i$ and $\mathbf{x} \in \underline{X}_{i} \cap \mathrm{~A}^{k}\left(B_{1}\right)$ and any vertex $\mathbf{x} \in \underline{X}_{i}$ with $\mathbf{x} \in \mathrm{A}^{k}\left(B_{1}\right)$ we must have

$$
\begin{equation*}
m_{X_{i}}(\mathbf{x})=m_{S_{i}^{\left(k+k_{2}\right)}}(\mathbf{x}) \tag{4.4}
\end{equation*}
$$

This would imply that $M_{i}^{(k)} \leq H_{i}^{\left(k+k_{2}\right)}$. As for the lower bound, it follows from Lemma 4.1 and the primitivity of S that there exists a constant $C_{2}^{\prime}$ such that $M_{i}^{(k)} \leq C_{2}^{\prime} \lambda(\mathrm{S})^{k+k_{2}}$. The upper bound is proved by taking $C_{2}=C_{2}^{\prime} \lambda(\mathrm{S})^{k_{2}}$.

To prove this claim we note that any $\mathbf{x} \in \underline{X}_{i} \cap \mathrm{~A}^{k}\left(B_{1}\right)$ must be in $S_{i}^{(l)}$ for some $l$ as a result of Lemma 3.1. Suppose that $l \geq k$. Then there exist some digits $\mathbf{d}_{1}, \mathbf{d}_{2}, \ldots, \mathbf{d}_{l}$ in $\mathcal{D}_{i j}$ 's such that

$$
\mathbf{x}=\mathrm{A}^{k} \mathbf{x}_{0}+\sum_{j=1}^{l} \mathbf{A}^{j-1} \mathbf{d}_{j}
$$

and

$$
\begin{equation*}
\mathbf{x}_{0}=\mathbf{A}^{-l} \mathbf{x}-\sum_{j=1}^{k} \mathrm{~A}^{-j} \mathbf{d}_{k+1-j} \tag{4.5}
\end{equation*}
$$

Observe that $\mathbf{x}_{0}$ is bounded since $l \geq k, \mathbf{A}^{-k} \mathbf{x} \in B_{1}$ and $\mathbf{A}$ is expanding. Hence there exists a constant $R_{0}$ independent of $k$ and $l$ such that $\left\|\mathbf{x}_{0}\right\| \leq R_{0}$. Let

$$
k_{2}=\sum_{i=1}^{n}\left|\underline{X}_{i} \cap B_{s} R_{0}(\mathbf{0})\right| .
$$

Assume that (4.4) is false. Then there exist some $k, i$ and $\mathbf{x} \in \underline{X}_{i} \cap \mathrm{~A}^{k}\left(B_{1}\right)$ such that $m_{X_{i}}(\mathbf{x})>m_{S_{i}^{\left(k+k_{2}\right)}}(\mathbf{x})$. This means there exists a cycle-free directed path in $\mathcal{G}(\mathcal{X})$ of length $l>k+k_{2}$ from an element $\mathbf{x}_{0}$ in the generating cycle $\mathcal{S}^{(0)}$ to $\mathbf{x}$, say $\mathbf{x}=\tau_{\mathbf{d}_{l}} \circ \cdots \circ \tau_{\mathbf{d}_{2}} \circ \tau_{\mathbf{d}_{1}}\left(\mathbf{x}_{0}\right)$. Let $\mathbf{x}_{j}=\tau_{\mathbf{d}_{j}}\left(\mathbf{x}_{j-1}\right), 1 \leq j \leq l$, be the vertices of this directed path. It follows that $\left\|\mathbf{x}_{j}\right\| \leq R_{0}$ for $0 \leq j \leq l-k$. But there are only $k_{2}$ vertices in $B_{R_{0}}(\mathbf{0})$ and $l-k>k_{2}$, hence there exist two identical vertices among $\left\{\mathbf{x}_{j}: 1 \leq j \leq l-k\right\}$. This contradicts the assumption that the directed path is cycle-free, proving (4.4).

Theorem 4.3. Let $\mathcal{X}$ be a weakly uniformly discrete $n$-multiset family satisfying $\psi(\mathcal{X})=\mathcal{X}$ for the data $\left(\mathbf{A}, \mathcal{D}_{i j}\right)$, with primitive subdivision matrix $\mathbf{S}$. Then $\lambda(\mathbf{S}) \leq|\operatorname{det}(\mathbf{A})|$.

Proof. We first assume that $\mathcal{X}$ is irreducible. The weak uniform discreteness of $\mathcal{X}$ implies that $M_{i}^{(k)} \leq C \operatorname{Vol}\left(\mathrm{~A}^{k} B_{1}\right)=C|\operatorname{det}(\mathrm{~A})|^{k}$ for some positive constant $C$. ¿From Theorem 4.2 it immediately follows that $|\operatorname{det}(\mathbf{A})| \geq \lambda(\mathbf{S})$.

If $\mathcal{X}$ is reducible, then $\mathcal{X}=\bigvee_{j=1}^{N} \mathcal{X}^{(j)}$. The argument above now applies to $\mathcal{X}^{(1)}$ to yield $\lambda(\mathbf{S}) \leq|\operatorname{det}(\mathbf{A})|$.

Proof of Theorem 2.3. Since $\mathcal{X}$ is a weak substitution Delone multiset family, it is weakly uniformly discrete. By theorem 4.3 this immediately gives $\lambda(\mathbf{S}) \leq|\operatorname{det}(\mathbf{A})|$.

To prove the other direction, let $\mathcal{X}=\bigvee_{j=1}^{N} \mathcal{X}^{(j)}$. Let $M_{i, j}^{(k)}$ be as in (4.2), but defined for $\mathcal{X}^{(j)}$. Then the relative denseness of each $X_{i}$ yields

$$
\sum_{j=1}^{N} M_{i, j}^{(k)} \geq C^{\prime} \operatorname{Vol}\left(\mathbf{A}^{k} B_{1}\right)=C^{\prime}|\operatorname{det}(\mathbf{A})|^{k}
$$

for some positive constant $C^{\prime}$. Hence $\max _{j} M_{i, j}^{(k)} \geq \frac{1}{N} C^{\prime}|\operatorname{det}(\mathrm{A})|^{k}$. Taking the $k$-th roots and letting $k \rightarrow \infty$ now yields $|\operatorname{det}(\mathbf{A})| \leq \lambda(\mathbf{S})$, using Theorem 4.2.

## 5. Uniformly Discrete Substitution Sets and Tilings

Our object in this section is to relate the existence of substitution Delone set families to that of self-replicating tilings, and to deduce Theorem 2.4.

In $\S 4$ we studied the Perron eigenvalue of $S$ in relation to the geometric properties of discrete solutions of a primitive inflation functional equation. We showed that the Perron eigenvalue condition $\lambda(\mathbf{S})=|\operatorname{det} \mathbf{A}|$ is necessary for it to have a solution that is a substitution Delone set family, and $\lambda(S) \leq|\operatorname{det} A|$ is necessary for it to have a weakly uniformly discrete solution. On the other hand, $\lambda(\mathrm{S}) \geq|\operatorname{det} \mathrm{A}|$ is necessary for the corresponding multi-tile equation to have a solution in which all $T_{i}$ have positive Lebesgue measure. To see this, taking the Lebesgue measure on both sides of the multi-tile equation (1.1) leads to

$$
|\operatorname{det} \mathbf{A}| \mu\left(T_{i}\right) \leq \sum_{j=1}^{n}\left|\mathcal{D}_{j i}\right| \mu\left(T_{j}\right),
$$

where $\left|\mathcal{D}_{j i}\right|$ denotes the cardinality of $\mathcal{D}_{j i}$ (counting multiplicity). Hence $\mathbf{e S} \geq|\operatorname{det} \mathrm{A}| \mathbf{e}$ where $\mathbf{e}=\left[\mu\left(T_{1}\right), \ldots, \mu\left(T_{n}\right)\right]$ is a positive row vector. This immediately yields $\lambda(\mathbf{S}) \geq|\operatorname{det} \mathbf{A}|$.

In this section the Perron eigenvalue condition is assumed as a hypothesis, then one can use a weaker assumption on $\mathcal{X}$, that of being uniformly discrete. We will derive Theorem 2.4 from the following result.

Theorem 5.1. Let $\psi(\mathcal{X})=\mathcal{X}$ be a inflation functional equation that has a primitive subdivision matrix that satisfies the Perron eigenvalue condition

$$
\lambda(\mathbf{S})=|\operatorname{det}(\mathbf{A})| .
$$

Then the following conditions are equivalent:
(i) For some $N>0$ there exists a weakly uniformly discrete multiset family $\hat{\mathcal{X}}$ such that $\psi^{N}(\hat{\mathcal{X}})=\hat{\mathcal{X}}$.
(ii) For some $N>0$ there exists a uniformly discrete multiset family $\hat{\mathcal{X}}$ such that $\psi^{N}(\hat{\mathcal{X}})=$ $\hat{\mathcal{X}}$.
(iii) For some $N>0$ there exists a self-replicating Delone multiset family $\hat{\mathcal{X}}$ such that $\psi^{N}(\hat{\mathcal{X}})=\hat{\mathcal{X}}$.
(iv) The unique compact solution $\left(T_{1}, \ldots, T_{n}\right)$ of the associated multi-tile functional equation consists of sets $T_{i}$ that have positive Lebesgue measure, $1 \leq i \leq n$.

We first show that Theorem 2.4 is a consequence of this result.

Proof of Theorem 2.4. (iii) $\Rightarrow$ (ii). This follows from Theorem 5.1 (iv) $\Rightarrow$ (iii).
(ii) $\Rightarrow$ (i). A self-replicating multitiling is a substitution Delone set family.
(i) $\Rightarrow$ (iii). By Theorem 2.3 the inflation functional equation satisfies the Perron eigenvalue condition. Since a substitution Delone set family is a uniformly discrete family, this follows from Theorem 5.1 (ii) $\Rightarrow$ (iv).

To prove Theorem 5.1, we need to first establish properties of the solutions of the multi-tile functional equation (5.1). We recall a basic result.

Proposition 5.2. The multi-tile functional equation

$$
\begin{equation*}
\mathrm{A}\left(T_{i}\right)=\bigcup_{j=1}^{n}\left(T_{j}+\mathcal{D}_{j i}\right), \quad 1 \leq i \leq n \tag{5.1}
\end{equation*}
$$

with primitive subdivision matrix S has a unique solution $\left(T_{1}, \ldots, T_{n}\right)$ in which each $T_{i}$ is compact and some $T_{j} \neq \phi$. In this solution all $T_{j}$ are nonempty, and

$$
\begin{equation*}
T_{i}=\left\{\sum_{k=1}^{\infty} \mathrm{A}^{-k} \mathbf{d}_{j_{k} j_{k-1}} \mid \mathbf{d}_{j_{k} j_{k-1}} \in \mathcal{D}_{j_{k} j_{k-1}},\left(j_{0}, j_{1}, j_{2}, \ldots\right) \in\{1,2, \ldots, N\}^{\mathbb{Z}^{+}}, j_{0}=i\right\} . \tag{5.2}
\end{equation*}
$$

Proof. We use Flaherty and Wang [1, Proposition 2.3], which proves without the primitivity assumption that (5.1) has a unique solution $\left(T_{1}, \ldots, T_{n}\right)$ in which all $T_{i}$ are nonempty compact sets, under the hypothesis that $\bigcup_{j=1}^{n} \mathcal{D}_{j i}$ is nonempty for each $i$. The primitivity assumption in our setting implies $\bigcup_{j=1}^{n} \mathcal{D}_{j i} \neq \emptyset$ for each $i$. Furthermore, it implies that if some $T_{j}$ is nonempty then all $T_{i}$ are nonempty. Thus the Flaherty and Wang result applies to yield the proposition.

We remark that (5.2) is equivalent to saying that for each point $\mathbf{x} \in T_{i}$ there exists an infinite directed path $\left(\mathbf{d}_{1}, \mathbf{d}_{2}, \mathbf{d}_{3}, \ldots\right)$ in the graph $\mathcal{G}(\mathcal{X})$ with $\mathbf{d}_{1} \in \mathcal{D}_{j i}$ for some $j$ such that

$$
\begin{equation*}
\mathbf{x}=\sum_{k=1}^{\infty} \mathrm{A}^{-k} \mathbf{d}_{k} \tag{5.3}
\end{equation*}
$$

and vice versa.
Now define the digit multisets

$$
\begin{equation*}
\mathcal{D}_{j i}^{(m)}:=\bigvee_{j_{1}, \ldots, j_{m-1}=1}^{n}\left(\mathcal{D}_{j j_{1}}+\mathbf{A} \mathcal{D}_{j_{1} j_{2}}+\cdots+\mathbf{A}^{m-1} \mathcal{D}_{j_{m-1} i}\right) \tag{5.4}
\end{equation*}
$$

in which the sum is interpreted as counting multiplicities. It is easy to check that iterating (5.1) yields

$$
\begin{equation*}
\mathrm{A}^{m}\left(T_{i}\right)=\bigcup_{j=1}^{n}\left(T_{j}+\underline{\mathcal{D}}_{j i}^{m}\right), \quad i=1, \ldots, n \tag{5.5}
\end{equation*}
$$

In (5.5) we do not count multiplicity, so it suffices to use $\mathcal{D}_{j i}^{m}$ instead of $\mathcal{D}_{j i}^{m}$.
Definition 5.1. A family of discrete multisets $\left\{\mathcal{E}_{\alpha}: \alpha \in I\right\}$ in $\mathbb{R}^{d}$ is equi-uniformly discrete if there exists an $\varepsilon_{0}>0$ such that each $\mathcal{E}_{\alpha}$ is uniformly discrete and any two distinct elements in $\mathcal{E}_{\alpha}$ are at least $\varepsilon_{0}$ distance apart. $\left\{\mathcal{E}_{\alpha}: \alpha \in I\right\}$ is called weakly equi-uniformly discrete if there exists an $M>0$ such that for each $\alpha \in I$ and ball $B$ of radius 1 in $\mathbb{R}^{d}$ the number of elements of $\mathcal{E}_{\alpha}$ in $B$ (counting multiplicity) is bounded by $M$.

The following theorem is an extention of a theorem in Sirvent and Wang [33].
Theorem 5.3. Let the compact sets $\left(T_{1}, \ldots, T_{n}\right)$ satisfy the multi-tile functional equation

$$
\mathrm{A}\left(T_{i}\right)=\bigcup_{j=1}^{n}\left(T_{j}+\mathcal{D}_{j i}\right), \quad i=1, \ldots, n
$$

with primitive subdivision matrix. Suppose that the sets $\left\{\underline{\mathcal{D}}_{j i}^{m}: 1 \leq i, j \leq n, m \geq 1\right\}$ are weakly equi-uniformly discrete, and some $T_{i}$ has positive Lebesgue measure. Then every $T_{i}$ has nonempty interior, and each $T_{i}=\overline{T_{i}^{o}}$.

The primitivity of the subdivision matrix implies that if a single $T_{i}$ has positive Lebesgue measure, then they all do. Before proving

Theorem 5.3 we first establish the following lemma.

Lemma 5.4. Under the assumption of Theorem 5.3, let $\delta_{m}$ be a sequence of positive numbers whose limit is 0 . Then there exist positive constants $R_{0}, k_{2}$ and subsets $\mathcal{E}_{1}^{m}, \ldots, \mathcal{E}_{n}^{m}$ of $\mathbb{R}^{d}$ contained in the ball $B_{R_{0}}(\mathbf{0})$ and cardinality bounded by $k_{2}$, such that

$$
\begin{equation*}
\mu\left(B_{1}(\mathbf{0}) \cap \Omega_{m}\right) \geq\left(1-5^{d+1} \delta_{m}\right) \mu\left(B_{1}(\mathbf{0})\right) \tag{5.6}
\end{equation*}
$$

where $\Omega_{m}:=\bigcup_{j=1}^{n}\left(T_{j}+\mathcal{E}_{j}^{m}\right)$.

Proof. Without loss of generality we assume $T_{1}$ has positive Lebesgue measure, and hence it has a Lebesgue point $\mathbf{x}^{*}$, i.e. there is a sequence $r_{m} \rightarrow 0$ such that

$$
\mu\left(B_{r_{m}}\left(\mathbf{x}^{*}\right) \cap T_{1}\right) \geq\left(1-\delta_{m}\right) \mu\left(B_{r_{m}}\left(\mathbf{x}^{*}\right)\right)
$$

It follows that

$$
\begin{equation*}
\mu\left(\mathrm{A}^{l}\left(B_{r_{m}}\left(\mathbf{x}^{*}\right) \cap T_{1}\right)\right) \geq\left(1-\delta_{m}\right) \mu\left(\mathrm{A}^{l}\left(B_{r_{m}}\left(\mathbf{x}^{*}\right)\right)\right), \text { for all } l \geq 0 \tag{5.7}
\end{equation*}
$$

We first show that for sufficiently large $l$, there exists a unit ball $B_{1}(\mathbf{y}) \subset \mathrm{A}^{l}\left(B_{r_{m}}\left(\mathbf{x}^{*}\right)\right)$ with

$$
\begin{equation*}
\mu\left(B_{1}(\mathbf{y}) \cap \mathrm{A}^{l}\left(T_{1}\right)\right) \geq\left(1-5^{d+1} \delta_{m}\right) \mu\left(B_{1}(\mathbf{0})\right) \tag{5.8}
\end{equation*}
$$

Indeed, since A is expanding $\mathrm{A}^{l}\left(B_{r_{m}}\left(\mathbf{x}^{*}\right)\right)$ is an ellipsoid $O_{l, m}$ whose shortest axis goes to infinity as $l$ goes to infinity. Let $O_{l, m}^{\prime}$ be the homothetically shrunk ellipsoid with shortest axis decreased in length by 2 , so that all points in it are at distant at least 1 from the boundary of $O_{l, m}$. By a standard covering lemma (cf. Stein [37, p. 9]) applied to $O_{l, m}^{\prime}$ there is a set of $\left\{B_{1}\left(\mathbf{y}^{\prime}\right)\right\}$ of disjoint union balls with centers in $O_{l, m}$ that cover volume at least $5^{-d} \mu\left(O_{l, m}^{\prime}\right)$. Also $5^{-d} \mu\left(O_{l, m}^{\prime}\right) \geq 5^{-d-1} \mu\left(O_{l, m}\right)$ since the shortest axis is of length at least $2(d+1)$. All these balls lie inside $O_{l, m}$. By (5.7) at most $\delta_{m} \mu\left(\mathrm{~A}^{l}\left(B_{r_{m}}\left(\mathbf{x}^{*}\right)\right)\right.$ ) of the volume of $\mathrm{A}^{l}\left(B_{r_{m}}\left(\mathbf{x}^{*}\right)\right)$ is uncovered by $\mathrm{A}^{l}\left(B_{r_{m}}\left(\mathbf{x}^{*}\right) \cap T_{1}\right)$, so at least one of the disjoint balls $\left\{B_{1}\left(\mathbf{y}^{\prime}\right)\right\}$ must satisfy (5.8).

By (5.4) we can rewrite the inequality (5.8) as

$$
\mu\left(B_{1}(\mathbf{y}) \cap\left(\bigcup_{j=1}^{n}\left(T_{j}+\underline{\mathcal{D}}_{j 1}^{l}\right)\right)\right) \geq\left(1-5^{d+1} \delta_{m}\right) \mu\left(B_{1}(\mathbf{y})\right)
$$

therefore

$$
\mu\left(B_{1}(\mathbf{y}) \cap\left(\bigcup_{j=1}\left(T_{j}+\underline{\mathcal{D}}_{j 1}^{l}-\mathbf{y}\right)\right)\right) \geq\left(1-5^{d+1} \delta_{m}\right) \mu\left(B_{1}(\mathbf{0})\right)
$$

This shows that if we choose

$$
\mathcal{E}_{j}^{m}=\left\{\mathbf{d}-\mathbf{y} \mid \mathbf{d} \in \underline{\mathcal{D}}_{j 1}^{l} \text { with }\left(T_{j}+\mathbf{d}-\mathbf{y}\right) \cap B_{1}(\mathbf{0}) \neq \emptyset\right\}
$$

then (5.6) holds. Since all $T_{j}$ are compact, all $\mathcal{E}_{j}^{m}$ are inside the ball $B_{R_{0}}(\mathbf{0})$ for $R_{0}:=$ $1+\max _{i} \operatorname{diam}\left(T_{i}\right)$. The cardinality of all $\mathcal{E}_{j}^{m}$ are bounded by some $k_{2}>0$ as a result of the weakly equi-uniformly discreteness of all $\underline{\mathcal{D}}_{i j}^{m}$.

Proof of Theorem 5.3. We apply the previous lemma and choose a subsequence $m_{k}$ so that $\left\{\mathcal{E}_{j}^{m_{k}}\right\}$ converges for all $j$, and we denote the limit by $\mathcal{E}_{j}^{\infty}$. This can always be done because $\left\{\mathcal{E}_{j}^{m}\right\}$ are uniformly bounded and have uniformly bounded cardinality. Clearly $\mathcal{E}_{j}^{\infty}$ has cardinality at most $K_{0}$. So

$$
\begin{aligned}
\mu\left(B_{1}(\mathbf{0}) \cap\left(\bigcup_{j=1}^{n}\left(T_{j}+\mathcal{E}_{j}^{\infty}\right)\right)\right) & \geq \liminf _{k \rightarrow \infty} \mu\left(B_{1}(\mathbf{0}) \cap\left(\bigcup_{j=1}^{n}\left(T_{j}+\mathcal{E}_{j}^{m_{k}}\right)\right)\right) \\
& \geq \liminf _{k \rightarrow \infty}\left(1-5^{d+1} \delta_{m_{k}}\right) \mu\left(B_{1}(\mathbf{0})\right) \\
& =\mu\left(B_{1}(\mathbf{0})\right) .
\end{aligned}
$$

Since each $T_{j}+\mathcal{E}_{j}^{\infty}$ is a closed set, we must have

$$
B_{1}(\mathbf{0}) \cap\left(\bigcup_{j=1}^{n}\left(T_{j}+\mathcal{E}_{j}^{\infty}\right)\right)=B_{1}(\mathbf{0})
$$

This means at least one of $T_{j}$ 's must have nonempty interior. But if so then the primitivity of the subdivision matrix implies that all $T_{j}$ must have nonempty interior. Let $T_{j}^{\prime}=\overline{T_{j}^{o}}$. Then ( $T_{1}^{\prime}, \ldots, T_{n}^{\prime}$ ) must also satisfy the same multi-tile equation (5.1). By the uniqueness $T_{j}=T_{j}^{\prime}$ for all $j$.

We now give several characterizations of the property $\mu\left(T_{i}\right)>0$.

Theorem 5.5. Assume that the family of compact sets $\left(T_{1}, \ldots, T_{n}\right)$ satisfies a primitive multitile functional equation with the data $\left(\mathbf{A}, \mathcal{D}_{i j}\right)$ satisfying the Perron eigenvalue condition $\lambda(\mathbf{S})=$ $|\operatorname{det}(\mathrm{A})|$. Then the following conditions are equivalent:
(i) The sets $\left\{\mathcal{D}_{j k}^{m}: 1 \leq i, j \leq n, m \geq 1\right\}$ are equi-uniformly discrete.
(ii) For some fixed $k, l$ the sets $\left\{\mathcal{D}_{k l}^{m}: m \geq 1\right\}$ are equi-uniformly discrete.
(iii) The sets $\left\{\mathcal{D}_{j k}^{m}: 1 \leq i, j \leq n, m \geq 1\right\}$ are weakly equi-uniformly discrete.
(iv) For some fixed $k, l$ the sets $\left\{\mathcal{D}_{k l}^{m}: m \geq 1\right\}$ are weakly equi-uniformly discrete.
(v) One set $T_{j}$ has $\mu\left(T_{j}\right)>0$.
(vi) Every set $T_{j}$ has $T_{j}=\overline{T_{j}^{0}}$, hence all $\mu\left(T_{j}\right)>0$.
(vii) All $T_{j}=\overline{T_{j}^{0}}$ and $\mu\left(\partial T_{k}\right)=0$.

Proof. The implications (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (iv), (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) are obvious.
(iv) $\Rightarrow(\mathrm{v})$. First it is easy to check that $\mathrm{S}^{m}=\left[\left|\mathcal{D}_{i j}^{m}\right|\right]$ where $\left|\mathcal{D}_{i j}^{m}\right|$ denotes the cardinality (counting multiplicity) of $\mathcal{D}_{i j}^{m}$. The primitivity hypothesis implies that there exists a $c_{0}>0$ such that, for all $m \geq 1,\left|\mathcal{D}_{i j}^{m}\right| \geq c_{0} \lambda(\mathbf{S})^{m}$ for all $1 \leq i, j \leq n$. Now for all $i$ let $T_{i}^{(0)}=\overline{B_{1}(\mathbf{0})}$ and

$$
T_{i}^{(m)}=\bigcup_{j=1}^{n} \mathrm{~A}^{-1}\left(T_{j}^{(m-1)}+\mathcal{D}_{j i}\right), \quad m>0
$$

Then $T_{i}^{(m)} \longrightarrow T_{i}$ in the Hausdorff metric (see [1]). We prove that $\mu\left(T_{l}\right)>0$. To see this, we notice that

$$
\mathrm{A}^{m}\left(T_{i}^{(m)}\right)=\bigcup_{j=1}^{n}\left(T_{j}^{(0)}+\underline{\mathcal{D}}_{j i}^{m}\right)
$$

So

$$
\begin{equation*}
\mathrm{A}^{m}\left(T_{l}^{(m)}\right)=\bigcup_{j=1}^{n}\left(T_{j}^{(0)}+\underline{\mathcal{D}}_{j l}^{m}\right) \supseteq T_{k}^{(0)}+\underline{\mathcal{D}}_{k l}^{m} . \tag{5.9}
\end{equation*}
$$

Since $\left\{\mathcal{D}_{k l}^{m}\right\}$ are weakly equi-uniformly discrete, there exists a constant $c_{1}>0$ such that $\left|\underline{\mathcal{D}}_{k l}^{m}\right| \geq c_{1}\left|\underline{\mathcal{D}}_{k l}^{m}\right|$ for all $m$. Now each unit ball contains at most $M$ elements of $\underline{\mathcal{D}}_{k l}^{m}$. Therefore each point in $T_{k}^{0}+\underline{\mathcal{D}}_{k l}^{m}$ can be covered by no more that $M$ copies of $T_{k}^{(0)}+\mathbf{d}, \mathbf{d} \in \underline{\mathcal{D}}_{k l}^{m}$. Therefore by (5.9)

$$
|\operatorname{det}(\mathbf{A})|^{m} \mu\left(T_{l}^{(m)}\right) \geq \frac{1}{M}\left|\underline{\mathcal{D}}_{k l}^{m}\right| \mu\left(T_{l}^{(0)}\right) \geq \frac{1}{M} \delta c_{0} c_{1} \lambda(\mathbf{S})^{m},
$$

where $\delta$ is the volume of the ball $B_{1}(\mathbf{0})$. This yields $\mu\left(T_{l}^{(m)}\right) \geq \frac{1}{M} \delta c_{0} c_{1}>0$ as $|\operatorname{det}(\mathbf{A})|=\lambda(\mathbf{S})$. It follows that

$$
\mu\left(T_{l}\right) \geq \limsup _{m \rightarrow \infty} \mu\left(T_{l}^{(m)}\right)>0
$$

(v) $\Rightarrow$ (i). First we note that $\mu\left(T_{j}\right)>0$ for all $j$ as a result of the primitivity of S . Let $\mathbf{e}=\left[\mu\left(T_{1}\right), \mu\left(T_{2}\right), \ldots, \mu\left(T_{n}\right)\right]$. Taking the Lebesgue measure on both side of the iterated multitile equation (5.5) we obtain

$$
\begin{equation*}
\lambda^{m}(\mathbf{S}) \mu\left(T_{i}\right) \leq \sum_{j=1}^{n} \mu\left(T_{j}\right)\left|\underline{\mathcal{D}}_{j i}^{m}\right| \leq \sum_{j=1}^{n} \mu\left(T_{j}\right)\left|\mathcal{D}_{j i}^{m}\right|, \quad 1 \leq i \leq n . \tag{5.10}
\end{equation*}
$$

In other words, $\lambda^{m}(\mathbf{S}) \mathbf{e} \leq \mathbf{e} \mathbf{S}^{m}$. But $\mathbf{S}^{m}$ is a primitive nonnegative matrix with PerronFrobenius eigenvalue $\lambda^{m}(\mathbf{S})$. So (5.10) can hold only when e is a left Perron-Frobenius eigenvector of $S$ and all inequalities in (5.10) are equalities. This immediately yields $\left|\mathcal{D}_{j i}^{m}\right|=\left|\mathcal{D}_{j i}^{m}\right|$, and so $\mathcal{D}_{j i}^{m}$ is an ordinary set. The equalities in (5.10) also implies that the unions $T_{j}+\underline{\mathcal{D}}_{j i}^{m}$ are all measure-wise disjoint. So all $\underline{\mathcal{D}}_{j i}^{m}$, and hence all $\mathcal{D}_{j i}^{m}$, are equi-uniformly discrete for some $\varepsilon>0$.
$(\mathrm{v}) \Leftrightarrow(\mathrm{vi})$. This is Theorem 5.3 since (v) also implies (i).
$(\mathrm{vi}) \Rightarrow($ vii $)$. We only need to prove that $\mu\left(\partial T_{j}\right)=0$ for all $j$. Let $\mathbf{e}=\left[\mu\left(\partial T_{1}\right), \ldots, \mu\left(\partial T_{n}\right)\right]$. We have for all $i$

$$
\mathbf{A}^{m}\left(\partial T_{i}\right)=\partial\left(\mathbf{A}^{m}\left(T_{i}\right)\right)=\partial\left(\bigcup_{j=1}^{n}\left(T_{j}+\underline{\mathcal{D}}_{j i}^{m}\right)\right) \subseteq \bigcup_{j=1}^{n}\left(\partial T_{j}+\underline{\mathcal{D}}_{j i}^{m}\right) .
$$

Similar to (5.10), taking the Lebesgue measure yields $\lambda^{m}(\mathbf{S}) \mathbf{e} \leq \mathbf{e} \mathbf{S}^{m}$. Again, this can occur only when $\mathbf{e}=\mathbf{0}$ or $\mathbf{e}$ is a Perron-Frobenius left eigenvector of $\mathbf{S}^{m}$. Assume that $\mathbf{e} \neq \mathbf{0}$ then all $\mu\left(\partial T_{j}\right)>0$, and

$$
\begin{equation*}
\mu\left(\partial\left(\mathbf{A}^{m}\left(T_{i}\right)\right)\right)=\sum_{j=1}^{n}\left|\mathcal{D}_{j i}^{m}\right| \mu\left(\partial T_{j}\right) \tag{5.11}
\end{equation*}
$$

But $T_{i}$ has nonempty interior, so for sufficiently large $m>0$ the inflated set $\mathrm{A}^{m}\left(T_{i}\right)$ will contain a sufficiently large ball in its interior. Since $\mathbf{A}^{m}\left(T_{i}\right)$ is the union of $T_{j}+\underline{\mathcal{D}}_{j i}^{m}, 1 \leq j \leq n$, there must be some $k$ and $\mathbf{d} \in \underline{\mathcal{D}}_{k i}$ such that $T_{k}+\mathbf{d}$ is completely contained in the interior of $\mathbf{A}^{m}\left(T_{i}\right)$. Hence

$$
\partial\left(\mathbf{A}^{m}\left(T_{i}\right)\right) \subseteq \bigcup_{j=1}^{n}\left(\partial T_{j}+\underline{\mathcal{D}}_{j i}^{m}\right) \backslash\left(T_{k}+\mathbf{d}\right)
$$

So (5.11) is impossible, a contradiction.
(vii) $\Rightarrow$ (vi). This is obvious.

Proof of Theorem 5.1. Iterating $N$ times the inflation functional equation $\mathcal{X}=\psi(\mathcal{X})$ on multisets gives a new inflation functional equation $\mathcal{X}=\psi^{N}(\mathcal{X})$, which corresponds to

$$
\begin{equation*}
X_{i}=\bigvee_{j=1}^{n}\left(\mathrm{~A}^{N}\left(X_{j}\right)+\mathcal{D}_{i j}^{(N)}\right), \quad 1 \leq i \leq n \tag{5.12}
\end{equation*}
$$

where the sums are interpreted as multiset sums.
(ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) are obvious.
(i) $\Rightarrow$ (iv). By assumption there exists an $M>0$ such that any unit ball in $\mathbb{R}^{d}$ contains at most $M$ elements (counting multiplicity) of each $X_{i}$. Taking $k=m N$ in (5.12) it follows that any unit ball contains at most $M$ elements of each $\mathcal{D}_{i j}^{m N}$. Observe that by (5.4) we have

$$
\mathcal{D}_{i j}^{k}=\bigvee_{l=1}^{n}\left(\mathcal{D}_{i l}^{k-1}+\mathrm{A}^{k-1} \mathcal{D}_{l j}\right)
$$

Therefore any unit ball contains at most $M$ elements of each $\mathcal{D}_{i j}^{k-1}$ if it contains at most $M$ elements of each $\mathcal{D}_{i j}^{k}$. This immediately yields the weakly equi-uniform discreteness of all the sets $\left\{\mathcal{D}_{i j}^{k}\right\}$. So $T_{i}^{o} \neq \emptyset$ by Theorem 5.5.
(iv) $\Rightarrow$ (iii). Since $T_{1}^{o} \neq \emptyset$ it follows from the observation (5.3) that there exists an infinite directed path $\left(\mathbf{d}_{1}, \mathbf{d}_{2}, \mathbf{d}_{3}, \ldots\right)$ in the graph $\mathcal{G}(\mathcal{X})$ with $\mathbf{d}_{1} \in \mathcal{D}_{j 1}$ for some $j$ such that

$$
\mathbf{x}_{0}=\sum_{k=1}^{\infty} \mathrm{A}^{-k} \mathbf{d}_{k} \in T_{1}^{o} .
$$

Since all $\mathcal{D}_{i j}$ are bounded, there exists an $N^{\prime}>0$ such that for all infinite directed paths $\left(\mathbf{d}_{1}^{\prime}, \mathbf{d}_{2}^{\prime}, \mathbf{d}_{3}^{\prime}, \ldots\right)$ with $\mathbf{d}_{j}^{\prime}=\mathbf{d}_{j}$ for $j \leq N^{\prime}$ we also have

$$
\mathbf{x}_{0}^{\prime}=\sum_{k=1}^{\infty} \mathrm{A}^{-k} \mathbf{d}_{k}^{\prime} \in T_{1}^{o} .
$$

The primitivity of the subdivision matrix $S$ now implies that we can find an infinite directed paths ( $\mathbf{d}_{1}^{*}, \mathbf{d}_{2}^{*}, \mathbf{d}_{3}^{*}, \ldots$ ) which has $\mathbf{d}_{j}^{*}=\mathbf{d}_{j}$ for $j \leq N^{\prime}$, and which is periodic for some period $N \geq N^{\prime}$ in the sense that $\mathbf{d}_{k+N}^{*}=\mathbf{d}_{k}^{*}$ for all $k$. Let $\mathbf{x}_{0}^{*}=\sum_{k=1}^{\infty} \mathrm{A}^{-k} \mathbf{d}_{k}^{*}$. Then $\mathbf{x}_{0}^{*} \in T_{1}^{o}$ and $\mathbf{A}^{N} \mathbf{x}_{0}^{*}=\mathbf{x}_{0}^{*}+\mathbf{d}$ for $\mathbf{d}=\sum_{j=0}^{N-1} \mathbf{A}^{j} \mathbf{d}_{N-j}^{*}$. Observe that $\mathbf{d} \in \mathcal{D}_{11}^{N}$. So we have

$$
\begin{equation*}
-\mathrm{x}_{0}^{*} \in \mathrm{~A}^{N}\left(-\mathrm{x}_{0}^{*}\right)+\mathcal{D}_{11}^{N} . \tag{5.13}
\end{equation*}
$$

Now consider the inflation functional equation $\mathcal{X}=\psi^{N}(\mathcal{X})$. Set $\mathcal{X}^{(0)}=\left(X_{1}^{(0)}, \ldots, X_{n}^{(0)}\right)$, with

$$
X_{1}^{(0)}=\left\{-\mathbf{x}_{0}^{*}\right\}, X_{2}^{(0)}=\emptyset, \ldots, X_{n}^{(0)}=\emptyset .
$$

Define $\mathcal{X}^{(m)}=\psi^{N}\left(\mathcal{X}^{(m-1)}\right)=\psi^{m N}\left(\mathcal{X}^{(0)}\right)$. Now set

$$
X_{1}^{(1)}=\bigvee_{j=1}^{n}\left(\mathrm{~A}^{N}\left(X_{j}^{(0)}\right)+\mathcal{D}_{1 j}^{N}\right)=\mathrm{A}^{N}\left(\left\{\mathbf{x}_{0}^{*}\right\}\right)+\mathcal{D}_{11}^{N}
$$

hence we have $X_{1}^{(0)} \subseteq X_{1}^{(1)}$. by (5.13), and obviously we have $\emptyset=X_{i}^{(0)} \subseteq X_{i}^{(1)}$ for $i \geq 2$. So the inclusion property $\mathcal{X}^{(0)} \subseteq \psi^{N}\left(\mathcal{X}^{(0)}\right)=\mathcal{X}^{(1)}$ holds. It follows that $\mathcal{X}^{(0)} \subseteq \mathcal{X}^{(1)} \subseteq \mathcal{X}^{(2)} \subseteq \cdots$. But notice that

$$
X_{i}^{(m)}=\bigvee_{j=1}^{n}\left(\mathbf{A}^{m N}\left(X_{j}^{(0)}\right)+\mathcal{D}_{i j}^{m N}\right)=\mathbf{A}^{m N}\left\{\mathbf{x}_{0}^{*}\right\}+\mathcal{D}_{i 1}^{m N}
$$

We conclude that each $X_{i}^{(m)}$ is an ordinary set and is $\varepsilon_{0}$-uniformly discrete. Let $X_{i}=$ $\bigcup_{m=0}^{\infty} X_{i}^{(m)}$ and $\hat{\mathcal{X}}=\left(X_{1}, \ldots, X_{n}\right)$. Then $\hat{\mathcal{X}}=\psi^{N}(\hat{\mathcal{X}})$, and $\hat{\mathcal{X}}$ is $\varepsilon_{0}$-uniformly discrete.

It remains to show that $\hat{\mathcal{X}}$ is a Delone family and to establish the tiling property of $\hat{\mathcal{X}}$. Observe that $\mathbf{0}$ is in the interior of $T_{1}-\mathbf{x}_{0}^{*}$. Now

$$
\mathrm{A}^{m N}\left(T_{1} \mathbf{x}_{0}^{*}\right)=\bigcup_{j=1}^{n}\left(T_{j}+\underline{\mathcal{D}}_{j 1}^{m N}-\mathrm{A}^{m N} \mathbf{x}_{0}^{*}\right)=\bigcup_{j=1}^{n}\left(T_{j}+X_{j}^{(m)}\right)
$$

Taking limit as $m \rightarrow \infty$ we see that $\bigcup_{j=1}^{n}\left(T_{j}+X_{j}\right)$ is a tiling of $\mathbb{R}^{d}$. So all $X_{j}$ must be relatively dense as a result of the primitivity of $S$. This completes the proof.

## 6. Self-Replicating Delone Sets

In this section we study self-replicating Delone set families as a subclass of substitution Delone set families.

Theorem 6.1. Let $\mathcal{X}$ be an irreducible substitution Delone set family satisfying $\psi(\mathcal{X})=\mathcal{X}$ for the data $\left(\mathbf{A}, \mathcal{D}_{i j}\right)$, where the subdivision matrix S is primitive and $\lambda(\mathrm{S})=|\operatorname{det}(\mathrm{A})|$. Suppose that the fundamental cycle of $\mathcal{X}$ has period 1 . Then $\mathcal{X}$ is a self-replicating Delone set family.

Proof. Since the fundamental cycle of $\mathcal{X}=\left(X_{1}, \ldots, X_{n}\right)$ has period 1 , it contains a single element $\left\{\mathbf{x}_{0}\right\}$. Without loss of generality we assume that $\mathbf{x}_{0} \in X_{1}$, so $\mathbf{x}_{0}=\mathbf{A} \mathbf{x}_{0}+\mathbf{d}$ for some
$\mathbf{d} \in \mathcal{D}_{11}$. Let $\mathcal{S}^{(0)}=\left(S_{1}^{(0)}, \ldots, S_{n}^{(0)}\right)$ such that $S_{1}^{(0)}=\left\{\mathbf{x}_{0}\right\}$ and all other $S_{i}^{(0)}=\emptyset$. Define $\mathcal{S}^{(m)}:=\psi\left(\mathcal{S}^{(0)}\right)=\left(S_{1}^{(m)}, \ldots, S_{n}^{(m)}\right)$. It follows from the expression for $\psi^{m}$ given in (5.12) that

$$
\begin{equation*}
S_{i}^{(m)}=\bigvee_{j=1}^{n}\left(\mathrm{~A}^{m}\left(S_{j}^{(0)}\right)+\mathcal{D}_{i j}^{m}\right)=\mathrm{A}^{m} \mathbf{x}_{0}+\mathcal{D}_{i 1}^{m} \tag{6.1}
\end{equation*}
$$

Suppose that $\left(T_{1}, \ldots, T_{n}\right)$ is the set of self-affine multi-tiles corresponding to ( $\mathrm{A}, \mathcal{D}_{i j}$ ). By Theorem 5.1 each $T_{i}$ satisfies $\overline{T_{i}^{o}}=T_{i}$. We have

$$
\begin{equation*}
\bigcup_{j=1}^{n}\left(T_{j}+S_{j}^{(m)}\right)=\bigcup_{j=1}^{n}\left(T_{j}+\mathbf{A}^{m} \mathbf{x}_{0}+\mathcal{D}_{j 1}^{m}\right) \tag{6.2}
\end{equation*}
$$

It follows from $\mathrm{A}^{m}\left(T_{1}\right)=\bigcup_{j=1}^{n}\left(T_{j}+\mathcal{D}_{j 1}^{m}\right)$ that

$$
\begin{equation*}
\mathrm{A}^{m}\left(T_{1}+\mathbf{x}_{0}\right)=\bigcup_{j=1}^{n}\left(T_{j}+S_{j}^{(m)}\right) \tag{6.3}
\end{equation*}
$$

The unions on the right side of (6.3) are all measure-wise disjoint. Taking limit $m \rightarrow \infty$ we see that $\Omega:=\bigcup_{j=1}^{n}\left(T_{j}+X_{j}\right)$ is a packing of $\mathbb{R}^{d}$.

It remains to prove that $\Omega$ is a tiling. The set $\Omega=\bigcup_{j=1}^{n}\left(T_{j}+X_{j}\right)$ is closed and satisfies

$$
\begin{aligned}
\mathrm{A}(\Omega) & =\bigcup_{j=1}^{n}\left(\mathbf{A}\left(T_{j}\right)+\mathbf{A}\left(X_{j}\right)\right) \\
& =\bigcup_{j=1}^{n}\left(\bigcup_{i=1}^{n}\left(T_{i}+\mathcal{D}_{i j}+\mathbf{A}\left(X_{j}\right)\right)\right) \\
& =\bigcup_{i=1}^{n}\left(T_{i}+\bigcup_{j=1}^{n}\left(\mathcal{D}_{i j}+\mathbf{A}\left(X_{j}\right)\right)\right) \\
& =\bigcup_{i=1}^{n}\left(T_{i}+X_{i}\right)=\Omega .
\end{aligned}
$$

We now argue by contradiction, and suppose $\Omega \neq \mathbb{R}^{d}$. Since $\Omega$ is closed, there exists a ball $B$ of radius $\varepsilon>0$ such that $B \cap \Omega=\emptyset$, which yields $\mathbf{A}^{N}(B) \cap \mathbf{A}^{N}(\Omega)=\mathrm{A}^{N}(B) \cap \Omega=\emptyset$. But A is expanding. So by taking $N$ sufficiently large $\mathrm{A}^{N}(B)$ contains a ball of arbitrarily large radius. This ball is disjoint from $\Omega$, so it is not filled by any translate $T_{j}+\mathbf{x}_{j}, \mathbf{x}_{j} \in X_{j}$ and $1 \leq j \leq n$. Therefore $X_{j}$ cannot be a Delone set, a contradiction. Thus we have a tiling.

Remarks. (1) The condition of Theorem 6.1 is sufficient but not necessary. There exist irreducible self-replicating Delone set families whose fundamental cycles have period exceeeding 1, see Example 7.7.
(2) Let $\mathcal{X}$ be an irreducible Delone set family which satisfies an inflation functional equation $\psi(\mathcal{X})=\mathcal{X}$ with primitive subdivision matrix S and is uniformly discrete and has a fundamental cycle of period $p$. By Theorem 2.3 and Theorem 5.1 the unique solution $\mathcal{T}:=\left(T_{1}, \ldots, T_{n}\right)$ consisting of compact sets of the associated multi-tile equation has $T_{i}$ of positive measure,
which are the closure of their interiors. The set $\mathcal{T}+\mathcal{X}:=\cup_{i=1}^{n}\left(T_{i}+X_{i}\right)$ gives a partial $q$ packing of $\mathbb{R}^{d}$ for some $q \leq p$, using the tiles $T_{i}$. That is, each point of $\mathbb{R}^{d}$ is covered with multiplicity at most $q$ off a set of measure zero, a set of infinite measure has multiplicity exactly $q$, and possibly another set of infinite measure has strictly smaller multiplicity. We propose the following problem:

Problem. Let $\mathcal{X}$ be an irreducible Delone set family which satisfies an inflation functional equation $\psi(\mathcal{X})=\mathcal{X}$ with primitive subdivision matrix $S$ and is uniformly discrete and has a fundamental cycle of period $p$. Is it true that $\mathcal{T}+\mathcal{X}:=\cup_{i=1}^{n}\left(T_{i}+X_{i}\right)$ is always a $q$-tiling for some $q \leq p$ ? If not, what are the extra conditions needed to ensure it?

## 7. Examples

Example 7.1. (Substitution multiset with unbounded multiplicity function) Let $\mathbf{A}=[3]$ and $\mathcal{D}_{1,1}=\{0,1,2, \ldots, m\}$, with $m \geq 3$. Take the seed $\mathcal{S}^{(0)}=S_{1}^{(0)}=\{0\}$. Then the inclusion property $\mathcal{S}^{(0)} \subseteq \mathcal{S}^{(1)}$ holds, hence

$$
X_{1}=\lim _{k \rightarrow \infty} S_{1}^{(k)}
$$

defines a multiset $X_{1}$. The multiset $X_{1} \subseteq \mathbb{Z}_{\geq 0}$ and each point $l \in \mathbb{Z}^{+}$occurs with finite multiplicity $m(l) \rightarrow \infty$ as $l \rightarrow \infty$. In fact $m(l)=l^{\log _{3} l+o(1)}$ as $l \rightarrow \infty$. This example corresponds to case (ii) in Theorem 3.4.

Example 7.2. (Discrete substitution set that is not uniformly discrete) Let $\mathbf{A}=[3]$ and $\mathcal{D}_{1,1}=$ $\{0,1, \pi\}$, with $\pi=3.14159 \ldots$ Take the seed $\mathcal{S}^{(0)}=S_{1}^{(0)}=\{0\}$. Then $\mathcal{S}^{(0)} \subseteq \mathcal{S}^{(1)}$ and the limit

$$
X_{1}=\lim _{k \rightarrow \infty} S_{1}^{(k)}
$$

exists. In this case the multiset $X_{1} \subseteq \mathbb{R}_{\geq 0}$ is discrete, and its elements all have multiplicity one. It is easy to show that it has linear growth. Indeed the $2 \cdot 3^{n-1}$ elements in $S_{1}^{(n)} \backslash S_{1}^{(n-1)}$ all satisfy

$$
3^{n} \leq x \leq \pi\left(3^{n}+3^{n-1}+\cdots+3+1\right) \leq 2 \pi \cdot 3^{n}
$$

The associated multitile functional equation is

$$
\mathrm{A}\left(T_{1}\right)=T_{1} \cup\left(T_{1}+1\right) \cup\left(T_{1}+\pi\right) .
$$

The compact solution $T_{1}$ to this equation has Lebesgue measure zero, see Kenyon [12] or Lagarias and Wang [16]. It follows from Theorem 5.1 that $X_{1}$ cannot be uniformly discrete.

Example 7.3. (Inflation functional equation having infinitely many discrete solutions) Let $\mathrm{A}=[2]$ and $\mathcal{D}_{1,1}=\{0,1\}$, which satisfies the hypotheses of Lemma 3.7. The associated tile is $T_{1}=[0,1]$. The allowed starting points for a cycle $Y$ of period $p$ are given by (3.17), which gives

$$
x_{0}=-\frac{m}{2^{p}-1} \quad \text { for } \quad 0 \leq m \leq 2^{p}-1 .
$$

Any such $x_{0}$ with $\operatorname{gcd}\left(m, 2^{p-1}-1\right)=1$ generates an irreducible multiset $\mathcal{X}_{m, p}=\left(X_{1}\right)$, which has all multiplicities one, and which corresponds to a multiple tiling of $\mathbb{R}$ with multiplicity $p$. (Note that the placement of the tile $T_{1}+x_{0}$ includes 0 in its interior, and the same holds for the other $p-1$ tiles in the periodic cycle.)

Example 7.4. (Inflation functional equation having no nonempty discrete multiset solution) Let $\mathrm{A}=[2]$ and $\mathcal{D}_{11}=\mathcal{D}_{12}=\mathcal{D}_{21}=\mathcal{D}_{22}=\{0,1\}$. This data satisfies the hypotheses of Lemma 3.7. Then the inflational equation $\psi(\mathcal{X})=\mathcal{X}$ has no discrete multiset solution. If fact, all elements in $\mathcal{X}$ must have infinite multiplicity. To see this we iterate the inflational equation to obtain

$$
X_{i}=\left(X_{1}+\mathcal{D}_{i 1}^{m}\right) \vee\left(X_{2}+\mathcal{D}_{i 2}^{m}\right), \quad i=1,2 .
$$

Now each $\mathcal{D}_{i j}^{m}$ has an underlying set $\underline{\mathcal{D}}_{i j}^{m}=\left\{0,1, \ldots, 2^{m-1}\right\}$ with each element having multiplicity $2^{m}$. Therefore the multiplicity of each element in $X_{i}$ is at least $2^{m}$. Hence no element in any $X_{i}$ can have a finite multiplicity.

Example 7.5. (Discrete substitution set with bounded multiplicities, with some multiplicities exceeding one) Consider the inflation functional equation $\mathbf{A}=[3]$ on $\mathbb{R}$, with $\mathcal{D}_{1,1}=\{\pi+3\}$, $\mathcal{D}_{1,2}=\{1\}, \mathcal{D}_{2,1}=\left\{-\frac{1}{3}, \pi\right\}, \mathcal{D}_{2,2}=\emptyset$. The subdivision matrix $S=\left[\begin{array}{ll}1 & 1 \\ 2 & 0\end{array}\right]$ is primitive, and its Perron eigenvalue $\lambda(S)=2$. We claim that the cycle $Y=\left\{0 \in X_{1}, 1 \in X_{2}\right\}$ of period $p=2$ generates a discrete irreducible multiset family $\mathcal{X}=\left(X_{1}, X_{2}\right)$. Outside the interval $[-2,2]$ all maps in the inflation functional equation are expanding by a factor at least 1.4. The cycle $Y$ are the only points of $\mathcal{X}$ inside $[-2,2]$ and the only point exiting from the cycle is $y=(\pi+3,1)$, which has multiplicity 2 ; thus $\mathcal{X}$ exists, and is irreducible and discrete. All other points are descendents of $y$, and they have the form $x=\left(3^{k} a_{k}+3^{k-1} a_{k-1}+\ldots+3 a_{1}+a_{0}\right) \pi+r$, in which each $a_{i}=0$ or 1 and $r$ is a rational number, and $k \geq 1$. The sequence ( $a_{k}, \ldots, a_{0}$ ) completely specifies the digit sequence leading to $x$, and every such digit sequence is legal. Since all seqences $3^{k} a_{k}+3^{k-1} a_{k-1}+\ldots+3 a_{1}+a_{0}$ give distinct integers, and $\pi$ is irrational, we conclude that all $x$ are distinct. Thus all points in $X_{1}$ and $X_{2}$ have multiplicity two, except the generating cycle $Y$, whose two points have multiplicity one.

Example 7.6. (Substitution Delone set that is not self-replicating) Let $A=[3]$ and $\mathcal{D}_{1,1}=$ $\{-1,0,1\}$. The associated tile $T_{1}=\left[-\frac{1}{2}, \frac{1}{2}\right]$, which corresponds to balanced all ternary expansions

$$
x=\sum_{j=1}^{\infty} \mathbf{d}_{j} 3^{-j}, \quad \mathbf{d}_{j} \in\{-1,0,1\}
$$

The values

$$
\mathcal{S}^{(0)}=\left\{x_{0}=-1 / 8, \quad x_{1}=-3 / 8\right\}
$$

has $\mathcal{S}^{(0)} \subseteq \psi\left(\mathcal{S}^{(0)}\right)=\{-17 / 8,-11 / 8,-9 / 8,-3 / 8,-1 / 8,5 / 8\}$, hence generates an irreducible discrete multiset family $\mathcal{X}=\left\{X_{1}\right\}$ which consists of the single set $X_{1}$ given by

$$
X_{1}=\sum_{k \rightarrow \infty} \mathcal{S}^{(k)}
$$

The set $X_{1}$ is irreducible and $\mathcal{S}_{0}$ is its generating cycle of period 2. A calculation gives

$$
X_{1}=\left(-\frac{3}{8}+\mathbb{Z}\right) \cup\left(-\frac{1}{8} \cup \mathbb{Z}\right)
$$

It is a Delone set, and $X_{1}+T$ is a multiple tiling of $\mathbb{R}$ of multiplicity 2 . The multiplicity equals the period of the generating cycle, since both elements $\mathcal{S}^{(0)}$ lie in the interior of the tile $T_{1}$. Thus $X_{1}$ is a substitution Delone set but not a self-replicating Delone set.

Example 7.7. (Self-replicating Delone set having a primitive cycle of order larger than one) Let $\mathrm{A}=[-2]$ and $\mathcal{D}_{11}=\{-2,-1\}$. Then $\mathcal{X}=\left(X_{1}\right)$ with $X_{1}=\mathbb{Z}$ is an irreducible substitution Delone set family whose fundamental cycle is $\{0,-1\}$ and has period 2 . The corresponding self-affine tile is $T_{1}=[0,1]$. So $\mathcal{T}+\mathcal{X}=T_{1}+X_{1}$ tiles $\mathbb{R}$, hence is a self-replicating Delone set.

Example 7.8. (Non-periodic self-replicating Delone sets) An example of a two-dimensional self-replicating Delone set, which not fully periodic, but has a one-dimensional lattice of periods, was given in Lagarias and Wang[17, Example 2.3]. Recently Lee and Moody [20] construct many self-replicating Delone sets which are non-periodic, including aperiodic examples, whose points are contained in a lattice in $\mathbb{R}^{d}$. They give such examples associated to non-periodic tilings including the sphinx tiling of Godreche [3] and the chair tiling.

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