DISK-LIKE SELF-AFFINE TILES IN $\mathbb{R}^2$

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Abstract. We give simple necessary and sufficient conditions for self-affine tiles in $\mathbb{R}^2$ to be homeomorphic to a disk.

1. Introduction

Throughout this note we consider integral self-affine tiles with standard digit sets. Such are tiles $T := T(A, \mathcal{D})$ satisfying

$$A(T) = T + \mathcal{D} \tag{1}$$

or

$$T = \bigcup_{d \in \mathcal{D}} A^{-1}(T + d), \tag{2}$$

where A is an expanding two by two matrix of integers, and $\mathcal{D} \subset \mathbb{Z}^2$ with $|\mathcal{D}| = |\det A|$ is a complete set of coset representatives for $\mathbb{Z}^2/A\mathbb{Z}^2$.

Moreover, we shall assume that $T(A, \mathcal{D})$ tiles by the lattice $\mathbb{Z}^2$, that is, $T + \mathbb{Z}^2$ is a tiling of $\mathbb{R}^2$. Such tiles will be called *self-affine $\mathbb{Z}^2$-tiles.* There are standard methods for checking this property [15, 18, 13, 14]. When the digit set $\mathcal{D}$ is primitive, only in special cases the corresponding tile $T(A, \mathcal{D})$ may not be a $\mathbb{Z}^2$-tile, see [14].

The simplest example of a self-affine $\mathbb{Z}^2$-tile is the unit square, divided into $n \times n$ small squares:

$$A = nI = \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix} \text{ and } \mathcal{D} = \left\{ \begin{bmatrix} i \\ j \end{bmatrix} \;\middle|\; i, j = 1, ..., n \right\}.$$

Figure 1 was obtained from this example, with $n = 2$, just replacing the residue $[1, 1]^T$ by $[-1, -1]^T$. Figures 2 and 3 were obtained from the 4×4 square by replacing two residues in an obvious manner. There are infinitely many other ways in which residues can be exchanged but nearly all of them lead to tiles with holes or with disconnected interior.

Question. Given a self-affine $\mathbb{Z}^2$-tile $T(A, \mathcal{D})$, under what conditions is $T(A, \mathcal{D})$ homeomorphic to a disk?

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Lattice tilings by topological disks must satisfy certain combinatorial properties. We state them here, and they are the keys to answering our question. Let us say that two tiles T' and T'' in a tiling are *neighbors* if $T' \cap T'' \neq \emptyset$. We call the tiles *vertex neighbors* if their intersection is a single point, and *edge neighbors* if their intersection contains uncountably many points. Note that there might be other types of neighbors.

If the tiles are topological disks, an edge will be an arc, as usual. The tile in Figure 1 will intersect an edge neighbor in a more complicated set (actually in a Sierpinski gasket).

It should be pointed out that for a given integral self-affine $\mathbb{Z}^2$-tile $T(A, \mathcal{D})$ there is a simple algorithm to determine its neighbors [17].

Proposition 1.1 ([3], Lemma 5.1)**.** *Let Ω be a topological disk which tiles $\mathbb{R}^2$ by lattice translates of the lattice $\mathcal{L}$. Then in the tiling $\Omega + \mathcal{L}$ one of the following must be true.*

(i) *Ω has no vertex neighbors and 6 edge neighbors $\Omega \pm \alpha, \Omega \pm \beta, \Omega \pm (\alpha + \beta)$ for some $\alpha, \beta \in \mathcal{L}$, and $\mathbb{Z}\alpha + \mathbb{Z}\beta = \mathcal{L}$.*

(ii) *Ω has 4 edge neighbors $\Omega \pm \alpha, \Omega \pm \beta$ and 4 vertex neighbors $\Omega \pm \alpha \pm \beta$ for some $\alpha, \beta \in \mathcal{L}$, and $\mathbb{Z}\alpha + \mathbb{Z}\beta = \mathcal{L}$.*

Now let $\mathcal{F}$ be a finite subset of $\mathbb{Z}^2$. We say a subset $\mathcal{E} \subset \mathbb{Z}^2$ is *$\mathcal{F}$–connected* if for any $u, v \in \mathcal{E}$ there exist $u_0 = u, u_1, ..., u_n = v \in \mathcal{E}$ with $u_{i+1} - u_i \in \mathcal{F}$.

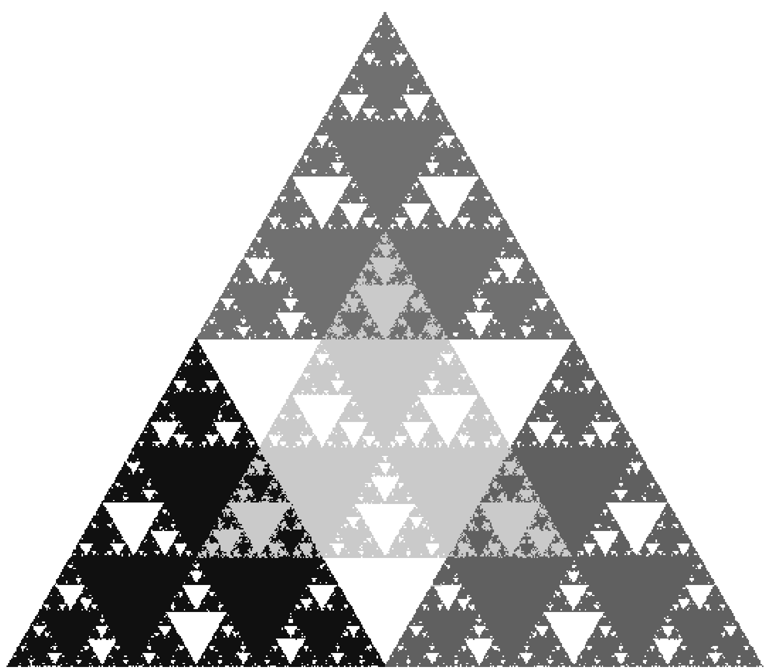

Figure 1

Proposition 1.2 (cf. [12]). *Let the self-affine $\mathbb{Z}^2$-tile $T = T(A, \mathcal{D})$ be a topological disk whose edge neighbors are $T + \mathcal{F}$, $\mathcal{F} \subset \mathbb{Z}^2$. Then $\mathcal{D}$ is $\mathcal{F}$-connected.*

Proof. Note that $A(T) = T + \mathcal{D}$ is a topological disk. Let $\mathcal{D}_1, \ldots, \mathcal{D}_k$ be the $\mathcal{F}$-connected components of $\mathcal{D}$ and assume that $k > 1$. Let $T_i = T + \mathcal{D}_i$. The set $T_1 \cap T_2$ is countable since $T + d_1$ and $T + d_2$ are not edge neighbors for $d_1 \in \mathcal{D}_1, d_2 \in \mathcal{D}_2$. The same is true for $T_i \cap T_j$ with $i \neq j$. Thus $A(T)$ becomes disconnected when a countable set is removed. This is not possible for a disk. ∎

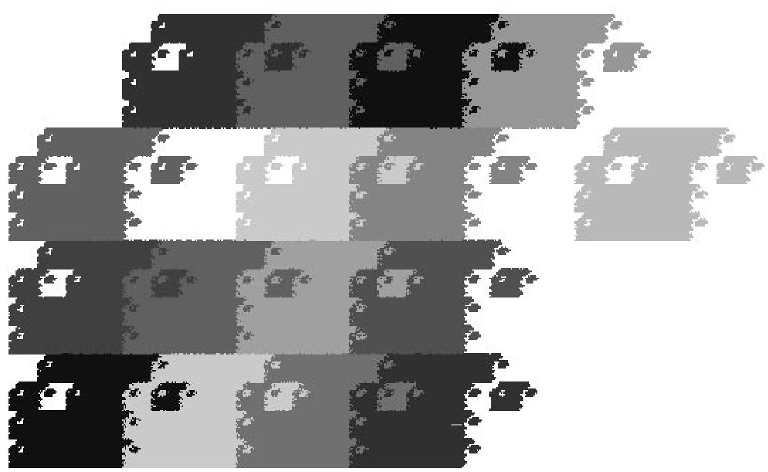

Figure 2

2. Main Results

The main contribution of this paper is to show that the necessary conditions given in Propositions 1.1 and 1.2 are also sufficient. These seem to be the first sufficient conditions for tiles to be disk-like, and they solve a problem in [12]. It turns out that the type of neighbors is not essential, only their number and relative lattice position.

Theorem 2.1. *Let $T(A, \mathcal{D})$ be a self-affine $\mathbb{Z}^2$–tile. Suppose that T has not more than 6 neighbors $T + \mathcal{F}$. Then T is a topological disk if and only if $\mathcal{D}$ is $\mathcal{F}$–connected.*

Theorem 2.2. *Let $T(A, \mathcal{D})$ be a self-affine $\mathbb{Z}^2$–tile. Suppose that T has 8 neighbors $T + \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha-\beta)\}$. Then T is a topological disk if and only if $\mathcal{D}$ is $\{\pm\alpha, \pm\beta\}$–connected.*

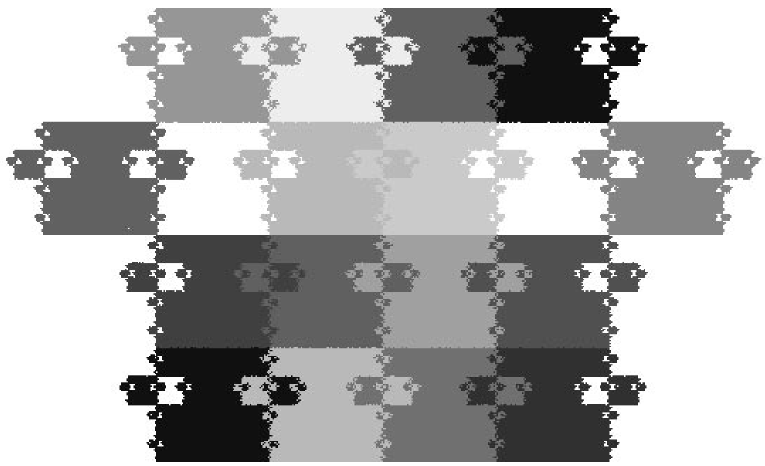

Figure 3

Let us give some examples to examine these conditions. The tile in Figure 1 has 6 edge neighbors, and $\mathcal{D}$ is $\mathcal{F}$-connected. However, it is not a disk since there are 6 other vertex neighbors. Figure 2 shows a tile with 6 neighbors which is disconnected because $\mathcal{D}$ is not $\mathcal{F}$-connected. In Figure 3 we have 8 neighbors as assumed in Theorem 2.2, and the tile is connected. It is not a disk, however, since $\mathcal{D}$ is not connected with respect to the edge neighbors only.

Figure 4 shows that the mere assumption of 8 neighbors in Theorem 2.2 would not suffice. Here $A = \left[\begin{smallmatrix} 0 & 3 \\ 1 & 1 \end{smallmatrix}\right]$ and $\mathcal{D} = \{[0,0]^T, [1,0]^T, [-1,0]^T\}$. The tile in the middle is T, and the three tiles of the middle row form $A(T)$. It is obvious that T has six edge neighbors $\pm\alpha = \pm[1,0]^T$, $\pm\beta = \pm[-2,1]^T$ and $\pm(\alpha+\beta)$. Moreover, the upper left and lower right neighbors $\pm\beta$ meet with their long narrow peaks in the centre of T. This is only indicated by the picture, for a proof see [3], 6.1. Thus T is not a topological disk, and T has two more vertex neighbors $\pm 2\beta$.

For small numbers $m = |\mathcal{D}|$ of pieces, all possible disk-like $\mathbb{Z}^2$-tiles have been classified up to affine conjugacy. For $m = 2$ there are three and for $m = 3$ seven non-isomorphic cases [3], for $m = 4$ their number is 29 [7, 12]. The proof that the tiles are disk-like was given "by inspection". Even for tiles like the twindragon which are well known to be topological disks, no proof of this property seems to be published. Theorems 2.1 and 2.2, together with the algorithm in [17], now provide rigorous arguments.

Example. We just indicate the proof for the twindragon where $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ and $\mathcal{D} = \{[0,0]^T, [1,0]^T, \}$. The neighbors are $\mathcal{F} = \{\pm[1,0]^T, \pm[0,1]^T, \pm[1,-1]^T\}$. Thus we have the six neighbor case of theorem 2.1, and it is enough to see the first neighbor in order to conclude that $\mathcal{D}$ is $\mathcal{F}$-connected. Similarly, all cases for $m \leq 4$ can be checked.

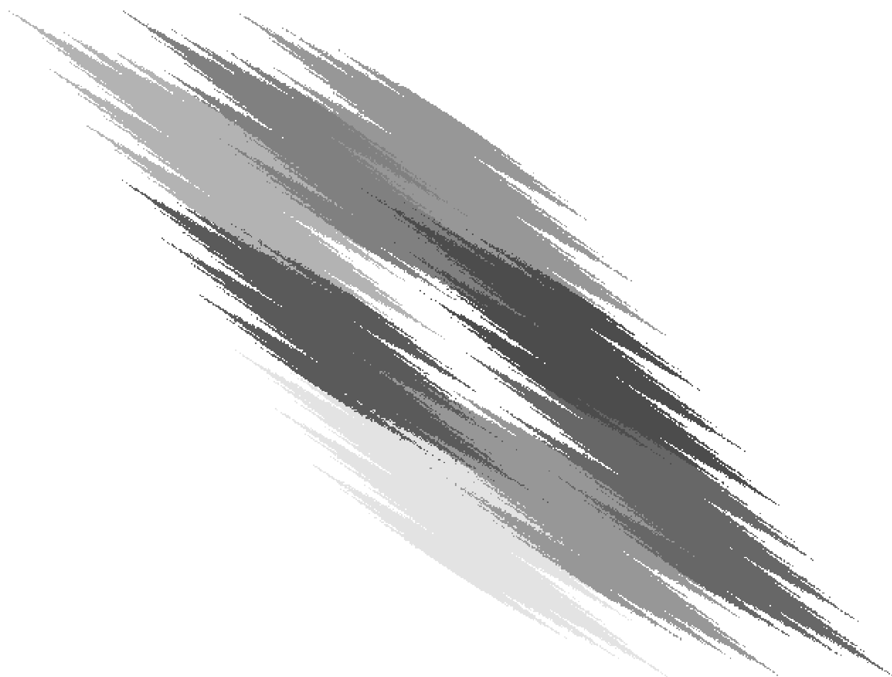

Figure 4

Our technique also allows to characterize the connectedness of a self-affine tile in n-dimensional space (cf. [8, 11]).

Theorem 2.3. *Let $T(A, \mathcal{D})$ be a self-affine set in $\mathbb{R}^n$ for an integer matrix $A \in M_n(\mathbb{Z})$ and $\mathcal{D} \subset \mathbb{Z}^n$. Let $T + \mathcal{F}$ be the neighbors of T where $\mathcal{F} \subset \mathbb{Z}^n$. Then T is connected if and only if $\mathcal{D}$ is $\mathcal{F}$-connected.*

We remark that in Theorem 2.3 we do not require $T(A, \mathcal{D})$ be a tile. For general data $(A, \mathcal{D})$ the self-affine set $T(A, \mathcal{D})$ given by (1) may not be a tile. A neighbor of T is nevertheless well defined for general self-affine sets.

Theorems 2.1 and 2.3 combine to give

Corollary 2.4. *Let $T(A, \mathcal{D})$ be a self-affine $\mathbb{Z}^2$-tile with not more than six neighbors. Then T is a topological disk if and only if T is connected.*

3. Proof of the Theorems

Proof of Theorem 2.3. It is clear that $T+d$ and $T+d'$ are neighbors if and only if $d-d' \in \mathcal{F}$. It is known that the connectedness of a self-affine set can be expressed as the connectedness of the graph which has the pieces as vertices and edges between neighbors ([4], Prop. 2, cf. [11]). For $A(T)$ this means that $\mathcal{D}$ is $\mathcal{F}$-connected. ∎

The following lemma, as well as Theorem 3.2 and Lemma 3.3, does not use any self-similarity, and the structure of the edges may be as complicated as in our Figure 3.

Lemma 3.1. *Let T be a $\mathbb{Z}^2$-tile with neighbors $T+\mathcal{F}$ for some $\mathcal{F} \subset \mathbb{Z}^2$. Let $\mathbb{Z}[\mathcal{F}]$ denote the subgroup of $\mathbb{Z}^2$ generated by $\mathcal{F}$. Then $\mathbb{Z}[\mathcal{F}] = \mathbb{Z}^2$.*

Proof. Call $\alpha_1, \alpha_2 \in \mathbb{Z}^2$ neighbors if $\alpha_1 - \alpha_2 \in \mathcal{F}$. Let $\mathcal{F}_0 = \{0\}$ and $\mathcal{F}_{n+1}$ be the neighbors of $\mathcal{F}_n$, $n \geq 0$. Define

$$\mathcal{G} = \bigcup_{n \geq 0} \mathcal{F}_n.$$

Clearly $\mathcal{G} \subseteq \mathbb{Z}[\mathcal{F}]$ (in fact they are equal). Assume that $\mathcal{G} \neq \mathbb{Z}^2$. Then $\mathcal{H} = \mathbb{Z}^2 \setminus \mathcal{G}$ is nonempty. Set $\Omega = T + \mathcal{G}$ and $\Omega' = T + \mathcal{H}$. It follows that $\Omega \cap \Omega' = \emptyset$. But both Ω and Ω' are closed sets and $\Omega \cup \Omega' = \mathbb{R}^2$. This contradicts the connectedness of $\mathbb{R}^2$. Therefore $\mathcal{G} = \mathbb{Z}^2$ and hence $\mathbb{Z}[\mathcal{F}] = \mathbb{Z}^2$. ∎

To prove the sufficiency of $\mathcal{F}$-connectedness in Theorem 2.1 and of the stronger $\{\pm\alpha, \pm\beta\}$-connectedness in Theorem 2.2, we can now assume that T is connected (and hence arcwise connected [4]). Our aim is to show also that the interior $\operatorname{int} T$ of T is connected. First we strengthen our assumptions in the case of not more than six neighbors.

Theorem 3.2. *Let T be a connected $\mathbb{Z}^2$-tile with at most six neighbors. Then there are α, β in $\mathbb{Z}^2$ such that the set of neighbors is $T + \mathcal{F}$ with $\mathcal{F} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$.*

Proof. Let $\Omega := T + \mathcal{F}$ and $\tilde{\Omega} := \mathbb{R}^2 \setminus (T \cup \Omega)$. We have $d(T, \tilde{\Omega}) = \delta > 0$ since T is separated from $\tilde{\Omega}$. For $\varepsilon > 0$ let $B_\varepsilon(z)$ denote the open disk of radius ε centered at z. The collection of open disks $\{B_\varepsilon(z) : z \in T\}$ covers T. So by compactness we may find $z_1, \ldots, z_k \in T$ such that $T_\varepsilon = \bigcup_{j=1}^k B_\varepsilon(z_j)$ covers T. T_ε is connected because T is. Now T_ε is a finite union of disks, so ∂T_ε consists of a finite number of simple piecewise smooth closed Jordan curves. Assume that C be the Jordan curve of the outer boundary. For each $y \in \mathbb{Z}^2 \setminus \{0\}$ let $z_y \in C$ such that $\langle z_y, y\rangle = \max\{\langle z, y\rangle : z \in C\}$. It is easy to see that $z_y + y \in C + y$ is outside C and $d(z_y + y, C) \geq 1$. Since there exists a point $z' \in T + y$ with $d(z', z_y + y) < \varepsilon$, the point z' must be outside C if $\varepsilon < 1/2$.

Choose $\varepsilon < \min\{\delta/2, 1/2\}$. Then for each $y \in \mathcal{F}$ the tile $T + y$ has points outside C. It also has points inside C because $T + y$ intersects T. Furthermore $d(T_\varepsilon, \tilde{\Omega}) > \delta/2$. So $C \subset \operatorname{int} \Omega$. Because each neighbor of T has both points inside and outside C, and because T is connected, C must intersect all neighbors of T. Parametrize C by $z(t)$, $t \in [0, 1]$ with $z(0) = z(1)$. We now partitition $[0, 1]$ by $0 = t_0 < t_1 < t_2 < \cdots < t_k = 1$ such that each segment $C_i = z([t_{i-1}, t_i])$ of the curve C has $\operatorname{diam}(C_i) < \delta/2$. This partitition yields a sequence

$$y_{11}, \ldots, y_{1j_1}, y_{11}, \ldots, y_{2j_1}, \ldots, y_{k1}, \ldots, y_{kj_k}$$

in $\mathcal{F}$ such that $\{y_{ij} :\ 1 \le j \le j_i\}$ consists of all $y \in \mathcal{F}$ such that $(T+y) \cap C_i \ne \emptyset$. Pruning the sequence so that any two adjacent elements in the sequence are distinct we obtain a new sequence $y_1, \ldots, y_m, y_{m+1} = y_1$. Since each $y \in \mathcal{F}$ appears at least once in (y_{ij}) it must appear also at least once in the new sequence (y_i). Furthermore points in two adjacent C_i's are less than δ apart so $d(T + y_i, T + y_{i+1}) < \delta$. Hence $y_{i+1} - y_i \in \mathcal{F}$.

Note that $\mathcal{F}$ must be centrally symmetric so $\mathcal{F}$ can only have 2, 4 or 6 elements. By Lemma 3.1 $\mathbb{Z}[\mathcal{F}] = \mathbb{Z}^2$. So $\mathcal{F}$ contains at least two linearly independent elements. This immediately rules out two elements for $\mathcal{F}$. If $\mathcal{F}$ has 4 elements then $\mathcal{F} = \{\pm\alpha, \pm\beta\}$ with α and β independent. Thus one of $\pm\alpha$ must be followed by one of $\pm\beta$ somewhere in the sequence, yielding one of $\pm\alpha \pm \beta$ in $\mathcal{F}$, a contradiction. Hence $\mathcal{F}$ must have 6 elements. Again, in the sequence (y_j) there must be two adjacent elements α_1 and α_2 that are independent, yielding $\alpha_1 - \alpha_2 \in \mathcal{F}$. Therefore

$$\mathcal{F} = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 - \alpha_2)\}.$$

The theorem is proved by setting $\alpha = \alpha_1$ and $\beta = -\alpha_2$. ∎

Lemma 3.3. *Let T be a connected $\mathbb{Z}^2$-tile with neighbors $T + \mathcal{F}$, $\mathcal{F} \subset \mathbb{Z}^2$. If $\mathcal{F} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha-\beta)\}$ then $T + \{\pm\alpha, \pm\beta\}$ are edge neighbors. If $\mathcal{F} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ then $T + \mathcal{F}$ are edge neighbors.*

Proof. Since $\mathbb{Z}\alpha + \mathbb{Z}\beta = \mathbb{Z}^2$ in both cases by Lemma 3.1 we may without loss of generality assume that $\alpha = [1, 0]^T$ and $\beta = [0, 1]^T$.

Let $\delta > 0$ denote the minimal distance between two disjoint tiles in the lattice tiling. Denote $S_1 = \operatorname{int}(T + \alpha\mathbb{Z})$. This is an open set near the x_1-axis which by the assumption of our lemma separates the set B_+ consisting of all tiles $T + m\beta + n\alpha$ with positive m from the set B_- consisting of all tiles with negative m. The distance between B_+ and B_- is $\ge \delta$. Take an integer k with $1/k < \delta/2$. Write $x = [x_1, x_2]^T$ and let

$$f(x_1) = \sup\{x_2 :\ d(x, B_-) < \delta/2\}$$

for all x with $x_1 = n/k$ with $n \in \mathbb{Z}$. Thus the points $z = [x_1, f(x_1)]^T$ fulfil $d(x, B_-) = \delta/2 \le d(x, B_+)$. We extend f linearly between these points and let

$$C_1 = \text{graph of } f = \{z(s) = [s, f(s)]^T :\ s \in \mathbb{R}\}.$$

Since $f(s+1) = f(s)$, the polygonal line C_1 is periodic: $C_1 = C_1 + \alpha$. Now we prove that $C_1 \subset S_1$. Take $z(s)$ on a line segment of C_1 and let $z(x_1)$ be that vertex of the line segment for which $f(x_1) \le f(s)$. For $x' = [x_1, f(s)]^T$ we have

$$\delta/2 \le d(x', B_-) \le |x_1 - s| + d(z(s), B_-) < \delta/2 + d(z(s), B_-),$$

which implies $d(z(s), B_-) > 0$. Similarly we see that $d(z(s), B_+) > 0$. The connectedness of T, and hence of B_- and B_+, now implies that all points of B_- lie below C_1 and all points of B_+ above. Hence $C_1 \subseteq S_1$.

Note that C_1 must cross from one tile into another, say from T to $T + m\alpha$. Clearly $m = \pm 1$, or the two tiles are disjoint. Say $m = 1$. So part of C_1 must lie in $\operatorname{int}(T \cup (T + \alpha))$. This can only happen if $T \cap (T + \alpha)$ contains uncountably many points. Therefore $T + \alpha$ is an edge neighbor.

The proofs for the other cases are identical. ∎

Let $T = T(A, \mathcal{D})$ be a self-affine tile satisfying (1). Iterating (1) yields

(3) $$A^k(T) = T + \mathcal{D}_k, \text{ where } \mathcal{D}_k := \mathcal{D} + A\mathcal{D} + \ldots + A^{k-1}\mathcal{D}.$$

Note that $\mathcal{D}_k = \mathcal{D}_{k-1} + A^k\mathcal{D}$, with $D_0 := \{0\}$.

Lemma 3.4. *Let $T(A, \mathcal{D})$ be a self-affine $\mathbb{Z}^2$-tile with neighbors $T + \mathcal{F}$, $\mathcal{F} \subset \mathbb{Z}^2$. If $\mathcal{F} = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$ and $\mathcal{D}$ is $\mathcal{F}$-connected then so is $\mathcal{D}_k$ for all $k \ge 0$. If $\mathcal{F} = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(\alpha - \beta)\}$ and $\mathcal{D}$ is $\{\pm\alpha, \pm\beta\}$-connected then so is $\mathcal{D}_k$ for all $k \ge 0$.*

Proof. In the 6 neighbors case note that $A^k(T) = T + \mathcal{D}_k$ and T is connected. By Theorem 2.3 $\mathcal{D}_k$ must be $\mathcal{F}$-connected.

In the 8 neighbors case let $\mathcal{F}_0 = \{\pm\alpha, \pm\beta\}$. We prove $\mathcal{F}_0$-connectedness of D_k by induction on k. Observe that $\mathcal{D}_0 = \{0\}$ is clearly $\mathcal{F}_0$-connected, and $\mathcal{D}_k = \mathcal{D} + A\mathcal{D}_{k-1}$. We assume that $\mathcal{D}_{k-1}$ is $\mathcal{F}$-connected and show that $\mathcal{D}_k$ is $\mathcal{F}$-connected.

It will be sufficient to show that for $u, u' \in \mathcal{D}_{k-1}$ with $u - u' \in \mathcal{F}$ there exist $d, d' \in \mathcal{D}$ such that $(d + Au) - (d' + Au')$ is also in $\mathcal{F}$. But $u - u' \in \mathcal{F}$ means that $T + u$ and $T + u'$ are edge neighbors. Hence the larger tiles $A(T) + Au$ and $A(T) + Au'$ are also edge neighbors: they have uncountably many common points. Since $A(T) = \bigcup_{d \in \mathcal{D}} T + d$, there must exist $d, d' \in \mathcal{D}$ such that $T + d + Au$ and $T + d' + Au'$ also have uncountably many points. Thus they are edge neighbors and the difference of the vectors is in $\mathcal{F}$ by our assumptions. Lemma 3.4 is proved. ∎

Lemma 3.5. *Under the assumptions of Theorem 2.1 or 2.2,* $\operatorname{int} T$ *is connected.*

Proof. We prove that $\operatorname{int} T$ is connected under the assumptions of Theorem 2.2. The other case is virtually identical (in fact a little simpler). Denote

$$\mathcal{F} = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(\alpha - \beta)\} \text{ and } \mathcal{F}_0 = \{\pm\alpha, \pm\beta\}.$$

Let z_1 and z_2 be two points in $\operatorname{int} T$. We shall construct an arc from z_1 to z_2 within $\operatorname{int} T$. Let $K_0 \in \mathbb{Z}$ such that $K_0 > \max\{|x| : x \in T\}$ and let $R > 5K_0$. Choose k sufficiently large so that $B_R(A^k z_i) \subseteq A^k(\operatorname{int} T)$. It follows from $A^k(T) = T + \mathcal{D}_k$ and the $\mathcal{F}_0$-connectedness of $\mathcal{D}_k$ that we may find $y_0, y_1, \ldots, y_N \in \mathcal{D}_k$ such that $y_{i+1} - y_i \in \mathcal{F}_0$ and $A^k z_1 \in T + y_0$, $A^k z_2 \in T + y_N$. Hence $|A^k z_1 - y_0| < K_0$ and $|A^k z_2 - y_N| < K_0$. We prove there exists an arc connecting $A^k z_1$ and $A^k z_2$ that lies within $\operatorname{int}(A^k T)$.

Let $\delta > 0$ be the minimal distance between two disjoint tiles in the $\mathbb{Z}^2$-tiling and let T_ε be as in the proof of Theorem 3.2 with $\varepsilon < \min(1, \delta/4)$. Then the set

$$\Omega = \bigcup_{y \in \mathbb{Z}^2 \setminus \{y_i\}} (T_\varepsilon + y) \setminus \left(\overline{B_R(A^k z_1) \cup B_R(A^k z_2)}\right)$$

is an open set whose boundary consists of finitely many circular arcs. Furthermore $\mathbb{R}^2 \setminus \overline{\Omega} \subseteq \operatorname{int}(A^k T)$. Assume that $B_R(A^k z_1)$ and $B_R(A^k z_2)$ belong to the same connected component of $\mathbb{R}^2 \setminus \overline{\Omega}$. Then we can find an arc in $\mathbb{R}^2 \setminus \overline{\Omega}$ that connects $A^k z_1$ and $A^k z_2$. This arc is in $\operatorname{int}(A^k T)$. So we may connect z_1 and z_2 by an arc in $\operatorname{int} T$.

Now assume that $B_R(A^k z_1)$ and $B_R(A^k z_2)$ belong to two different connected components of $\mathbb{R}^2 \setminus \overline{\Omega}$, say Ω_1 and Ω_2, respectively. We derive a contradiction. Choose a simple closed curve $C \subseteq \partial\Omega_1$ such that $B_R(A^k z_1)$ and $B_R(A^k z_2)$ are on separate sides of C, and without loss of generality assume that $B_R(A^k z_1)$ is on the inside of C. We parametrize C by $x(t)$ where $t \in [0,1]$ with $x(0) = x(1)$. As t varies from 0 to 1 the curve wraps around $B_R(A^k z_1)$. Take points $x_i = x(t_i)$ for $0 \le i \le m$ where $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $|x_{i+1} - x_i| < \delta/4$. Each x_i is in the closure of $T_\varepsilon + w_i$ for some $w_i \notin \{y_j\}$ with $w_0 = w_m$. It is easy to see that $d(T + w_{i+1}, T + w_i) < \delta$ for $0 \le i < m$. Without loss of generality we may assume that $w_{i+1} \ne w_i$ for all $0 \le i \le m$, or we may remove the redundant vertices. It follows that $w_{i+1} - w_i \in \mathcal{F}$.

Let C_1 be the closed piecewise linear curve with vertices $w_0, w_1, \ldots, w_m$, which is a closed curve. Since each $|x_i - w_i| \le K_0 + \varepsilon < 2K_0$, we must have $|w_i - A^k z_1| \ge 3K_0$ and $|w_i - A^k z_1| \ge 3K_0$. Therefore $d(A^k z_i, C_1) > 2K_0$. It follows that C_1 must wrap around $B_{K_0}(A^k z_1)$ as it traverses w_0 through w_m while leaving $B_{K_0}(A^k z_2)$ outside. Hence any path from y_0 to y_N must cross C_1. In particular, the piecewise arc C_2 with vertices $y_0, y_1, \ldots y_N$ must intersect C_1. This means some line segment $\overline{w_i w_{i+1}}$ must intersect some line segment $\overline{y_j y_{j+1}}$. But $y_{j+1} - y_j \in \mathcal{F}_0$ and $w_{i+1} - w_i \in \mathcal{F}_0$. It is easy to check that the only way the two line segments can intersect is that they share at least one common vertex. This contradicts the assumption that the y_j and the w_i are disjoint.

Therefore $\operatorname{int} T$ must be connected. ∎

Proof of Theorems 2.1 and 2.2. We first observe that Lemma 3.3 implies that $\operatorname{int} T$ is simply connected. That is, each simple closed curve $C \subset \operatorname{int} T$ contracts to a point within the set $\operatorname{int} T$, briefly, $\operatorname{int} T$ contains no holes.

If there are holes, they must contain points of another tile T', and even interior points of T'. By the lemma, the interior of T' is connected and must therefore be completely surrounded by T. This is not possible since in a lattice tiling $T' = T + x$ for some x.

By the Riemann mapping theorem there is a homeomorphism (even a conformal map) f from $\operatorname{int} T$ to the open disk U. This map extends to a homeomorphism from T to the closed disk $\overline{U}$ if each boundary point x of T is simple (cf. [16], 14.20). That means that for each sequence (x_n) in $\operatorname{int} T$ converging to x there is a ray (homeomorphic image of $[0, \infty)$) connecting x_1 with $x_2, x_3, ...$ which also converges to x. Our proof will be complete when we are able to construct such rays.

In the proof of Lemma 3.3 we constructed arcs $[x, y]$ inside $\operatorname{int} T$ between any two points x, y of $\operatorname{int} T$. Moreover, when x, y belonged to the same piece $A^{-k}(T')$ of T, or to two neighboring pieces, then the arc could be chosen inside that piece, or those neighboring pieces, respectively. So let us take the ray as the composition of arcs $[x_i, x_{i+1}]$ where $i \geq 1$. We show that this ray converges to x when the rays ar properly chosen.

Let us consider k-th level pieces of T for some fixed k. The boundary point x of T can belong to at most two pieces in the hexagonal case and to at most three pieces in the octagonal case, since x is also contained in a piece of a neighbor of T. The union of these pieces forms a neighborhood V_k of x. Since $x_n \to x$, all x_n with $n \geq n_k$ belong to these two or three pieces of T. However, since these pieces are edge neighbors, the arcs $[x_n, x_{n+1}]$ can be chosen inside the union of these pieces, that is, inside V_k for $n \geq n_k$. Since this holds for all k and the V_k shrink down to x, the ray converges to x. The proof is finished. ∎

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