GENERATION OF FINITE TIGHT FRAMES BY HOUSEHOLDER THRANSFORMATIONS

DE-JUN FENG, LONG WANG, AND YANG WANG

ABSTRACT. Finite tight frames are used widely for many applications. An important problem is to construct finite frames with prescribed norm for each vector in the tight frame. In this paper we provide a fast and simple algorithm for such purpose. Our algorithm employs the Householder transformations. For a finite tight frame consisting m vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ only $O(nm)$ operations are needed. In addition, we also study the following question: Given a set of vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$, how many additional vectors, with possibly constraints, do you need to add in order to obtain a tight frame?

1. INTRODUCTION

Let $\mathbf{H}$ be a Hilbert space. A set of elements $\{\mathbf{u}_n\}$ in $\mathbf{H}$ (counting multiplicity) is called a *frame* if there exist two positive constants C_* and C^* such that for any $\mathbf{v} \in \mathbf{H}$ we have

$$C_*\|\mathbf{v}\|^2 \leq \sum_n |\langle \mathbf{v}, \mathbf{u}_n \rangle|^2 \leq C^*\|\mathbf{v}\|^2. \tag{1.1}$$

The constants C_* and C^* are called the *lower frame bound* and the *upper frame bound*, respectively. A frame is called a *tight frame* if $C_* = C^*$, and it is called a *Parseval frame* if $C_* = C^* = 1$.

Frames were first introduced by Duffin and Schaeffer [6] in a study of nonharmonic Fourier series. The study of frames has exploded in recent years, partly because of their applications in digital signal procession. They are an integral part of time-frequency analysis. For more on frames we refer the readers to Gröchenig [10] and the references therein.

This paper is primarily concerned with *finite frames*, i.e. frames in a finite dimensional Hilbert space. There has been a surge in interest in finite tight frames recently, mainly as a

1991 *Mathematics Subject Classification.* Primary 42C15.

Key words and phrases. Frames, tight frame, tight frame matrix, Householder matrix, condition number.

The first author is supported in part by the Special Funds for Major State Basic Research Projects in China.

The third author is supported in part by the National Science Foundation, grants DMS-0070586 and DMS-0139261.

result of several important applications. They have been used for internet coding, wireless communication, quantum detection theory, and more. Each new application requires a new class of tight frames. It is important to construct tight frames to fit particular applications. One such problem is to find a tight frame with prescribed norms. Casazza, Leon and Tremain [4] established the existence condition for finite frames in the form of an inequalities. Nevertheless, fast and efficient algorithms are needed to produce "custom" tight frames. This remained a challenging problem, as stated in [4].

Since we are working in a finite dimensional Hilbert space we may without loss of generality assume that $\mathbf{H} = \mathbb{H}^n$ where $\mathbb{H} = \mathbb{R}$ or $\mathbb{H} = \mathbb{C}$. Let $M_{n,m}(\mathbb{H})$ denote the set of all $n \times m$ matrices with entries in $\mathbb{H}$.

Definition 1.1. A matrix $A \in M_{n,m}(\mathbb{H})$ is called a *frame matrix* (FM) if $\mathrm{rank}(A) = n$. A is called a *tight frame matrix* (TFM) if $AA^* = \lambda I_{n\times n}$ for some $\lambda > 0$; if additionally all columns of A have the same norm then A is called an *equi-norm TFM.*

When A is a frame matrix (resp. TFM) the column vectors of A form a frame (resp. tight frame) of $\mathbb{H}^n$, and vice versa. Suppose that $A = [\mathbf{a}_1, \cdots, \mathbf{a}_m]$ be a FM in $M_{n,m}(\mathbb{H})$ with columns $\mathbf{a}_1, \cdots, \mathbf{a}_m$. It is straighforward to check that for any $\mathbf{x} \in \mathbb{H}^n$ we have

$$\mathbf{x}^* AA^* \mathbf{x} = \sum_{j=1}^m |\langle \mathbf{a}_j, \mathbf{x}\rangle|^2.$$

Let $\lambda_{\max}$ and $\lambda_{\min}$ be the maximal and minimal eigenvalues of AA^*, respectively. It follows that for all $\mathbf{x} \in \mathbb{H}^n$,

$$\lambda_{\min}\|\mathbf{x}\|^2 \le \sum_{j=1}^m |\langle \mathbf{a}_j, \mathbf{x}\rangle|^2 \le \lambda_{\max}\|\mathbf{x}\|^2. \tag{1.2}$$

Hence $\lambda_{\max}$ and $\lambda_{\min}$ are the upper and lower frame bounds for the frame defined by A. Denote

$$c(A) = \frac{\lambda_{\max}}{\lambda_{\min}}$$

and call it *the condition number* of A. For most applications it is better to make the condition number $c(A)$ as small as possible. If $c(A) = 1$ then A is a TFM. But this is not always possible. A natural question is: Given vectors $\mathbf{a}_1, \ldots, \mathbf{a}_p \in \mathbb{H}^n$, how many vectors $\mathbf{a}_{p+1}, \ldots, \mathbf{a}_m \in \mathbb{H}^n$ do we need to add to them in order to obtain a tight frame? If only a fixed number $m-p$ of vectors are allowed to be added, how small can we make the condition number $c(A)$ to be for $A = [\mathbf{a}_1, \cdots, \mathbf{a}_m]$? For this question we have the following theorem:

Theorem 1.1. *For any $\mathbf{a}_1, \ldots, \mathbf{a}_p \in \mathbb{H}^n$, let $V = [\mathbf{a}_1, \cdots, \mathbf{a}_p]$. Suppose that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are all the eigenvalues of VV^*. Then for any vectors $\mathbf{a}_{p+1}, \mathbf{a}_{p+2}, \ldots, \mathbf{a}_m$, the matrix $A = [\mathbf{a}_1, \cdots, \mathbf{a}_m]$ satisfies*

$$c(A) \geq \frac{\lambda_1}{\lambda_{n-k}}, \tag{1.3}$$

where $k = m - p$ (if $n - k \leq 0$, we define $\lambda_{n-k} = \lambda_1$). Furthermore the equality in (1.3) can be attained by some $\mathbf{a}_{p+1}, \mathbf{a}_{p+2}, \ldots, \mathbf{a}_m$. In particular, at most $n-1$ vectors $\mathbf{a}_{p+1}, \mathbf{a}_{p+2}, \ldots, \mathbf{a}_{p+n-1}$ are needed to make $A = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_{p+n-1}]$ a TFM.

Let $\mathbb{S}^n$ denote the set of vectors in $\mathbb{H}^n$ with norm 1. A similar question for equi-norm tight frames can be asked. Given vectors $\mathbf{a}_1, \ldots, \mathbf{a}_p \in \mathbb{S}^n$, how many vectors $\mathbf{a}_{p+1}, \cdots, \mathbf{a}_m \in \mathbb{S}^n$ do we need to add in order to obtain a equi-norm tight frame? For this question, we have

Theorem 1.2. *For any $\mathbf{a}_1, \ldots, \mathbf{a}_p \in \mathbb{S}^n$ let $V = [\mathbf{a}_1, \cdots, \mathbf{a}_p]$. Suppose that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are all the eigenvalues of VV^*. Denote by d the smallest integer $\geq \lambda_1 + 1$. Then one can find $\mathbf{a}_{p+1}, \ldots, \mathbf{a}_{nd} \in \mathbb{S}^n$ such that the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_{nd}$ form a equi-norm tight frame for $\mathbb{H}^n$.*

However, the main result in this paper is a very fast and efficient algorithm for generating TFMs whose columns have prescribed norms. This algorithm employs Householder transformations, with at most m transformations needed to generate a TFM in $M_{n,m}(\mathbb{H})$. For a given sequence $a_1 \geq a_2 \cdots \geq a_m > 0$, is it possible to find a TFM $A = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_m]$ such that $\|\mathbf{a}_j\| = a_j$? If so, how do you construct such a TFM A? The following is proved in a recent paper [4]:

Theorem 1.3 (Casazza, Leon and Treimain [4]). *Let $a_1 \geq a_2 \cdots \geq a_m > 0$ and $n \leq m$. Then there exists a TFM $A = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_m]$ such that $\|\mathbf{a}_j\| = a_j$ for all $1 \leq j \leq m$ if and only if*

$$\sum_{i=1}^{m} a_i^2 \geq n a_1^2. \tag{1.4}$$

Condition (1.4) is called the *fundamental inequality* in [4]. The proof given in [4] for the sufficiency of (1.4) is an *existence* proof. In [3] Casazza and Leon provide an algorithm for the construction of TFMs whose columns have prescribed lengths. However, their algorithm uses a lots of intermediate parameters and appears to be rather complicated. As pointed in [4], "much more work needs to be done in this direction." Our alternative algorithm for constructing TFMs using Householder transformations is direct and simpler.

2. Proof of Theorem 1.1 and Theorem 1.2

We prove the theorems in this section by first establishing several lemmas.

Lemma 2.1. *Let A and B be $n \times n$ postive semi-definite Hermitian matrices. Let $\mu_1 \geq \ldots \geq \mu_n \geq 0$ be the eigenvalues of A and $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ be the eigenvalues of B. Suppose that $A - B$ is positive semi-definite with* $\mathrm{rank}(A-B) \leq k$ *for some integer $k < n$. Then $\mu_1 \geq \lambda_1$ and $\mu_n \leq \lambda_{n-k}$.*

Proof. Let $\mathbf{x}_0$ be any eigenvector of B associated to λ_1. Then $\mathbf{x}_0^* B \mathbf{x}_0 = \lambda_1 \|\mathbf{x}_0\|^2$ and $\mathbf{x}_0^* A \mathbf{x}_0 \leq \mu_1 \|\mathbf{x}_0\|^2$. Therefore

$$0 \leq \mathbf{x}_0^*(A-B)\mathbf{x}_0 \leq \mu_1 \|\mathbf{x}_0\|^2 - \lambda_1 \|\mathbf{x}_0\|^2.$$

This implies $\mu_1 \geq \lambda_1$. We prove $\mu_n \leq \lambda_{n-k}$ by contradiction. Assume on the contrary that $\mu_n > \lambda_{n-k}$. Then

$$\mu_n > \lambda_{n-k} \geq \cdots \geq \lambda_n.$$

Consider the subspace $\mathbf{V}$ of $\mathbb{H}^n$ given by

$$\mathbf{V} = \mathbf{E}_{\lambda_{n-k}} + \cdots + \mathbf{E}_{\lambda_n},$$

where $\mathbf{E}_{\lambda_j}$ denotes the eigenspace of B associated to eigenvalue λ_j. Then $\dim(\mathbf{V}) \geq k+1$. Observe that $\mathbf{V}$ is an invariant subspace of B, and the largest eigenvalue of B on $\mathbf{V}$ is λ_{n-k}. Hence for any $\mathbf{x} \in \mathbf{V}$ with $\mathbf{x} \neq \mathbf{0}$, we have

$$\mathbf{x}^*(A-B)\mathbf{x} = \mathbf{x}^* A \mathbf{x} - \mathbf{x}^* B \mathbf{x} \geq \mu_n \|\mathbf{x}\|^2 - \lambda_{n-k} \|\mathbf{x}\|^2 > 0. \tag{2.1}$$

However, from the fact that $\mathrm{rank}(A-B) \leq k$ we have $\dim(\ker(A-B)) \geq n-k$. This implies that $\mathbf{V} \cap \ker(A-B) \neq \{\mathbf{0}\}$, since $\dim(\mathbf{V}) + \dim(\ker(A-B)) > n$. So there exists a vector $\mathbf{x}_1 \in \mathbf{V} \cap \ker(A-B)$ with $\mathbf{x}_1 \neq \mathbf{0}$. For $\mathbf{x}_1$ we have $\mathbf{x}_1^*(A-B)\mathbf{x}_1 = \mathbf{0}$, contradictiing (2.1). ∎

Lemma 2.2. *Let B be an $n \times n$ positive semi-definite Hermitian matrix with* $\mathrm{rank}(B) \leq k$. *Then there exists a matrix $W \in M_{n,k}(\mathbb{H})$ such that $B = WW^*$.*

Proof. This is well known, see e.g. [12, p. 98]. ∎

Proof of Theorem 1.1. Suppose that $\mathbf{a}_{p+1}, \cdots, \mathbf{a}_m$ are any given vectors in $\mathbb{H}^n$. To prove (1.3) we assume $k := m - p < n$ (The case $k \geq n$ is covered by the case $k = n - 1$ by choosing $\mathbf{a}_j = \mathbf{0}$ for $j \geq n$.) Set

$$W = [\mathbf{a}_{p+1}, \cdots, \mathbf{a}_m] \quad \text{and} \quad A = [V, W].$$

Then $AA^* = VV^* + WW^*$. Note that WW^* is positive semi-definite with rank at most $k = m-p$. Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0$ be the eigenvalues of AA^*. By Lemma 2.1, $\mu_1 \geq \lambda_1$ and $\mu_n \leq \lambda_{n-k}$. Hence $c(A) = \frac{\mu_1}{\mu_n} \geq \frac{\lambda_1}{\lambda_{n-k}}$.

To prove the existence of $\mathbf{a}_{p+1}, \ldots, \mathbf{a}_m$ such that $c(A) = \frac{\lambda_1}{\lambda_{n-k}}$, choose a unitary matrix U such that UVV^*U^* is the diagonal matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. For each $n-k+1 \leq i \leq n$, define $\alpha_i = \lambda_{n-k} - \lambda_i$. Now set $Q = \mathrm{diag}(\underbrace{0, \ldots, 0}_{n-k}, \alpha_{n-k+1}, \ldots, \alpha_n)$. It is clear UQU^* is positive semi-definite with rank at most k. By Lemma 2.2, there exists a matrix $W \in M_{n\times k}(\mathbb{H})$ such that $U^*QU = WW^*$. Take $\mathbf{a}_{p+1}, \cdots, \mathbf{a}_m$ to be the column vectors of W. Then one can check that

$$UAA^*U^* = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n-k}, \ldots, \lambda_{n-k}),$$

and thus $c(A) = \frac{\lambda_1}{\lambda_{n-k}}$. ∎

We now focus on Theorem 1.2, which is much more difficult.

Lemma 2.3. *Let a, b be two real numbers with $a \geq b \geq 0$ and let $r > 0$. Set $d := \max\{a, b+r\}$. Then for any $\theta \in [d, a+r]$ there exist $x, y \in \mathbb{R}$ with $x^2 + y^2 = c$ such that the eigenvalues of the matrix*

$$\begin{bmatrix} a + x^2 & xy \\ xy & b + y^2 \end{bmatrix} \tag{2.2}$$

are exactly θ and $a + b + r - \theta$.

Proof. For any $[x, y]^T \in \mathbb{R}^2$ with $x^2 + y^2 = r$, the eigenvalues of the matrix (2.2) are

$$\frac{1}{2}\left[(a+b+r) \pm \sqrt{(a+b+r)^2 - 4ab - 4ay^2 - 4bx^2}\right].$$

Note that when (x, y) varies over the circle $x^2 + y^2 = r$ the range of the function $4ay^2 + 4bx^2$ is $[4br, 4ar]$. Therefore the range of

$$\sqrt{(a+b+r)^2 - 4ab - 4ay^2 - 4bx^2}$$

is $[|a-b-r|, a-b+r]$, which implies the required result. In fact for the given $\theta \in [d, a+r]$ we may take

$$x = \sqrt{\frac{ar+ab-\theta(a+b+r-\theta)}{a-b}} \quad \text{and} \quad y = \sqrt{\frac{\theta(a+b+c-\theta)-bc-ab}{a-b}}.$$

∎

Lemma 2.4. *Let A be an $n \times n$ positive semi-definite Hermitian matrix with eigenvalues $\mu_1 \geq \cdots \geq \mu_n$. Let $r > 0$ and $k \in \mathbb{N}$ with $1 \leq k \leq n-1$. Then for any $\theta \in [\max\{\mu_{k+1} + r, \mu_k\}, \mu_k + r]$, one may construct a $\mathbf{v} \in \mathbb{H}^n$ with $\|\mathbf{v}\| = \sqrt{r}$ such that the eigenvalues of the matrix $A + \mathbf{v}\mathbf{v}^*$ are exactly*

$$\mu_1, \ \ldots \ \mu_{k-1}, \ \theta, \ \mu_k + \mu_{k+1} + r - \theta, \ \mu_{k+2}, \ \ldots, \mu_n. \tag{2.3}$$

Proof. Construct a unitary matrix P such that $P^*AP = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n)$. Since

$$\theta \in [\max\{\mu_{k+1} + r, \mu_k\}, \mu_k + r],$$

it follows from Lemma 2.3 that one may construct a vector $\mathbf{v} = [x, y]^T \in \mathbb{R}^2$ with $\|\mathbf{v}\|^2 = c$ such that the eigenvalues of $\operatorname{diag}(\mu_k, \mu_{k+1}) + \mathbf{v}\mathbf{v}^*$ are exactly θ and $\mu_k + \mu_{k+1} + r - \theta$. Define $\mathbf{u} \in \mathbb{H}^n$ so that the k-th and $(k+1)$-th entries of $\mathbf{u}$ are x, y respectively and all other entries are 0. Then the eigenvalues of $P^*AP + \mathbf{u}\mathbf{u}^*$ (as well as $A + P\mathbf{u}\mathbf{u}^*P^{-1}$) are given by (2.3). Take $\mathbf{v} = P\mathbf{u}$, then $\mathbf{v}$ is what we desired. This completes the proof of the lemma. ∎

Proof of Theorem 1.2. We will construct $\mathbf{a}_{p+1}, \ldots, \mathbf{a}_{nd} \in \mathbb{S}^n$ recursively by the following steps.

Step 1. Pick an integer k_1 and construct $\mathbf{a}_{p+1}, \ldots, \mathbf{a}_{p+k_1} \in \mathbb{S}^n$ such that the eigenvalues of $[\mathbf{a}_1, \ldots, \mathbf{a}_{p+k_1}][\mathbf{a}_1, \ldots, \mathbf{a}_{p+k_1}]^*$ are exactly $d, \hat{\lambda}_2, \lambda_3, \ldots, \lambda_n$, where $\hat{\lambda}_2 := \lambda_1 + \lambda_2 + k_2 - d$ satisfies $\lambda_2 \leq \hat{\lambda}_2 < d$.

To do this we first choose a unitary matrix P_1 such that $P_1^* V V^* P_1 = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. By the definition of d we know that $d - 2 \leq \lambda_1 < d - 1$. Take $k_1 = 2$. If $\lambda_1 = d - 2$, we take $\mathbf{a}_{p+1} = \mathbf{a}_{p+2} = P_1[1, 0, \cdots, 0]^*$. If $\lambda_1 > d - 2$, we take $\mathbf{a}_{p+1} = P_1[1, 0, \cdots, 0]^*$ and $\mathbf{a}_{p+2} = P_1[x, y, 0, \ldots, 0]$, where x, y are constructed so that $|x|^2 + |y|^2 = 1$ and the eigenvalues of

$$\operatorname{diag}(\lambda_1 + 1, \lambda_2) + [x, y]^*[x, y]$$

are d and $\lambda_1 + \lambda_2 + 2 - d$.

Step 2. Pick an integer k_2 and construct $\mathbf{a}_{p+k_1+1}, \dots, \mathbf{a}_{p+k_1+k_2} \in \mathbb{S}^n$ such that the eigenvalues of $[\mathbf{a}_1, \dots, \mathbf{a}_{p+k_1+k_2}][\mathbf{a}_1, \dots, \mathbf{a}_{p+k_1+k_2}]^*$ are exactly $d, d, \hat{\lambda}_3, \lambda_4, \dots, \lambda_n$, where $\hat{\lambda}_3 := \lambda_1 + \lambda_2 + \lambda_3 + k_1 + k_2 - 2d$ satisfies $\lambda_3 \le \hat{\lambda}_3 < d$.

To realize this constuction, we adopt a method similar to what we have used in Step 1. First we construct a unitary matrix P_2 such that

$$P_2^*[\mathbf{a}_1, \dots, \mathbf{a}_{p+k_1}][\mathbf{a}_1, \dots, \mathbf{a}_{p+k_1}]^* P_2 = \operatorname{diag}(d, \hat{\lambda}_2, \lambda_3, \dots, \lambda_n).$$

Assume that $d - \ell - 1 \le \hat{\lambda}_2 < d - \ell$ for some integer ℓ. We take $k_2 = \ell + 1$. If $\hat{\lambda}_2 = d - \ell - 1$, we let $\mathbf{a}_{p+k_1+1} = \cdots = \mathbf{a}_{p+k_1+k_2} = P_2[0, 1, 0, \cdots, 0]^*$. Otherwise we let $\mathbf{a}_{p+k_1+1} = \cdots = \mathbf{a}_{p+k_1+k_2-1} = P_2[0, 1, 0, \cdots, 0]^*$ and let $\mathbf{a}_{p+k_1+k_2} = P_2[0, x, y, 0, \dots, 0]^*$, where x, y are constructed so that $|x|^2 + |y|^2 = 1$ and the eigenvalues of $\operatorname{diag}(\hat{\lambda}_2 + \ell, \lambda_3) + [x, y]^*[x, y]$ are d and $\hat{\lambda}_2 + \ell + \lambda_3 + 1 - d$.

Continuing the above procedure, we can construct $\mathbf{a}_{p+1}, \dots, \mathbf{a}_{nd} \in \mathbb{S}^n$ so that the eigenvalues of $[\mathbf{a}_1, \dots, \mathbf{a}_{nd}][\mathbf{a}_1, \dots, \mathbf{a}_{nd}]^*$ are exacly $d, \dots, d$. Thus by setting $A = [\mathbf{a}_1, \dots, \mathbf{a}_{nd}]$ we have $AA^* = dI_{n\times n}$. This completes the proof of the theorem. ∎

The result of Theorem 1.2 is not sharp. Nevertheless there is a clear lower bound on how many columns from $\mathbb{S}^n$ we must add to V to get a TFM. Note that by adding columns from $\mathbb{S}^n$ to V to form the matrix A we have $\rho(AA^*) \ge \rho(VV^*)$, and the trace of the new matrix AA^* is m, where m is the number of columns of A. If A is a TFM then $AA^* = \lambda I_n$, so $\lambda = \frac{m}{n}$. Thus $\frac{m}{n} \ge \lambda_1$, and $m \ge \lceil n\lambda_1 \rceil$. There is a gap between our result and the lower bound. At this time we do not even have a reasonable conjecture as to what is a true lower bound. It appears to be a very difficult problem. We do know, however, that $\lceil n\lambda_1 \rceil$ is not a sharp lower bound. The following is an example.

Example 2.1. Take $\mathbf{a}_1 = [1, 0]^*$ and $\mathbf{a}_2 = [\sin\theta, \cos\theta]^*$. Then the eigenvalues of $V = [\mathbf{a}_1, \mathbf{a}_2]$ are $1 + \sin\theta$ and $1 - \sin\theta$. Take $\theta > 0$ sufficiently small. Then the lower bound predicted by $m \ge n\lambda_1$ is $m = 3$. However, it is impossible to add one vector to V to make it a TFM. At least two vectors in $\mathbb{S}^n$ are needed to make it a TFM.

3. Algorithm For Generating TFM Using Householder Transformations

In this section, we will develop an algorithm to produce TFMs whose columns have prescribed norms. More precisely, for $m \ge n$ and positive numbers $a_1 \ge a_2 \ge \cdots \ge a_m > 0$, we give an $O(nm)$ algorithm to construct an $n \times m$ matrix $A = [\mathbf{a}_1, \ldots, \mathbf{a}_m]$ such that A is a TFM and $\|\mathbf{a}_i\| = a_i$ for all $1 \leq i \leq m$. Without loss of generality we may normalize $\{a_j\}$ so that $\sum_{i=1}^m a_i^2 = n$; in this case $AA^* = I_{n\times n}$. The fundamental inequality states that such a A exists if and only if $|a_1| \leq 1$.

Definition 3.1. A *Householder matrix* is a matrix of the form $H = I_{n\times n} - 2\mathbf{x}\mathbf{x}^*$, where $\mathbf{x} \in \mathbb{H}^n$ and $\|\mathbf{x}\| = 1$.

It is well known that any Householder matrix is unitary. For any TFM A and any unitary matrix P, the matrix AP is again a TFM. We have

Lemma 3.1. *For any TFM $A \in M_{n,m}(\mathbb{H})$ there exist Householder matrices $H_0, H_1, \ldots, H_{m-2}$ and a diagonal matrix $D = \operatorname{diag}(c_1, \ldots, c_n)$ with $c_j \in \mathbb{H}$ and $|c_j| = 1$ for all $1 \leq j \leq n$ such that*

$$A = \lambda D[I_n, 0_{n\times(m-n)}]H_0H_1\cdots H_{m-2} = \lambda[D, 0_{n\times(m-n)}]H_0H_1\cdots H_{m-2}. \tag{3.1}$$

Conversely, any A of the form (3.1) is a TFM.

Proof. Suppose that A is a TFM then $AA^* = cI_n$. Let $\lambda = \sqrt{c}$. Then the rows of $\lambda^{-1}A$ are orthonormal. Therefore we may augment it to a unitary $m \times m$ matrix P by adding $m - n$ rows to A. This forces $AP^* = \lambda[I_n, 0]$, which yields $A = \lambda[I_n, 0]P$. But it is well known that any unitary $m \times m$ matrix P can be expressed as

$$P = \operatorname{diag}(c_1, c_2, \ldots, c_m)H_0H_1\cdots H_{m-2}$$

where H_j are Householder matrices and $c_j \in \mathbb{H}$ with $|c_j| = 1$ for all j, cf. Householder [11, page 7]. (3.1) is now proved by setting $D = \operatorname{diag}(c_1, \ldots, c_n)$.

The converse of the lemma is clearly true. ∎

The above lemma is not constructive. The objective of this section is to devise a way to find these Householder matrices $\{H_j\}$ so that $A = \lambda[I_n, 0_{n\times(m-n)}]H_0H_1\cdots H_{m-2}$, which is a TFM, has the desired norm for each of its columns. We develop two additional lemmas, which will be needed for our algorithm.

Lemma 3.2. *Let $\mathbf{a}, \mathbf{b} \in \mathbb{H}^n$ and $a = \|\mathbf{a}\|, b = \|\mathbf{b}\|, a \leq b$. Then for any c with $a \leq c \leq b$, we may find $x, y \in \mathbb{R}$ (hence $x, y \in \mathbb{H}$) such that $x^2 + y^2 = 1$ and $\|x\mathbf{a} + y\mathbf{b}\| = c$.*

Proof. The existence is rather obvious. We omit the derivation. Instead we will give the explicit formula for the required x, y. One can easily check it. Set $u = \sqrt{\frac{1}{4}(b^2 - a^2)^2 + (\mathbf{a}\cdot\mathbf{b})^2}$. Then the required x, y are given by

$$\begin{aligned} x &= \sqrt{\frac{\alpha\gamma + \beta\sqrt{1-\gamma^2} + 1}{2}}, \\ y &= \frac{\alpha\sqrt{1-\gamma^2} + \gamma\beta}{\sqrt{2(\alpha\gamma + \beta\sqrt{1-\gamma^2} + 1)}}, \end{aligned}$$

where $\alpha = u^{-1}(\frac{b^2-a^2}{2})$, $\beta = u^{-1}(\mathbf{a}\cdot\mathbf{b})$ and $\gamma = u^{-1}(c^2 - \frac{a^2+b^2}{2})$. ∎

Lemma 3.3. *Let $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{H}^n$ and $V = [\mathbf{a}_1, \ldots, \mathbf{a}_m]$. For any $1 \le i < j \le m$ and any θ between $\|\mathbf{a}_i\|$ and $\|\mathbf{a}_j\|$, i.e.*

$$\min(\|\mathbf{a}_i\|, \|\mathbf{a}_j\|) \le \theta \le \max(\|\mathbf{a}_i\|, \|\mathbf{a}_j\|),$$

we may construct a Householder matrix H such that the column vectors $\mathbf{b}_1, \ldots, \mathbf{b}_m$ of the matrix VH satisfy

$$\|\mathbf{b}_i\| = \theta, \qquad \|\mathbf{b}_j\| = \sqrt{\|\mathbf{a}_i\|^2 + \|\mathbf{a}_j\|^2 - \theta^2},$$

and $\mathbf{b}_k = \mathbf{a}_k$ for all $k \ne i, j$.

Proof. Let i, j and θ be given. By Lemma 3.2 we may construct x, y in $\mathbb{R}$ such that $x^2 + y^2 = 1$ and $\|x\mathbf{a}_i + y\mathbf{a}_j\| = \theta$. Now set

$$u = \sqrt{(1-x)/2}, \quad \text{and} \quad v = \begin{cases} \frac{-y}{\sqrt{2(1-x)}}, & \text{if } x \ne 1, \\ 1, & \text{if } x = 1. \end{cases}$$

One can check that $u^2 + v^2 = 1$ and $1 - 2u^2 = x$, $-2uv = y$. Define $\mathbf{x} \in \mathbb{S}^n$ to be the vector whose i-th and j-th entries are u, v, respectively, and all other entries 0. Let $H = I_m - 2\mathbf{x}\mathbf{x}^*$. Then $VH = [\mathbf{b}_1, \ldots, \mathbf{b}_m]$ such that

$$\mathbf{b}_i = (1-2u^2)\mathbf{a}_i - 2uv\mathbf{a}_j = x\mathbf{a}_i + y\mathbf{a}_j, \qquad \mathbf{b}_j = -2uv\mathbf{a}_i + (1-2v^2)\mathbf{a}_j,$$

and $\mathbf{b}_k = \mathbf{a}_k$ for all $k \ne i, j$. Observe that $\|\mathbf{b}_i\| = \theta$. One may check directly that $\|\mathbf{b}_i\|^2 + \|\mathbf{b}_j\|^2 = \|\mathbf{a}_i\|^2 + \|\mathbf{a}_j\|^2$, and it follows that $\|\mathbf{b}_j\| = \sqrt{\|\mathbf{a}_i\|^2 + \|\mathbf{a}_j\|^2 - \theta^2}$. This completes the proof. ∎

Algorithm for Constructing TFMs with Presicribed Column Norms.

For a given sequence $a_1 \geq a_2 \geq \cdots a_m \geq 0$ satisfying the fundamental inequality $a_1 \leq 1$ and $\sum_{j=1}^m a_j^2 = n$ we construct a TFM $A = [\mathbf{a}_1, \ldots, \mathbf{a}_m]$ such that $AA^* = I_n$ and $\|\mathbf{a}_j\| = a_j$ for all j.

Staring with $A_0 = [I_n, 0_{n\times(m-n)}]$, we accomplish the above task by constructing a sequence of $m-1$ Householder matrices $\{H_j : 0 \leq j \leq m-2\}$ such that the matrices $A_k = \tilde{A}_{k-1}H_{k-1}$, where $\tilde{A}_{k-1}$ is simply A_{k-1} with possibly some columns interchanged and $1 \leq k \leq m-1$, have the following properties:

(i) $A_k A_k^* = I_{n\times n}$.

(ii) If we denote $A_k = [\mathbf{b}_{k,1}, \mathbf{b}_{k,2}, \ldots, \mathbf{b}_{k,m}]$ then $\|\mathbf{b}_{k,i}\| = a_i$ for $i \leq k$, and $\sum_{i=k+1}^m \|\mathbf{b}_{k,i}\|^2 = \sum_{i=k+1}^m a_i^2$. Furthermore for any $j \geq k+2$, either $\|\mathbf{b}_{k,j}\| \geq a_{k+1}$ or $\mathbf{b}_{k,j} = \mathbf{0}$.

Clearly in the end, the matrix A_{m-1} is what we are looking for.

First, we construct A_1. Note that $A_0 = [\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{0}, \ldots, \mathbf{0}]$. If $n = m$ then all $a_j = 1$, and we are done. So we may without loss of generality assume that $m > n$. By a proper permutation of the column vectors of A_0 we obtain an $n \times m$ matrix $\tilde{A}_0 = (\mathbf{d}_{0,1}, \mathbf{d}_{0,2}, \cdots, \mathbf{d}_{0,m})$ such that $\mathbf{d}_{0,1} = \mathbf{e}_1$ and $\mathbf{d}_{0,2} = \mathbf{0}$. Clearly, $\tilde{A}_0\tilde{A}_0^* = I_{n\times n}$. Note that $0 = \|\mathbf{d}_{0,2}\| \leq a_1 \leq \|\mathbf{d}_{0,1}\| = 1$. By Lemma 3.3, we can find a Householder matrix H_0 such that the column vectors of $\tilde{A}_0H_0$, denoted by $\mathbf{b}_{1,1}, \mathbf{b}_{1,2}, \cdots, \mathbf{b}_{1,m}$, satisfy $\|\mathbf{b}_{1,1}\| = a_1$, $\|\mathbf{b}_{1,2}\| = \sqrt{1-a_1^2}$ and $\mathbf{b}_{1,j} = \mathbf{d}_{0,j}$ for $j \geq 3$. Set $A_1 := \tilde{A}_0H_0$. It is clear that A_1 satisfies the conditions (i) and (ii).

Now assume that $A_k = [\mathbf{b}_{k,1}, \mathbf{b}_{k,2}, \ldots, \mathbf{b}_{k,m}]$ satisfying the conditions (i) and (ii) has been constructed for some $1 \leq k \leq m-2$. We will construct A_{k+1} for three possible senarios for $\|\mathbf{b}_{k,k+1}\|$:

$$(a) \quad \|\mathbf{b}_{k,k+1}\| = a_{k+1}; \qquad (b) \quad \|\mathbf{b}_{k,k+1}\| < a_{k+1}; \qquad (c) \quad \|\mathbf{b}_{k,k+1}\| > a_{k+1}.$$

For scenario (a), we simply do nothing by taking $A_{k+1} = A_k$.

For scenario (b), we claim that there exists a $j_0 \geq k+2$ such that $\|\mathbf{b}_{k,j_0}\| \geq a_{k+1}$. Assume this is not true. Then according to (ii), $\mathbf{b}_{k,j} = \mathbf{0}$ for all $j \geq k+2$. Thus $\sum_{i=k+1}^m \|\mathbf{b}_{k,i}\|^2 = \|\mathbf{b}_{k,k+1}\|^2 < a_{k+1}^2 \leq \sum_{i=k+1}^m a_i^2$, a contradiction. Pick such a j_0. Making a proper permutation of the column vectors of A_k, we obtain the matrix

$$\tilde{A}_k = [\mathbf{d}_{k,1}, \mathbf{d}_{k,2}, \ldots, \mathbf{d}_{k,m}]$$

such that $\mathbf{d}_{k,i} = \mathbf{b}_{k,i}$ for $i \leq k+1$ and $\mathbf{d}_{k,k+2} = \mathbf{b}_{k,j_0}$. It is clear that $\tilde{A}_k \tilde{A}_k^* = I_n$. Since $\|\mathbf{d}_{k,k+1}\| < a_{k+1}$ and $\|\mathbf{d}_{k,k+2}\| \geq a_{k+1}$, it follows from Lemma 3.3 that we may construct a Householder matrix H_k such that the column vectors of $\tilde{A}_k H_k$, denoted as $\mathbf{b}_{k+1,1}, \ldots, \mathbf{b}_{k+1,m}$, satisfy $\mathbf{b}_{k+1,j} = \mathbf{b}_{k,j}$ for $j \neq k+1, k+2$, and $\|\mathbf{b}_{k+1,k+1}\| = a_{k+1}$. Define $A_{k+1} := \tilde{A}_k H_k$. It is easily checked that A_{k+1} satisfies the conditions (i) and (ii).

For scenario (c), we claim that there exists a $j_0 \geq k+2$ such that either $\|\mathbf{b}_{k,j_0}\| = a_{k+1}$ or $\mathbf{b}_{k,j_0} = \mathbf{0}$. Assume this is false. Then $\|\mathbf{b}_{k,j}\| > a_{k+1}$ for all $j \geq k+2$. Thus $\sum_{i=k+1}^m \|\mathbf{b}_{k,i}\|^2 > \sum_{i=k+1}^m a_{k+1}^2 > \sum_{i=k+1}^m a_i^2$, a contradiction. The rest of the construction process for A_{k+1} is the same as that for scenario (b).

Hence we have constructed A_{k+1} satisfying the conditions (i) and (ii). By continuing this process we reach in $m-1$ steps A_{m-1}. Set $A = A_{m-1}$ then it has the desired property. And this completes our construction. ∎

Remark. Although we obtain a single TFM A with the desired norms for the columns, our algorithm offers also some flexibilities. It is rather easy to see that the algorithm works if one starts with $A_0 = [Q, 0]$ for any $n \times n$ unitary matrix Q. The algorithm can easily be adapted also to allow some flexibility in intermediate steps, but the process will be more complicated. The advantage of the current algorithm is that it is very fast and easy to implement. Only $O(nm)$ operations are needed to complete the process. We give an numerical example to illustrate our algorithm.

Example 3.1. We wish to construct a TFM $A \in M_{4,6}(\mathbb{R})$ whose columns have prescribed norms $2, 2, 2, \sqrt{3}, \sqrt{2}, 1$. These norms satsify the fundamental inequality (1.4), so a TFM whose columns have these norms exists. Our algorithm yields the following Householder matrices $H_j = I_6 - 2\mathbf{x}_j \mathbf{x}_j^*$, $j = 0, 1, \ldots, 4$, in steps:

$$\begin{aligned}
\mathbf{x}_0 &= [0.169102, -0.985599, 0, 0, 0, 0]^T, \\
\mathbf{x}_1 &= [0, 0.179702, -0.983721, 0, 0, 0]^T, \\
\mathbf{x}_2 &= [0, 0, 0.192588, -0.98128, 0, 0]^T, \\
\mathbf{x}_3 &= [0, 0, 0, 0.382683, -0.92388, 0]^T \\
\mathbf{x}_4 &= [0, 0, 0, 0, 0.302905, -0.953021]^T.
\end{aligned}$$

The TFM A in the end is

$$\begin{aligned} A &= [I_4, 0]H_0H_1H_2H_3H_4 \\ &= \begin{bmatrix} 2 & 0.25 & -0.25 & 0.433013 & -0.353553 & -0.25 \\ 0 & 0 & 0 & 1.5 & 1.22474 & 0.866025 \\ 0 & 1.98431 & 0.283473 & -0.49099 & 0.400892 & 0.283473 \\ 0 & 0 & 1.96396 & 0.566947 & -0.46291 & -0.327327 \end{bmatrix} \end{aligned}$$

The resulting AA^* is within 2×10^{-16} of the theoretical result $4.5I_4$.

Acknowledgement

Throughout this research the first author is a post-doc at Georgia Tech while the second author is visiting Georgia Tech from Floyd College on a Faculty Development Program grant. Both wish to thank the School of Mathematics, Georgia Institute of Technology for its support.

References

[1] J. J. Benedetto and M. Fickus, Finite normalized tight frames, Preprint.
[2] P. G. Casazza, Mordern tools for Weyl-Heisenberg (Gabor) frame theory, *Adv. in Image. and Electron Physics*, Vol. 115(2000)1-127.
[3] P. G. Casazza and M. Leon, Existence and construction of finite tight frames, Preprint
[4] P. G. Casazza, M. Leon and J. C. Tremain, Custom built tight frames, Preprint.
[5] P. G. Casazza and J. Kovačević, Uniform tight frames with erasures, Preprint.
[6] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.* **72** (1952), 341–366.
[7] Y. Elder and G. D. Forney, Optimal tight frames and quantum measurement, Preprint.
[8] V. K. Goyal, J. Kovačević and J. A. Kelner, Quantized frame expansions with erasures, *Appl. Comput. Harmon. Anal.* **10** (2001), 203–233.
[9] V. K. Goyal, M. Vetterli and N. T. Thao, Quantized overcomplete expansions in $\mathbb{R}^N$: analysis, synthesis, and algorithms, *IEEE Trans. Inform. Theory* **44** (1998), 16–31.
[10] K. Gröchenig, *Foundation of time-frequency Analysis*, Birkhauser, Boston(2001).
[11] A. S. Householder, *The theory of matrices in numerical analysis*, Blaisdell Publishing Co. Ginn and Co. New York-Toronto-London, 1964.
[12] W. V. Parker and J. C. Eaves, *Matrices*, The Ronald Press Co., New York, 1960.
[13] D. Serre, *Matrices. Theory and applications*, Translated from the 2001 French original. Graduate Texts in Mathematics, 216. Springer-Verlag, New York, 2002.

Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, P. R. China
E-mail address: dfeng@math.tsinghua.edu.cn

Division of Science and Math., Floyd College, Rome, GA 30162
E-mail address: lwang@floyd.edu

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA.
E-mail address: wang@math.gatech.edu