THE EXISTENCE OF GABOR BASES AND FRAMES

DEGUANG HAN AND YANG WANG

ABSTRACT. For an arbitrary full rank lattice Λ in $\mathbb{R}^{2d}$ and a function $g \in L^2(\mathbb{R}^d)$ the Gabor (or Weyl-Heisenberg) system is $\mathbf{G}(\Lambda, g) := \{e^{2\pi i \langle \ell, x\rangle} g(x-\kappa) \mid (\kappa, \ \ell) \in \Lambda\}$. It is well-known that a necessary condition for $\mathbf{G}(\Lambda, g)$ to be an orthonormal basis for $L^2(\mathbb{R}^d)$ is that the density of Λ has $D(\Lambda) = 1$. However, except for symplectic lattices it remains an unsolved question whether $D(\Lambda) = 1$ is sufficient for the existence of a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis. We investigate this problem and prove that this is true for some of the important cases. In particular we show that this is true for $\Lambda = M\mathbb{Z}^d$ where M is either a block triangular matrix or any rational matrix with $|\det M| = 1$. Moreover, if M is rational we prove that there exists a compactly supported g such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis. We also obtain similar results for Gabor frames when $D(\Lambda) \geq 1$.

1. INTRODUCTION

Let Λ be a full rank lattice in $\mathbb{R}^{2d}$ and $g \in L^2(\mathbb{R}^d)$. The *Gabor (or Weyl-Heisenberg) system associated with* Λ *and* g is the following family of functions in $L^2(\mathbb{R}^d)$:

$$\mathbf{G}(\Lambda, g) := \left\{ e^{2\pi i \langle \ell, x\rangle} g(x-\kappa) \;\middle|\; (\kappa, \ \ell) \in \Lambda \right\}. \tag{1.1}$$

Gabor systems were introduced for the purpose of signal processing, and they are closely related to the representation of the Heisenberg group. In this paper we consider Gabor orthonormal basis and Gabor frames. Recall that a family of functions $\{f_j\}$ in $L^2(\mathbb{R}^d)$ is a frame if there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|f\|_2^2 \leq \sum_j |\langle f, f_j\rangle|^2 \leq C_2 \|f\|_2^2 \tag{1.2}$$

for all $f \in L^2(\mathbb{R}^d)$. The constants C_1, C_2 are called the *frame bounds* for the frame. A frame $\{f_j\}$ is called a *tight frame* if $C_1 = C_2$, and a *Parseval tight frame* if $C_1 = C_2 = 1$. Frames are natural generalizations of bases by allowing redundancies. Another generalization of orthonormal basis is *Riesz basis* $\{f_j\}$ in $L^2(\mathbb{R}^d)$, which means $\{f_j\}$ is complete in $L^2(\mathbb{R}^d)$ and there exist postive constants C_1 and C_2 such that

$$C_1 \sum_j |c_j|^2 \leq \| \sum_j c_j f_j\|^2 \leq C_2 \sum_j |c_j|^2 \tag{1.3}$$

The second author is supported in part by the National Science Foundation, grants DMS-0070586 and DMS-0139261.

for all $\{c_j\} \in l^2$. A fundamental question in the study of Gabor bases and frames is to find conditions on g and Λ such that $\mathbf{G}(\Lambda, g)$ is a basis or frame. Well known is the following theorem, often refered to as the Density Theorem for Gabor systems:

The Density Theorem. *Let Λ be a full rank lattice in $\mathbb{R}^{2d}$.*

(A) *If there exists a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is complete in $L^2(\mathbb{R}^d)$, then $D(\Lambda) \geq 1$.*
(B) *If there exists a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is a Riesz basis for $L^2(\mathbb{R}^d)$, then $D(\Lambda) = 1$.*

Part (A) of the above theorem was proved in dimension $d = 1$ for separable lattices $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ by Daubechies ([Da1], [Da2]) under the additional assumption that $\alpha\beta$ is rational. She in fact gave a constructive proof in that setting. As a corollary of a result about von Neumann algebras associated with lattices, M. Rieffle indeed proved the theorem earlier in [Rie] for any separable lattice $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ with any α and β. In higher dimensions the Density Theorem was proved by Ramanathan and Steger [RSt], who also proved that under the assumption that $\mathbf{G}(\Lambda, g)$ is a frame then $D^-(\Lambda) \geq 1$, where $D^-(.)$ denotes the lower Beurling density, even when the lattice condition on Λ is relaxed. Part(B) of the theorem was also proved in [RSt], and without the lattice condition on Λ. There have been a great deal of research related to the density of Λ in a Gabor system in various context using various different approaches, see e.g. [RSh], [CDH], [FS1], [GH1], [Gro], [Wa] and the references therein.

This paper concerns the converse of the Density Theorem. We consider the following questions on the existence of Gabor bases or frames: Let Λ be a full rank lattice in $\mathbb{R}^{2d}$. If $D(\Lambda) = 1$ can we always find a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis or a Riesz basis for $L^2(\mathbb{R}^d)$? If $D(\Lambda) \geq 1$ can we always find a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is a frame or even a tight frame for $L^2(\mathbb{R}^d)$?

The existence questions were answered for separable lattices $\Lambda = \mathcal{K} \times \mathcal{L}$ in which both $\mathcal{K}$ and $\mathcal{L}$ are full rank lattices in $\mathbb{R}^d$. For $d = 1$ the answer is trivial. With $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ we may simply choose $g = \frac{1}{\sqrt{|\alpha|}}\chi_{[0,|\alpha|)}$. Then it is an easy exercise to check that $\mathbf{G}(\Lambda, g)$ is a tight frame for $L^2(\mathbb{R})$ if $D(\Lambda) \geq 1$ and an orthonormal basis if $D(\Lambda) = 1$. The same results also hold in higher dimensions, although the problem became highly nontrivial. If $D(\Lambda) = 1$ then there exists a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. If $D(\Lambda) \geq 1$ then there exists a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is a tight frame for $L^2(\mathbb{R}^d)$, see Han and Wang [HW]. They proved the existence results by by studying the existence of domains that tile simultaneously by two different lattices.

However, for non-separable lattices Λ the existence questions remain unsolved for $d > 1$. For $d = 1$ the existence results hold, and can be proved easily by transforming the lattice Λ into a separable lattice using symplectic matrices, see Gröchenig [Gro]. We shall give a short review of the subject in Section 3. Key to this result is the fact that every lattice in $\mathbb{R}^2$ is symplectic. This is no longer true for $\mathbb{R}^{2d}$ with $d > 1$. While the existence results hold for symplectic lattices in any dimension, they appear to be difficult questions for non-symplectic lattices. In this paper we prove the existence results for a large class of non-symplectic lattices. A useful fact is that the existence of bases and frames are all equivalent to the existence of complete Gabor systems.

Theorem 1.1. *Let Λ be a full rank lattice in $\mathbb{R}^{2d}$ with $D(\Lambda) = 1$. The following are equivalent:*

(A) *There exists a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is complete in $L^2(\mathbb{R}^d)$.*
(B) *There exists a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is a Riesz basis for $L^2(\mathbb{R}^d)$.*
(C) *There exists a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$.*

A similar result holds for the existence of frames.

Theorem 1.2. *Let Λ be a full rank lattice in $\mathbb{R}^{2d}$ with $D(\Lambda) \geq 1$. The following are equivalent:*

(A) *There exists a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is complete in $L^2(\mathbb{R}^d)$.*
(B) *There exists a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is a frame for $L^2(\mathbb{R}^d)$.*
(C) *There exists a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is a tight frame for $L^2(\mathbb{R}^d)$.*

Note that any full rank lattice Λ in $\mathbb{R}^{2d}$ can be written as $\Lambda = R\mathbb{Z}^{2d}$ where R is a non-singular $2d \times 2d$ matrix. The density of Λ is the $D(\Lambda) = |\det(R)|^{-1}$. We shall call $|\det(R)|$ the *volume* of Λ and denote it by $\mathrm{v}(\Lambda)$. We prove the existence of bases and frames for certain non-symplectic lattices:

Theorem 1.3. *Let Λ be a full rank lattice in $\mathbb{R}^{2d}$, $\Lambda = R\mathbb{Z}^{2d}$. Suppose that $R = TM$ where T is symplectic and $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with either $B = 0$ or $C = 0$, where A, B, C, D are all $d \times d$ matrices. Then*

(A) *There exists a $g(x) \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ if and only if $D(\Lambda) = 1$.*
(B) *There exists a $g(x) \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is a tight frame of $L^2(\mathbb{R}^d)$ if and only if $D(\Lambda) \geq 1$.*

A particularly interesting case is when Λ is rational, i.e. $\Lambda = R\mathbb{Z}^{2d}$ and R has rational entries. We obtain a stronger conclusion with the following theorem:

Theorem 1.4. *Let that $\Lambda = R\mathbb{Z}^{2d}$ for a rational matrix R. If $D(\Lambda) = 1$ then there exists a compactly supported $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. If $D(\Lambda) \geq 1$ then there exists a compactly supported $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is a tight frame for $L^2(\mathbb{R}^d)$.*

We point out that the existence of a compactly supported $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is a basis (frame) is not solved even for separable lattices for $d > 1$, or for non-separable lattices in dimension $d = 1$. We conjecture that for some lattices Λ with $D(\Lambda) = 1$ there exists no compactly supported g such that $\mathbf{G}(\Lambda, g)$ is a basis even in dimension $d = 1$. In particular, we conjecture that for $\Lambda = R\mathbb{Z}^2$ with $R = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ and $b \notin \mathbb{Q}$ there exists no compactly supported g such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis for $L^2(\mathbb{R})$.

The rest of this paper is organized as follows: In Section 2 we briefly review our lattice tiling results and discuss how they can be used to the separable lattice case. These results are also needed in Section 4. Section 3 is a short review on symplectic lattices and the Stone-von Neumann theorem. Both sections are provided for the convenience of the readers. Section 4 is devoted to proving Theorems 1.1-1.4.

The authors wish to thank J. P. Gabardo and Chris Heil for helpful comments.

2. Separable Lattices

This section is devoted to reviewing some of the results in Han and Wang [HW] on the existence of Gabor bases and frames for separable lattices. We shall be using these results to prove the new results stated in Section 1. Let Λ be a full rank separable lattice, $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$. Motivated by the one dimensional case, one simple idea to prove the existence of a Gabor basis $\mathbf{G}(\Lambda, g)$ is to test whether g can be chosen as a characteristic function. This naturally leads to a tiling problem for two different lattices. To state the problem and the results we need to recall some notations.

Let Ω be a measurable set in $\mathbb{R}^d$ (not necessarily bounded), and let $\mathcal{L}$ be a full rank lattice in $\mathbb{R}^d$. We say Ω *tiles* $\mathbb{R}^d$ by $\mathcal{L}$, or Ω is a *fundamental domain* of $\mathcal{L}$, if

(i) $\bigcup_{\ell\in\mathcal{L}}(\Omega + \ell) = \mathbb{R}^d$ a.e.;
(ii) $(\Omega + \ell) \cap (\Omega + \ell')$ has Lebesgue measure 0 for any $\ell \neq \ell'$ in $\mathcal{L}$.

We say that Ω *packs* $\mathbb{R}^d$ by $\mathcal{L}$ if only (ii) holds. Equivalently, Ω tiles $\mathbb{R}^d$ by $\mathcal{L}$ if and only if

$$\sum_{\ell\in\mathcal{L}} \chi_\Omega(x - \ell) = 1 \text{ for a.e. } x \in \mathbb{R}^d, \tag{2.1}$$

and Ω packs $\mathbb{R}^d$ by $\mathcal{L}$ if and only if

$$\sum_{\ell\in\mathcal{L}} \chi_\Omega(x - \ell) \leq 1 \text{ for a.e. } x \in \mathbb{R}^d. \tag{2.2}$$

Clearly, $\mu(\Omega) = \mathrm{v}(\mathcal{L})$ if Ω tiles by $\mathcal{L}$, and $\mu(\Omega) \leq \mathrm{v}(\mathcal{L})$ if Ω packs by $\mathcal{L}$. Furthermore, if Ω packs $\mathbb{R}^d$ by $\mathcal{L}$ and $\mu(\Omega) = \mathrm{v}(\mathcal{L})$, then Ω necessarily tiles $\mathbb{R}^d$ by $\mathcal{L}$.

Now let $\Lambda = \mathcal{K} \times \mathcal{L}$ be a full rank lattice in $\mathbb{R}^{2d}$ with $\mathrm{v}(\Lambda) = \mathrm{v}(\mathcal{K}) \cdot \mathrm{v}(\mathcal{L}) = 1$. Write $\mathcal{K} = A\mathbb{Z}^d$ and $\mathcal{L} = B\mathbb{Z}^d$. Then $|\det A| = |\det(B^T)^{-1}|$. It not hard to check that the question of whether there exists a characteristic function which generates a orthogonal Gabor basis is equivalent to the question of whether the two lattices $A\mathbb{Z}^d$ and $(B^T)^{-1}\mathbb{Z}^d$ have a common fundamental domain. Similarly, if $\mathrm{v}(\Lambda) \leq 1$, then the question of whether there exists a characteristic function which generates a tight Gabor frame is equivalent to the question of whether there exists a measurable set Ω such that it tiles $\mathbb{R}^d$ by the lattice $A\mathbb{Z}^d$ and packs by the lattice $(B^T)^{-1}\mathbb{Z}^d$. The tiling question has been studied in the area of tiling theory for many years (cf. [Ko]) and is closely related to a well-known open problem of Steinhaus' which asks whether there exists a common fundamental domain for all the lattices $R_\theta\mathbb{Z}^2$, where R_θ is the rotation matrix by angle θ. The following result was proved in Han and Wang [HW]:

Theorem 2.1 ([HW])**.** *Let $\mathcal{L}, \mathcal{K}$ be two full rank lattices in $\mathbb{R}^d$ such that $\mathrm{v}(\mathcal{L}) \geq \mathrm{v}(\mathcal{K})$. Then there exists a measurable set Ω in $\mathbb{R}^d$ such that Ω tiles $\mathbb{R}^d$ by $\mathcal{K}$ and packs $\mathbb{R}^d$ by $\mathcal{L}$. In particular, when $\mathrm{v}(\mathcal{L}) = \mathrm{v}(\mathcal{K})$, then there exists a measurable set Ω in $\mathbb{R}^d$ such that Ω tiles $\mathbb{R}^d$ by both $\mathcal{K}$ and $\mathcal{L}$.*

If in addition both $\mathcal{K}$ and $\mathcal{L}$ are rational then Ω can be chosen to be compact, see [HW], Corollary 2.4. This yields the following:

Theorem 2.2 ([HW])**.** *Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a full rank lattice in $\mathbb{R}^{2d}$ with $\mathrm{v}(\Lambda) \leq 1$. Suppose that A and B are in $\mathcal{M}_n(\mathbb{Q})$. Then there exists a bounded set Ω such that $\mathbf{G}(\Lambda, g)$ is a tight Gabor frame for $L^2(\mathbb{R}^d)$ when $g = \chi_\Omega$.*

These two results yield the following existence result for Gabor bases and frames:

Theorem 2.3 ([HW])**.** *Let $\mathcal{L}$, $\mathcal{K}$ be two full rank lattices in $\mathbb{R}^d$ and let $\Lambda = \mathcal{K} \times \mathcal{L}$.*

(i) *If $\mathrm{v}(\mathcal{L})\mathrm{v}(\mathcal{K}) = 1$ then there exists a $g(x) \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$.*
(ii) *If $\mathrm{v}(\mathcal{L})\mathrm{v}(\mathcal{K}) \leq 1$ then there exists a $g(x) \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is a tight frame for $L^2(\mathbb{R}^d)$.*

3. Symplectic Lattices

This section is devoted to a short review of symplectic lattices and the Stone-von Neumann Theorem. A matrix $M \in \mathcal{M}_{2d}(\mathbb{R})$ is *symplectic* if $M^T J M = J$ where $J = \begin{bmatrix} 0 & -I_d \\ I_d & 0 \end{bmatrix}$. If we write $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ then M is symplectic if and only if $AC^T = A^T C$, $BD^T = B^T D$ and $A^T D - C^T B = I_d$. We shall use $Sp(d, \mathbb{R})$ to denote the set of all symplectic matrices in $\mathcal{M}_{2d}(\mathbb{R})$. A lattice Λ in $\mathbb{R}^{2d}$ is *symplectic* if $\Lambda = \alpha M \mathbb{Z}^{2d}$ for some $\alpha \in \mathbb{R} \setminus \{0\}$ and $R \in Sp(d, \mathbb{R})$.

Symplectic matrices arise from the study of the Heisenberg group $\mathbb{H}$ and the Schrödinger representation on $\mathbb{H}$. The Hesenberg group is $\mathbb{H} = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ with the group multiplication given by

$$(x, y, \tau)(u, v, \eta) = (x + u, y + v, \tau\eta e^{i\pi(\langle x,v\rangle - \langle y,u\rangle)})$$

for $x, y \in \mathbb{R}^d$ and $\tau, \eta \in \mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. The Schrödinger (irreducible) unitary representation π of $\mathbb{H}$ on $L^2(\mathbb{R})$ is given by

$$\pi(x, y, \tau) f(t) = \tau e^{\pi i x y} e^{2\pi i y t} f(t + x), \qquad f \in L^2(\mathbb{R}).$$

Stone-von Neumann Theorem. *Let ρ be any irreducible unitary representation of $\mathbb{H}$ on a Hilbert space $\mathcal{H}$ such that*

$$\rho(0, 0, \tau) h = \tau h, \qquad \tau \in \mathbb{T}, \quad h \in \mathcal{H}.$$

Then ρ is unitarily equivalent to the Schrödinger representation π, i.e., there is a unitary operator $U : \mathcal{H} \to L^2(\mathbb{R}^d)$ such that

$$\rho(x, y, \tau) = U^* \pi(x, y, \tau) U, \qquad (x, y, \tau) \in \mathbb{H}.$$

Note that any $M \in Sp(d, \mathbb{R})$ preserves the symplectic form $[Mz, Mw] = [z, w]$ for all $z, w \in \mathbb{R}^{2d}$, where $[z, w] = \langle x, v\rangle - \langle u, y\rangle$ with $z = [x, y]^T$ and $w = [u, v]^T$. Hence $\rho(x, y, \tau) = \pi(M[x, y]^T, \tau)$ defines an irreducible unitary representation for $\mathbb{H}$ and satisfies the condition

$$\rho(0, 0, \tau)h = \pi(0, 0, \tau)h = \tau h, \qquad h \in L^2(\mathbb{R}^d),\ \tau \in \mathbb{T}.$$

It follows from the Stone-von Neumann theorem that there exists a unitary operator, say $\sigma(M)$, on $L^2(\mathbb{R}^d)$ such that

$$\pi(M[x, y]^T, \tau) = \sigma(M)\rho(x, y, \tau)\sigma(M)^{-1}, \qquad x,\ y \in \mathbb{R}^d,\ \tau \in \mathbb{T}.$$

This leads to the following:

Lemma 3.1. *Let Λ be a full rank lattice in $\mathbb{R}^{2d}$ and let $M \in Sp(d, \mathbb{R})$. Then there is a function $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis (resp. Riesz basis, frame, tight frame) for $L^2(\mathbb{R}^d)$ if and only if there is a function $h \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(M\Lambda, h)$ is an orthonormal basis (resp. Riesz basis, frame, tight frame) for $L^2(\mathbb{R}^d)$.*

In general it is hard to find σ. However, for certain M, $\sigma(M)$ are familiar operators. For example, if $M = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ then $\sigma(M) = i^{1/2}\mathcal{F}^{-1}$, where $\mathcal{F}$ denotes the Fourier transform; if $M = \begin{bmatrix} I & 0 \\ C & I \end{bmatrix}$ with $C = C^T$ then $(\sigma(M)f)(t) = e^{-\pi i\langle t, Ct\rangle} f(t)$. For more details about Stone-von Neumann Theorem and representations of $\mathbb{H}$ we refer to [Fol] or [Gro].

For any 2×2 real matrix M, it is obvious that $M \in Sp(1, \mathbb{R})$ if and only if $\det(M) = 1$. Now any full rank lattice Λ in $\mathbb{R}^2$ can be expressed as $\Lambda = M(\alpha\mathbb{Z}^2)$ with $\alpha \in \mathbb{R}$ and $\det(M) = 1$. Since the existence results hold for the lattice $\alpha\mathbb{Z}^2$, Lemma 3.1 now yields the following theorem, see [Gro]:

Theorem 3.2. *Let Λ be a full rank symplectic lattice in $\mathbb{R}^2$.*

- (A) *Suppose that $D(\Lambda) = 1$ then there exists a $g \in L^2(\mathbb{R})$ such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$.*
- (B) *Suppose that $D(\Lambda) \geq 1$ then there exists a $g \in L^2(\mathbb{R})$ such that $\mathbf{G}(\Lambda, g)$ is a tight frame for $L^2(\mathbb{R}^d)$.*

The factorization of a matrix into a product of a scalar and a symplectic matrix can no longer be done for all matrices in higher dimensions. This is the main obstacle in proving the existence of Gabor bases and frames for a given lattice Λ.

4. Proof of Theorems

We first prove Theorem 1.3, which we divide up as two lemmas. Note that the phase τ in $\pi(x, y, \tau) \in \mathbb{H}_d$ does not affect the basis or frame property of a Gabor system, we will simply let $\tau = 1$ and denote $\pi(x, y, 1)$ by $\pi(x, y)$.

Lemma 4.1. *Let Λ be a full rank lattice in $\mathbb{R}^{2d}$ such that $\Lambda = M\mathbb{Z}^{2d}$ with*

$$M = \begin{bmatrix} A & 0 \\ B & D \end{bmatrix}.$$

If $\mathrm{v}(\Lambda) = 1$ (resp. $\mathrm{v}(\Lambda) \leq 1$), then there exists a function $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis (resp. tight frame) for $L^2(\mathbb{R}^d)$. Moreover g can be chosen such that $|g(t)|$ is the scalar multiple of a characteristic function.

Proof. We will apply a matrix T (which is not necessarily a symplectic matrix, and hence the Stone-von Neumann Theorem does not apply here) to Λ such that $T\Lambda$ is a separable lattice. We then apply Theorem 2.2 to obtain g. Let $C = -BA^{-1}$ and

$$T = \begin{bmatrix} I & 0 \\ C & I \end{bmatrix}.$$

Then

$$TM = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}.$$

A simple calculation shows that for any $x, y \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$,

$$\begin{aligned} \pi(T[x,y]^T)f(t) &= \pi(x, Cx + y)f(t) \\ &= e^{2\pi i\langle Cx+y,t\rangle + \pi i\langle x,Cx+y\rangle} f(t+x) \\ &= e^{\pi i\langle x,(C^T-C)t\rangle} e^{-\pi i\langle t,Ct\rangle} e^{2\pi\langle y,t\rangle+\pi\langle x,y\rangle} e^{\pi i\langle t+x,C(t+x)\rangle} f(t+x) \\ &= e^{\pi i\langle x,(C^T-C)t\rangle} U\pi(x,y)U^{-1}f(t), \end{aligned}$$

where $(Uf)(t) = e^{-\pi i\langle t,Ct\rangle} f(t)$.

We first consider the case $D(\Lambda) \geq 1$, which is equivalent to $\mathrm{v}(\Lambda) = |\det(AD)| \leq 1$. Hence $|\det A| \leq |\det(D^T)^{-1}|$. By Theorem 2.1 there exists a measurable set Ω in $\mathbb{R}^d$ such that Ω tiles $\mathbb{R}^d$ by $A\mathbb{Z}^d$ and packs $\mathbb{R}^d$ by $(D^T)^{-1}\mathbb{Z}^d$. Thus for any $f \in L^2(\mathbb{R}^d)$ we have

$$\|f\|^2 = \sum_{m\in\mathbb{Z}^d} \|P_m f\|^2$$

and

$$\|P_m f\|^2 = \sum_{n\in\mathbb{Z}^d} |\langle f, \pi(Am, Dn)h(t)\rangle|^2$$

where $h(t) = \frac{1}{\sqrt{|\det A|}}\chi_\Omega$ and P_m is the orthogonal projection onto $L^2(\Omega + Am)$. Now we define $g(t) = U^{-1}h(t)$. We claim that $\{\pi(x, y)g : (x, y) \in \Lambda\}$ is a tight frame for $L^2(\mathbb{R}^d)$

(and hence $\mathbf{G}(\Lambda, g)$ is a tight frame). Indeed, for any $f \in L^2(\mathbb{R}^d)$ we have

$$\begin{aligned}
\sum_{(x,y)\in\Lambda} |\langle f, \pi(x,y)g\rangle|^2 &= \sum_{(x,y)\in\Lambda} |\langle f, e^{\pi i\langle x,(C-C^T)t\rangle} U^{-1}\pi(T(x,y)^T)Ug\rangle|^2 \\
&= \sum_{(x,y)\in\Lambda} |\langle e^{-\pi i\langle x,(C-C^T)t\rangle} Uf, \pi(T(x,y)^T)h\rangle|^2 \\
&= \sum_{(m,n)\in\mathbb{Z}^d\times\mathbb{Z}^d} |\langle e^{-\pi i\langle Am,(C-C^T)t\rangle} Uf, \pi(Am, \tilde{D}n)^T)h\rangle|^2 \\
&= \sum_{(x,y)\in\Lambda} |\langle e^{-\pi i\langle x,(C-C^T)t\rangle} Uf, \pi(T(x,y)^T)h\rangle|^2 \\
&= \sum_{m\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} |\langle e^{-\pi i\langle Am,(C-C^T)t\rangle} Uf, \pi(Am, \tilde{D}n)^T)h\rangle|^2 \\
&= \sum_{m\in\mathbb{Z}^d} \|P_m e^{-\pi i\langle Am,(C-C^T)t\rangle} Uf\|^2 \\
&= \sum_{m\in\mathbb{Z}^d} \|P_m f\|^2 = \|f\|^2.
\end{aligned}$$

Thus $\{\pi(x,y)g : (x,y) \in \Lambda\}$ is a tight frame as claimed. In fact it is a Parseval tight frame. This shows that $\mathbf{G}(\Lambda, g)$ is a tight frame for $L^2(\mathbb{R}^d)$.

In the case of $D(\Lambda) = 1$ we follow the exact same procedure. However in this case Ω tiles by both $A\mathbb{Z}^d$ and $(D^T)^{-1}\mathbb{Z}^d$. This yields a Gabor orthonormal basis $\mathbf{G}(\Lambda, g)$ for $L^2(\mathbb{R}^d)$, see also [HW].

We remark that in both cases $|g| = \frac{1}{\sqrt{|\det(A)|}\chi_\Omega}$. ∎

Lemma 4.2. *Let Λ be a full rank lattice in $\mathbb{R}^{2d}$ such that $\Lambda = M\mathbb{Z}^{2d}$ with*

$$M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}.$$

If $\mathrm{v}(\Lambda) = 1$ (resp. $\mathrm{v}(\Lambda) \le 1$), then there exists a function $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis (resp. tight frame) for $L^2(\mathbb{R}^d)$. Moreover g can be chosen such that $|\widehat{g}(t)|$ is the scalar multiple of a characteristic function.

Proof. Note that the Fourier transform of $g_{\kappa,\ell} := e^{2\pi\langle \ell,x\rangle} g(x-\kappa)$ is $e^{2\pi\langle \ell,\kappa\rangle}\widehat{g}_{\ell,-\kappa}$. Hence $\mathbf{G}(\Lambda, g)$ is an orthonormal basis (resp. tight frame) if and only if $\mathbf{G}(\tilde{\Lambda}, \widehat{g})$ is, where $\tilde{\Lambda} = \tilde{M}\mathbb{Z}^{2d}$ with

$$\tilde{M} = \begin{bmatrix} D & 0 \\ -B & -A \end{bmatrix}.$$

The lemma now follows from Lemma 4.1. ∎

Proof of Theorem 1.3. Let $\Lambda = R\mathbb{Z}^{2d}$ with $R = TM$ where T is symplectic and M is block triangular. Let $\Gamma = M\mathbb{Z}^{2d}$. We have already established the existence of an orthonormal

Gabor basis (resp. tight Gabor frame) if $D(\Gamma) = 1$) (resp. $D(\Gamma) \geq 1$). The theorem now follows directly from Lemma 3.1. ∎

Corollary 4.3. *Let Λ be a full rank lattice in $\mathbb{R}^{2d}$ such that $\Lambda = M\mathbb{Z}^{2d}$ with*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

and assume that $\mathrm{v}(\Lambda) = 1$ (resp. $\mathrm{v}(\Lambda) \leq 1$). If either CA^{-1} or BD^{-1} is symmetric, then there exists a function $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis (resp. tight frame) for $L^2(\mathbb{R}^d)$.

Proof. If CA^{-1} is symmetric, we define $T = \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}$; whereas if BD^{-1} is symmetric we let $T = \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix}$. The matrix T is in $Sp(d, \mathbb{R})$. Furthermore let $\Gamma = TM\mathbb{Z}^{2d}$ then there exists an $h \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Gamma, h)$ is an orthonormal basis (resp. a tight frame) because TM is block triangular and $\mathrm{v}(\Gamma) = \mathrm{v}(\Lambda) = 1$ (resp. $\mathrm{v}(\Gamma) = \mathrm{v}(\Lambda) \leq 1$). Since $\Lambda = T^{-1}\Gamma$ and T^{-1} is symplectic, the corollary now follows directly from Lemma 3.1. ∎

We now prove Theorem 1.4. To do so we need to establish the following lemma.

Lemma 4.4. *Let $M \in \mathcal{M}_n(\mathbb{Z})$. Then there exists a unmodular $P \in \mathcal{M}_n(\mathbb{Z})$ such that MP is lower triangular.*

Proof. Let $[a_{11}, a_{12}, \cdots, a_{1n}]$ be the first row of M with $\gcd(a_{1j}) = d_1$. It follows from elementary number theory that

$$x_1 a_{11} + x_2 a_{12} + \cdots + x_n a_{1n} = d_1$$

for some integers $x_1, x_2, \ldots, x_n$ with $\gcd(x_j) = 1$. Let P_1 be the unimodular integer matrix with $[x_1, x_2, \cdots, x_n]^T$ as its first column. Such a matrix is well known to exist, see Newman [Ne], Theorem II.1, page 13. Observe that

$$M_1 := MP_1 = \begin{bmatrix} d_1 & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}.$$

Since each b_{1j} is a linear combination of a_{1j}'s, $d_1 \,|\, b_{1j}$ for $j = 2, ..., n$. Therefore we may use column Gaussian elimination to reduce M_1 to

$$M_2 = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}.$$

Note that $M_2 = M_1 P_2$ with a unimodular $P_2 \in \mathcal{M}_n(\mathbb{Z})$. Now The lemma is proved by induction on n. ∎

Proof of Theorem 1.4. We have $\Lambda = R\mathbb{Z}^{2d}$ with $R \in \mathcal{M}_{2d}(\mathbb{Q})$. Write $R = \frac{1}{q}\tilde{R}$ such that $\tilde{R} \in \mathcal{M}_{2d}(\mathbb{Z})$ and $q \in \mathbb{Z}$. It follows from Lemma 4.4 that $\tilde{R} = \tilde{T}P$ where $\tilde{T} \in \mathcal{M}_{2d}(\mathbb{Z})$ is a lower triangular integral matrix and P is unimodular integral matrix. Thus $\tilde{R}\mathbb{Z}^{2d} = \tilde{T}P\mathbb{Z}^{2d} = \tilde{T}\mathbb{Z}^{2d}$. Set $T = \frac{1}{q}\tilde{T}$ and thus $\Lambda = T\mathbb{Z}^{2d}$.

Now T is lower triangular, and write $T = \begin{bmatrix} A & 0 \\ B & D \end{bmatrix}$. Therefore in the case $D(\Lambda) = 1$ it follows from Theorem 1.3 and its proof that there exists a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is an orthonormal basis, with $|g| = \frac{1}{\sqrt{|\det(A)|}}\chi_\Omega$ for any domain Ω that tiles by both $A\mathbb{Z}^d$ and $(D^T)^{-1}\mathbb{Z}^d$. Since both A and D are rational, we may choose a bounded domain Ω ([HW], Corollary 2.4). Hence we may choose a compactly supported g. In the case $D(\Lambda) \geq 1$ the same argument shows that we may choose a g with $|g| = \frac{1}{\sqrt{|\det(A)|}}\chi_\Omega$, where Ω can be any domain that tiles by $A\mathbb{Z}^d$ and packs by $(D^T)^{-1}\mathbb{Z}^d$. Again a bounded such Ω exists since both A and D are rational ([HW]). The theorem is now proved. ∎

We conclude the paper by proving Theorems 1.1 and 1.2.

Proof of Theorem 1.1. To Be Filled. ∎

Proof of Theorem 1.2. To Be Filled. ∎

References

[CDH] O. Christensen, B. Deng and C. Heil, Density of Gabor frames, *Appl. Comput. Harm. Anal* **7**(1999), 292–304.

[Da1] I. Daubechies, The wavelet transformation, time-frequency localization and s ignal processing, *IEEE Trans. Inform. Theory*, **36** (1990), 961-1005.

[Da2] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF regional conference series in Applied Math., Philadelphia (1992)

[DLL] I. Daubechies, H. Landau and Z. Landau, Gabor time frequency lattices and the Wexler-Raz identity *J. Fourier. Anal. Appl.* **1** (1995), 437-478.

[DS] R. Duffin and A.Shaeffer, A class of non-harmonic Fourier series, *Trans. Amer. Math. Soc.* 72 (1952) 341-366

[FS1] H. G. Feichtinger and T. Strohmer (eds), *Gabor Analysis and Algorithms: Theory and Applications,* Applied and Numerical Harmonic Analysis, Birkhuser, 1998.

[FS2] H. G. Feichtinger and T. Strohmer (eds), *Advances in Gabor Analysis,* Applied and Numerical Harmonic Analysis, Birkhuser, 2002.

[Fol] G. B. Folland, *Harmonic Analysis in Phase Space,* Annals of Math. Studies, Princeton Univ. Press, Princeton (NJ), 1989.

[GH1] J. P. Gabardo and D. Han, Frame representations for group-like unitary operator systems, *J. Operator Theory*, to appear.

[GH2] J. P. Gabardo and D. Han, Subspace Weyl-Heisenberg frames, *J. Fourier Analysis and Appl.*, **7** (2001), 419–433.

[GH3] J. P. Gabardo and D. Han, Aspects of Gabor analysis and operator algebras, *Advances in Gabor Analysis,* Eds: H. G. Feichtinger and T. Strohmer, Birkhuser, 2002, 129-150.

[Ga] D. Gabor, Theory of Communication, *J. Inst. Elec. Eng.* (London) **93** (1946), 429–457.

[Gro] K. H. Gröchenig, *Foundations of Time-Frequency Analysis,* Applied and Numerical Harmonic Analysis, Birkhuser, 2001.

[HL] D. Han and D. Larson, Frames, bases and group parametrizations, *Memoirs Amer. Math. Soc.,* **697** (2000).

[HW] D. Han and Y. Wang, Lattice tiling and Gabor frames, *Geometric and Functional Analysis,* 11(2001), 742–758.

[Ko] M. Kolountzakis, Multi-lattice tiles, *Internat. Math. Res. Notices* **19** (1997), 937-952.

[Ne] M. Newman, *Integral Matrices*, Academic Press, 1972.

[Rie] M. A. Rieffel, von Neumann algebras associated with pairs of lattices in Lie groups, *Math. Ann.* **257**(1981), 403–413.

[RSh] A. Ron and Z. Shen, Weyl-Heisenberg frames and Riesz bases in $L^2(\mathbb{R}^d)$, *Duke Mathematical Journal* **89** (1997), 237–282.

[RSt] J. Ramanathan and T. Steger, Incompleteness of sparse coherent states, *Appl. Comp. Harm. Anal.* **2** (1995), 148–153.

[Wa] Y. Wang, Sparse complete Gabor systems on a lattice, preprint.

Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA.

E-mail address: `wang@math.gatech.edu`