Exploring Lag Diversity in the High-Order Ambiguity Function for Polynomial Phase Signals*

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SUMMARY

Polynomial phase signals (PPS) have been studied extensively recently as a means of modeling certain nonstationary processes. Applications have appeared e.g., in SAR imaging of moving targets [3] and time-varying motion estimation [1].

A single component Mth-order PPS is modeled in discrete-time as

$$y(t) = A\ e^{j\sum_{m=0}^{M} a_m t^m}, \qquad t = 0, 1, \ldots, T-1. \tag{1}$$

Define $y^{(*q)}(t) = y(t)$ for q even, and $y^{(*q)}(t) = y^*(t)$ for q odd. Peleg and Porat introduced the high-order instantaneous moment (see e.g., [2, Ch. 12]),

$$\mathcal{P}_M[y(t);\tau] \stackrel{\triangle}{=} \prod_{q=0}^{M-1} [y^{(*q)}(t-q\tau)]^{\binom{M-1}{q}}, \tag{2}$$

which reduces the Mth-order PPS in (1) to a single harmonic at frequency $\tilde{\omega} = M!\ a_M \tau^M$. The generalized FS coefficient of (2), $P_M[y;\alpha,\tau] \stackrel{\triangle}{=} \lim_{T\to\infty} T^{-1} \sum_{t=0}^{T-1} \mathcal{P}_M[y(t);\tau]\ e^{-j\alpha t}$, is called the high-order ambiguity function (HAF). It peaks at $\alpha = \tilde{\omega}$ for the signal in (1). To avoid the aliasing effect, one has to limit $\tilde{\omega}$ to within $[-\pi,\pi)$ which means that the range of a_M is drastically reduced if a large τ is used.

In practice, we observe $x(t) = y(t) + v(t)$, where $v(t)$ is the zero-mean additive noise. We estimate a_M by replacing $y(t)$ by $x(t)$ in the HAF expression and then peak-picking $|P_M[x;\alpha,\tau]|$. Once a_M is found, we demodulate $x(t)$ by $\exp\{-ja_M t^M\}$ to reduce the order of the PPS by one, and repeat the procedure to find all remaining polynomial phase coefficients successively.

Peleg has shown that the optimal lag, in the sense of minimizing the asymptotic variance of the $\hat{a}_M$ estimate, is $\tau = T/M$ for $M = 2, 3$ and $\tau = T/(M+2)$ for $4 \le M \le 10$ [2, Ch. 12].

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We therefore reach a paradox here: τ's as large as T/M benefit estimation accuracy but are impractical to use due to the severe limitation on the range of a_M.

We propose to use two co-prime lags τ_1 and τ_2. We claim that as long as $|M!\ a_M| < \pi$, we can identify a_M uniquely based on two HAFs $P_M[x;\alpha,\tau_1]$ and $P_M[x;\alpha,\tau_2]$. Note that our identifiability condition does not explicitly depend on τ, and is a *much less severe* condition than the original one with a single τ. The HAF will peak at (likely to be aliased) locations $[M!\ a_M\tau_1^M \bmod (2\pi)]$ and $[M!\ a_M\tau_2^M \bmod (2\pi)]$, from which an Euclidean algorithm is used to recover a_M. The advantage of this two-lag approach is that estimation accuracy is warranted without stringent limitation on the parameters. Moreover, the extra lag provides additional averaging effect and therefore further improves accuracy.

Example 1: We have $T = 512$ samples of $x(t) = \exp(ja_2t^2) + v(t)$, where $a_2 = 0.1$, and $v(t)$ is the zero-mean i.i.d. complex Gaussian noise with variance 0.5. For a given lag τ, we form $P_2[x;\alpha,\tau]$ by simply taking the normalized DFT of the product process $x^*(t-\tau)x(t)$. Method 1 uses a single lag $\tau = 1$ and the corresponding $2a_2\tau = 0.2$ is free from the aliasing effect. The bias and standard deviation (std) of the $\widehat{a}_2$ estimates calculated from 100 independent realizations were 4.5219×10^{-6} and 5.7433×10^{-5}, respectively. Method 2 uses two lags $\tau_1 = T/2 = 256$ and $\tau_2 = 201$ and aliasing occurs for both lags. However, we can reconstruct a_2 based on these "dislocated" peaks using the Euclidean algorithm. The bias and std from the same 100 independent realizations were -1.1679×10^{-7} and 1.5475×10^{-6}, respectively. With the two-lag approach, we have gained an order of magnitude improvement in both the bias and the std here.

With a minimum of two (co-prime) lags, we can estimate a_M uniquely as long as $|a_M| < \pi/M!$. Additional lags are not necessary but may help improve the accuracy even further.

An more important application of the two-lag approach is encountered when we have multi-component PPS with similar leading coefficients. Consider as an example,

$$x(t) = A_1\ e^{j(a_{1,2}t^2 + a_{1,1}t + a_{1,0})} + A_2\ e^{j(a_{2,2}t^2 + a_{2,1}t + a_{2,0})} + v(t),$$

with $a_{1,2} \approx a_{2,2}$. The HAF of order 2, $P_2[x;\alpha,\tau]$, should show peaks at $2a_{1,2}\tau$ and $2a_{2,2}\tau$ (cross terms do not contribute much, see [4]). But if τ is small, the peaks may not be resolved, especially when the data length is small (recall that the fundamental resolution of DFT based algorithms is limited by $2\pi/T$). If we observe a single peak, we may conclude erroneously that a single chirp (in the context of SAR, only one target) is present.

For these applications, we propose to use large τ's to *magnify* the difference among the polynomial phase coefficients. By using two co-prime τ's, we will be able to conquer the wrap-around effect, improve the estimation accuracy, and correctly determine the number of components.

Example 2: We have here $T = 512$ points of

$$x(t) = 1.2\ e^{j(0.4t^2 - 0.3t)} + e^{j(0.41t^2 - t)} + v(t),$$

where $v(t)$ is zero-mean i.i.d. complex Gaussian with variance $\sigma_v^2 = 0.5$. Figure 1 illustrates the magnitude of $P_2[x;\alpha,\tau]$ for different τ's and Table 1 shows the corresponding peak locations. A small lag such as $\tau = 1$ reveals a single peak, but larger τ's have the capability of splitting the peak into two indicating that there are actually two chirps. By solving $[2a_{l,2}\tau_1 \bmod (2\pi)]$ and $[2a_{l,2}\tau_2 \bmod (2\pi)]$ jointly, we can recover the true chirp coefficients $a_{l,2}$ from the aliased positions.

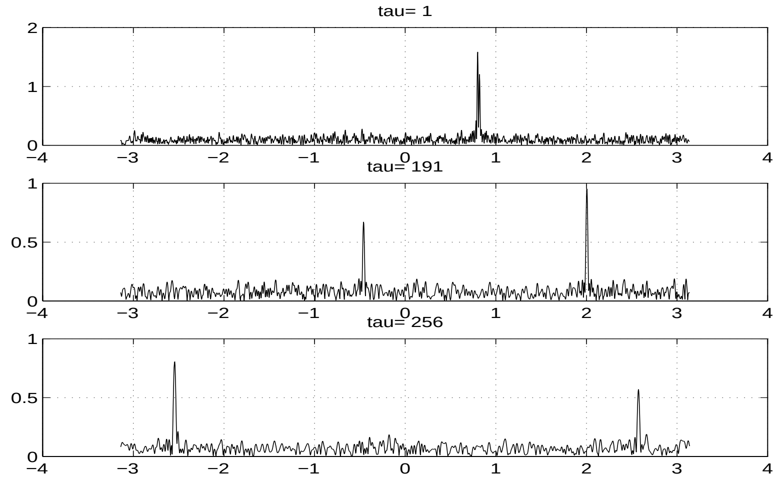

Figure 1: The ambiguity function (HAF of order 2) of a two-component chirp for various τ's.

$a_{1,2} = 0.40$ $a_{2,2} = 0.41$	Peak locations		
	$\tau_1 = 1$	$\tau_2 = 191$	$\tau_3 = 256$
$2a_{1,2}\tau_i \bmod (2\pi)$	0.8000	2.0036	-2.5451
$2a_{2,2}\tau_i \bmod (2\pi)$	0.8200	-0.4596	2.5749

Table 1: Peak locations of Figure 1.

Other advantages of lag diversification will be discussed in the full paper. Additional simulations will also be shown.

References

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