Haar–Type Orthonormal Wavelet Bases In R^2

Jeffrey C. Lagarias
AT&T Bell Laboratories
600 Mountain Avenue
Murray Hill, NJ 07974

Yang Wang*
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332

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Abstract

K.-H. Gröchenig and A. Haas asked whether for every expanding integer matrix $A \in \mathbf{M}_n(\mathbf{Z})$ there is a Haar-type orthonormal wavelet basis having dilation factor A and translation lattice $\mathbf{Z}^n$. They proved that this is the case when the dimension $n = 1$. This paper shows that this is also the case when the dimension $n = 2$.

Keywords: Haar–type wavelet basis, self–affine tile, multiresolution analysis, primitive digit set, $\mathbf{Z}^n$–tiling.

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1 Introduction

The Haar basis of $L^2(\mathbf{R})$ constructed in 1910 by A. Haar [10] consists of step functions on dyadic intervals. This construction has recently been viewed as the simplest example of a compactly supported wavelet basis of $L^2(\mathbf{R})$ constructed by a multiresolution analysis, see Mallat [17].

Gröchenig and Madych [9] raised the question of constructing multidimensional analogues of the Haar basis. They defined a *Haar–type orthonormal wavelet basis* to be those compactly supported orthonormal wavelet bases of $L^2(\mathbf{R}^n)$ given by a multiresolution analysis generated from a scaling function of the form $c\chi_Q(x)$ where $\chi_Q(x)$ is the characteristic function of a compact set Q. Recall that a multiresolution analysis has two basic ingredients, which are:

(i). An *admissible pair* (A, Γ) consisting of an $n \times n$ matrix $A \in \mathbf{M}_n(\mathbf{R})$ which is expanding, i.e. all eigenvalues λ of A have $|\lambda| > 1$, together with a full rank lattice Γ in $\mathbf{R}^n$ which is A–invariant, i.e. $A(\Gamma) \subseteq \Gamma$.

(ii). A *scaling function* $\phi(x) \in L^2(\mathbf{R}^n)$ which satisfies a dilation equation

$$\phi(x) = \sum_{\gamma \in \Gamma} c_\gamma \phi(Ax - \gamma)$$

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having the matrix A as the dilation factor. Furthermore,

$$\int_{\mathbf{R}^n} \phi(x-\beta)\phi(x-\gamma)\,dx = \delta_{\beta,\gamma}.$$

A scaling function $\phi(x)$ must satisfy some mild further conditions to actually give a multiresolution analysis, see Mallat [17] and Daubechies [7], Chapter 5. For the special case of Haar–type bases, Gröchenig and Madych [9] found necessary and sufficient conditions for a compact set Q to yield a function $c\chi_Q(x)$ which gives a multiresolution analysis with respect to a given admissible pair (A, Γ). This is stated as Proposition 2.1 in Section 2.

This paper deals with the problem of finding Haar bases for specific pairs (A, Γ). The question of which admissible pairs (A, Γ) have a Haar–type wavelet basis was first raised by Gröchenig and Haas [8]. They asked whether every admissible pair (A, Γ) has a Haar–type basis. They proved that this was so in the one–dimensional case, and also in the two–dimensional case whenever the expanding matrix A has two rational eigenvalues or two conjugate complex eigenvalues. The object of this paper is to extend the approach of Gröchenig and Haas to prove the following result.

Theorem 1.1 *Every two–dimensional admissible pair (A, Γ) has a Haar–type orthonormal wavelet basis.*

The proof follows Gröchenig and Haas in reducing to the case of (A, Γ) with $\Gamma = \mathbf{Z}^n$, and uses the criterion of Gröchenig and Madych that Haar bases for $\Gamma = \mathbf{Z}^n$ correspond to self–affine tiles of measure one. The main technical innovation over the work of Gröchenig and Haas is a Fourier–analytic sufficient condition for a Haar–type orthonormal wavelet basis given as Theorem 3.1 in Section 3. As explained at the end of Section 3, this sufficient condition is effective in dimension two only, and gives little information in dimensions $n \geq 3$.

What happens in dimensions $n \geq 3$? We consider this in a separate paper [15]. In the special case $|\det(A)| = 2$ we show there that necessary conditions for the existence of Haar bases include that the class number of certain algebraic number fields must be one. These conditions provide a strong hint that in sufficiently high dimensions there exists an admissible pair (A, Γ) not possessing any Haar–type orthonormal wavelet bases. However we know of no counterexamples at this time.

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2 Preliminary Reductions

In this section we reduces to the case of admissible pairs (A, Γ) with $\Gamma = \mathbf{Z}^n$, and reformulate the main result using the criterion of Gröchenig and Madych. We first observe that in studying Haar bases for admissible pairs (A, Γ), one can always simplify the problem by a suitable linear transformation $x \mapsto Px$ of $\mathbf{R}^n$ which takes the admissible pair (A, Γ) to $(PAP^{-1}, P\Gamma)$. We may choose this to make $P\Gamma = \mathbf{Z}^n$, in which case $\tilde{A} = PAP^{-1} \in \mathbf{M}_n(\mathbf{Z})$ is an integer matrix. Therefore in the rest of this paper we suppose that $A \in \mathbf{M}_n(\mathbf{Z})$ is an expanding integer matrix and that $\Gamma = \mathbf{Z}^n$.

We now state the criterion of Gröchenig and Madych [9] for a function $c\chi_Q(x)$ of a compact Q to be the scaling function of a multiresolution analysis for the admissible pair $(A, \mathbf{Z}^n)$.

Proposition 2.1 (Gröchenig and Madych) *Let Q be a compact set in $\mathbf{R}^n$ of positive measure. Then necessary and sufficient conditions for $\phi(x) = c\chi_Q(x)$ to be the scaling function of a multiresolution analysis for a pair $(A, \mathbf{Z}^n)$, where $A = \mathbf{M}_n(\mathbf{Z})$, are as follows:*

(i) *There is a set $\mathcal{D} \subset \mathbf{Z}^n$ which is a complete set of residues of $\mathbf{Z}^n/A\mathbf{Z}^n$ such that the unique solution to the set–valued functional equation*

$$A(T) = \bigcup_{d\in\mathcal{D}} (T + d) \tag{2.1}$$

has $T = Q$ up to a set of measure zero.

(ii) *The set T has Lebesgue measure $\mu(T) = 1$.*

Proof. This follows from Theorems 1 and 3 of [9]. Condition (ii) implies that the normalizing constant $c = 1$ in $c\chi_Q(x)$. ∎

The solution to the functional equation (2.1) in the general case when $\mathcal{D} \subset \mathbf{R}^n$ is finite are well–known to be unique and to be explicitly given by

$$T(A, \mathcal{D}) = \Big\{ \sum_{k=1}^{\infty} A^{-k} d_k \ : \ \text{ each } d_k \in \mathcal{D} \Big\}. \tag{2.2}$$

Such sets $T(A, \mathcal{D})$ are called *self–affine tiles* when $|\mathcal{D}| = |\det(A)|$ and $\mu(T(A, \mathcal{D})) > 0$. Results of Bandt [1] (Theorem 1 and 3) state that if $\mathcal{D}$ is a complete set of residues of $\mathbf{Z}^n/A\mathbf{Z}^n$ then $\mu(T(A, \mathcal{D})) > 0$ and that $T(A, \mathcal{D})$ is the closure of its interior. Gröchenig and Haas [8] show that there exists a subset Λ of $\mathbf{Z}^n$ such that $T(A, \mathcal{D}) + \Lambda$ is a tiling of $\mathbf{R}^n$. This implies that $\mu(T(A, \mathcal{D})) \geq 1$. The criterion (ii) of Gröchenig and Madych is therefore equivalent to:

(ii') The set $T(A, \mathcal{D})$ tiles $\mathbf{R}^n$ with the lattice $\mathbf{Z}^n$.

In view of these facts, Theorem 1.1 is an immediate consequence of the following result.

Theorem 2.2 *Let A be an expanding matrix in $\mathbf{M_2}(\mathbf{Z})$. Then there exists $\mathcal{D} \subset \mathbf{Z}^2$ which is a complete set of residues of $\mathbf{Z}^n/A\mathbf{Z}^n$ such that $\mu(T(A, \mathcal{D})) = 1$ or, equivalently, such that $T(A, \mathcal{D})$ tiles $\mathbf{R}^2$ with the lattice $\mathbf{Z}^2$.*

The rest of the paper proves Theorem 2.2. It is done in two parts, Theorem 4.1 in Section 4 treats the cases where $|\det(A)| \geq 3$ and Theorem 5.1 in Section 5 treats the cases where $|\det(A)| = 2$. In Section 3 we develop a sufficient condition for $\mu(T(A, \mathcal{D})) = 1$, stated as Theorem 3.1.

We remark that Theorem 2.2 is an existence result which does not give information about the topological structure of the tile $T(A, \mathcal{D})$. In contrast, Gröchenig and Haas [8] develop methods which allow them to prove that certain $T(A,\mathcal{D})$ are connected. Bandt and Gelbrich [2] study all such tiles in $\mathbf{R}^n$ which are topological disks. They prove that in each dimension this set is finite, and find them all when $n=2$ and $|\det(A)|=2$ or 3. For general background on wavelets and fractals see [3], [4], [6], [7], and for self–affine tiles see [11]–[15], [21], [22].

3 A Sufficient Condition for a $\mathbf{Z}^n$–tiling

In this section A is an expanding matrix in $\mathbf{M_n}(\mathbf{Z})$ and $\mathcal{D}$ is a complete residue system modulo A, i.e. a complete set of residues of $\mathbf{Z}^n/A\mathbf{Z}^n$. We first observe that if $\mathcal{D}'=\mathcal{D}+v$ for some vector $v\in\mathbf{Z}^n$ then

$$T(A,\mathcal{D}')=T(A,\mathcal{D})+\sum_{j=1}^{\infty}A^{-j}v.$$

Since $\mathcal{D}'$ is still a complete residue system modulo A and $T(A,\mathcal{D}')$, $T(A,\mathcal{D})$ have the same Lebesgue measure, from now on we may without loss of generality assume that $0\in\mathcal{D}$.

We call $\mathcal{D}$ *primitive*, in the terminology of Lagarias and Wang ([13]), if $\mathbf{Z}[A,\mathcal{D}]=\mathbf{Z}^n$ where $\mathbf{Z}[A,\mathcal{D}]$ is the minimal A–invariant sublattice of $\mathbf{Z}^n$ containing

$$\mathcal{D}-\mathcal{D}:=\{d-d':\ d,d'\in\mathcal{D}\}.$$

It is known that in order for $\mu(T(A,\mathcal{D}))=1$ the digit set $\mathcal{D}$ must be primitive ([8], [13]). However, $\mathcal{D}$ being primitive is not sufficient for $\mu(T(A,\mathcal{D}))=1$, as illustrated by the following example:

$$A=\begin{bmatrix}2&1\\0&2\end{bmatrix}\quad\text{and}\quad\mathcal{D}=\left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}3\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}3\\1\end{bmatrix}\right\}.$$

It is easy to check that $\mathcal{D}$ is a primitive complete residue system modulo A. However, $\mu(T(A,\mathcal{D}))=3$ (see [12]).

To prove Theorem 2.2 we follow the Fourier analytic approach used in [8]. Let $\mathbf{T}^n=\mathbf{R}^n/\mathbf{Z}^n$ be the n–torus. Define

$$m_{\mathcal{D}}(x)=\frac{1}{|\mathcal{D}|}\sum_{d\in\mathcal{D}}\exp(2\pi i\langle d,x\rangle).\tag{3.1}$$

The main ingredient used in the proof of Theorem 2.2 is the following sufficient condition for $\mu(T(A,\mathcal{D}))=1$ to hold in $\mathbf{R}^n$.

Theorem 3.1 *Let $A\in\mathbf{M}_n(\mathbf{Z})$ be an expanding matrix and let $\mathcal{D}\subset\mathbf{Z}^n$ be a primitive complete residue system modulo A. Suppose that $m_{\mathcal{D}}(x)$ has only finitely many zeros in $\mathbf{T}^n$. Then $\mu(T(A,\mathcal{D}))=1$.*

To prove Theorem 3.1 we will need several lemmas. Let $B=A^T$ and let $B_*:\mathbf{T}^n\to\mathbf{T}^n$ be the canonical map induced by B. Then we have the following useful lemma.

Lemma 3.2 *For all $x \in \mathbf{T}^n$,*

$$\sum_{y \in B_*^{-1}(x)} |m_{\mathcal{D}}(y)|^2 = 1. \tag{3.2}$$

Proof. See [8], Lemma 5.1. ∎

We now define the following linear operator[1] $\hat{C}_{\mathcal{D}} : L^2(\mathbf{T}^n) \longrightarrow L^2(\mathbf{T}^n)$,

$$(\hat{C}_{\mathcal{D}} f)(x) := \sum_{y \in B_*^{-1}(x)} |m_{\mathcal{D}}(y)|^2 f(y). \tag{3.3}$$

It follows from Lemma 3.2 that $\hat{C}_{\mathcal{D}} 1 = 1$. The following lemma is essentially due to Lawton and Resnikoff ([16]).

Lemma 3.3 *$\mu(T(A,\mathcal{D})) \neq 1$ if and only if there exists a nonconstant trigonometric polynomial $f(x)$ of the form*

$$f(x) = \sum_{j=1}^{m} a_j \cos(2\pi\langle v_j, x\rangle) \tag{3.4}$$

where all $a_j > 0$ and $v_j \in \mathbf{Z}^n$ such that $\hat{C}_{\mathcal{D}} f = f$.

Proof. See [8], Proposition 5.3. ∎

Let $f(x)$ be any continuous function on $\mathbf{T}^n$. Define

$$E_f^+ = \Big\{ y \in \mathbf{T}^n \;:\; f(y) = \max_{x \in \mathbf{T}^n} f(x) \Big\} \;\; \textit{and} \;\; E_f^- = \Big\{ y \in \mathbf{T}^n \;:\; f(y) = \min_{x \in \mathbf{T}^n} f(x) \Big\}.$$

Lemma 3.4 *Let $\mathcal{D}$ be primitive. Suppose that there exists a nonconstant $f(x) \in C^0(\mathbf{T}^n)$ such that $\hat{C}_{\mathcal{D}} f = f$. Then $E^+(f) \cup E^-(f)$ is an infinite subset of $\mathbf{T}^n$.*

Proof. Since both E_f^+ and E_f^- are nonempty, one of which, say E_f^-, must not contain 0. Now, for any $x_0 \in E_f^-$, $\hat{C}_{\mathcal{D}} f = f$ implies

$$\sum_{y \in B_*^{-1}(x_0)} |m_{\mathcal{D}}(y)|^2 \Big(f(y) - f(x_0) \Big) = 0. \tag{3.5}$$

Because $x_0 \in E_f^-$, $f(y) - f(x_0) \geq 0$. So we have either $m_{\mathcal{D}}(y) = 0$ or $y \in E_f^-$ for any $y \in B_*^{-1}(x_0)$, and (3.2) implies that there exists at least one $y_0 \in B_*^{-1}(x_0)$ such that $y_0 \in E_f^-$. Since $B_*(y_0) = x_0$ and $x_0 \in E_f^-$ is arbitrarily chosen, this implies that

$$B_*(E_f^-) \supseteq E_f^-. \tag{3.6}$$

We suppose that $E^+(f) \cup E^-(f)$ is finite, and argue by contradiction. Now E_f^- is finite, so it follows from (3.6) that

$$B_*(E_f^-) = E_f^-, \tag{3.7}$$

[1] This is the Fourier transform of the linear operator C studied in [8], see [8], Lemma 5.2.

which implies also that for any $x \in E_f^-$ there is a *unique* $y \in B_*^{-1}(x)$ such that $y \in E_f^-$.

So for any $x \in E_f^-$, we have $B_*(x) \in E_f^-$ and x is the only pre–image of $B_*(x)$ in E_f^-. Hence $m_{\mathcal{D}}(y) = 0$ for all $y \in B_*^{-1}(B_*(x))$ except $y = x$. This gives $|m_{\mathcal{D}}(x)| = 1$. Because $0 \in \mathcal{D}$, we must have

$$1 = m_{\mathcal{D}}(x) = \frac{1}{|\mathcal{D}|} \sum_{d \in \mathcal{D}} \exp(2\pi i \langle d, x \rangle).$$

Therefore $\langle d, x \rangle \equiv 0 \pmod 1$ for all $d \in \mathcal{D}$. Let

$$\Lambda = \Big\{ u \in \mathbf{Z}^n : \ \langle u, x \rangle \equiv 0 \ (\text{mod } 1) \text{ for all } x \in E_f^- \Big\}.$$

Clearly Λ contains $\mathcal{D}$ and is a subgroup of $\mathbf{Z}^n$. Λ is A–invariant because for any $u \in \Lambda$ and $x \in E_f^-$,

$$\langle Au, x \rangle \equiv \langle u, B_* x \rangle \equiv 0 \pmod 1.$$

Hence $\mathbf{Z}[A, \mathcal{D}] \subseteq \Lambda$. But because $x \not\equiv 0 \pmod 1$ for all $x \in E_f^-$, this implies that $\Lambda \neq \mathbf{Z}^n$. This contradicts the primitiveness of $\mathcal{D}$. ∎

Proof of Theorem 3.1: First let $f(x)$ be nonconstant and $f(x) \in C^0(\mathbf{T}^n)$. Suppose that $\hat{C}_{\mathcal{D}} f = f$. We show that $E_f^+ \cup E_f^- \subset \mathbf{T}^n$ must be finite. Assume this is false. Then either E_f^+ or E_f^-, say E_f^+, is infinite. Because $\mathbf{T}^n$ is compact, E_f^+ has a limit point x^*.

Claim: *$B_*^{-1}(x^*) \subseteq E_f^+$ and every $y \in B_*^{-1}(x^*)$ is a limit point of E_f^+.*

To prove the claim, let $\{x_k\} \subseteq E_f^+$ converge to x^*. We have $(\hat{C}_{\mathcal{D}} f)(x_k) = f(x_k)$. So

$$\sum_{y \in B_*^{-1}(x_k)} |m_{\mathcal{D}}(y)|^2 \Big(f(y) - f(x_k) \Big) = 0.$$

Hence either $y \in E_f^+$ or $m_{\mathcal{D}}(x_k) = 0$ for all $y \in B_*^{-1}(x_k)$. But $m_{\mathcal{D}}(x)$ has only finitely many zeros in $\mathbf{T}^n$. Thus there exists a constant K_0 such that $m_{\mathcal{D}}(y) \neq 0$ holds for all $y \in B_*^{-1}(x_k)$ and $k \geq K_0$. So

$$B_*^{-1}(\{x_k : \ k \geq K_0\}) \subseteq E_f^+.$$

But every $y \in B_*^{-1}(x^*)$ is a limit point in $B_*^{-1}(\{x_k : \ k \geq K_0\})$. It follows from the continuity of $f(x)$ that $B_*^{-1}(x^*) \subseteq E_f^+$, and thus the claim is true.

Now it follows from the claim that for all $m > 0$, $B_*^{-m}(x^*) \subseteq E_f^+$ and every $y \in B_*^{-m}(x^*)$ is a limit point in E_f^+. So

$$E_f^+ \supseteq \bigcup_{m=1}^{\infty} B_*^{-m}(x^*).$$

But $\cup_{m=1}^{\infty} B_*^{-m}(x^*)$ is dense in $\mathbf{T}^n$. This implies $E_f^+ = \mathbf{T}^n$ and hence $f(x)$ is a constant, a contradiction.

Therefore $E_f^+ \cup E_f^- \subseteq \mathbf{T}^n$ must be finite, proving the claim.

We can now complete the proof of Theorem 3.1. Suppose that $\mu(T(A, \mathcal{D})) \neq 1$. Then there exists a nonconstant trigonometric polynomial $f(x)$ of the form (3.4) such that $\hat{C}_{\mathcal{D}} f =$

f. Hence $E_f^+ \cup E_f^- \subseteq \mathbf{T}^n$ is finite. But this contradicts Lemma 3.4 because $\mathcal{D}$ is primitive. Hence we must have $\mu(T(A, \mathcal{D})) = 1$. ∎

Remark. The criterion of Theorem 3.1 is effective for studying most self–affine tiles in $\mathbf{R}^2$ but not for studying such tiles in $\mathbf{R}^n$ for $n \geq 3$. This is because the real and imaginary parts of the equation $m_{\mathcal{D}}(x) = 0$ then gives two equations in which the number of unknowns is n. In two dimensions the "generic" situation is for the zero–set of $m_{\mathcal{D}}(x)$ on $\mathbf{T}^2$ to be zero–dimensional, hence for "generic" $\mathcal{D}$ the hypothesis of Theorem 3.1 are satisfied, as the proofs in Section 4 indicate. (There are "exceptional" $\mathcal{D}$, however.) In dimension $n \geq 3$ there are still two equations but n unknowns, and the hypotheses of Theorem 3.1 are not satisfied for "generic" $\mathcal{D}$.

4 Haar Bases in $\mathbf{R}^2$: Case $|\det(A)| \geq 3$

Our object in this section is to prove:

Theorem 4.1 *Let $A \in \mathbf{M}_2(\mathbf{Z})$ be an expanding matrix with $|\det(A)| \geq 3$. Then there exists a primitive complete residue system $\mathcal{D}$ modulo A such that $\mu(T(A, \mathcal{D})) = 1$.*

We accomplish this by constructing a primitive complete residue system modulo A such that $m_{\mathcal{D}}(x)$ has only finitely many zeros in $\mathbf{T}^2$.

We first introduce some notation. Let $\gcd(A)$ denote the greatest common divisor of the entries of A, and for any $d \in \mathbf{Z}^2$ let $\gcd(d)$ denote the greatest common divisor of the entries of d. For any $d_1,\, d_2 \in \mathbf{Z}^2$, let $\det([d_1, d_2])$ denote the determinant of the matrix whose first and second columns are d_1 and d_2 respectively. The following is a well–known result:

Lemma 4.2 *Let $d = \gcd(A)$ and $q = |\det(A)|$. Then $\mathbf{Z}^2/A\mathbf{Z}^2$ is isomorphic to $\mathbf{Z}_d \oplus \mathbf{Z}_{q/d}$.*

Proof. We make use of the Smith normal form for A: there exist integer unimodular matrices $U,\, V \in \mathbf{GL}_2(\mathbf{Z})$ such that

$$UAV = \operatorname{diag}(s_1, s_2) = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

where $s_1|s_2$. (See [18], Theorem II.9.) Since $\gcd(A) = \gcd(UAV) = d$, it follows that $|s_1| = d$ and $|s_2| = q/d$. So

$$\mathbf{Z}^n/A\mathbf{Z}^n \cong \mathbf{Z}^n/UAV(\mathbf{Z}^n) \cong \mathbf{Z}_d \oplus \mathbf{Z}_{q/d}.$$

∎

Lemma 4.3 *Let $d_0,\, d_1,\, \ldots,\, d_N \in \mathbf{Z}^2$ be fixed and let $l_1^*,\, l_2^* \in \mathbf{Z}^2$ be linearly independent. Then for sufficiently large $\lambda \in \mathbf{Z}$ the solutions to*

$$\begin{cases} \cos(2\pi\langle d_0 + \lambda l_1^*, x\rangle) + \sum_{j=1}^{N} \cos(2\pi\langle d_j, x\rangle) = 0, \\ \langle l_2^*, x\rangle = c \end{cases} \tag{4.1}$$

is a discrete set for any $c \in \mathbf{R}$.

Proof. Since we can always find a unimodular integer matrix $P \in \mathbf{GL}_2(\mathbf{Z})$ such that Pl_2^* is parallel to e_1 ($e_1 = [1,0]^T$), we may assume, without loss of generality, that $l_2^* = p_0 e_1$ for some $p_0 \in \mathbf{Z}$. So $\langle l_2^*, x\rangle = c$ gives $p_0 x_1 = c$, $x_1 = c/p_0$. Let $d_0 = [b_1, b_2]^T$ and $l_1^* = [t_1, t_2]^T$. Then $t_2 \neq 0$ because l_1^*, l_2^* are linearly independent. We have

$$\cos 2\pi\langle d_0 + \lambda l_1^*, x\rangle + \sum_{j=1}^{N} \cos 2\pi\langle d_j, x\rangle =$$

$$\cos 2\pi\Big(\frac{(b_1+\lambda t_1)c}{p} + (b_2+\lambda t_2)x_2\Big) + \sum_{j=1}^{N} \cos 2\pi\langle d_j, x\rangle.$$

The above expression is a trigonometric polynomial with respect to the variable x_2. By choosing a sufficiently large λ, neither $\cos 2\pi(b_2+\lambda t_2)x_2$ nor $\sin 2\pi(b_2+\lambda t_2)x_2$ can be cancelled out by terms in $\sum_{j=1}^{N}\cos 2\pi\langle d_j, x\rangle$. Hence

$$\cos 2\pi\Big(\frac{(b_1+\lambda t_1)c}{p} + (b_2+\lambda t_2)x_2\Big) + \sum_{j=1}^{N} \cos 2\pi\langle d_j, x\rangle \not\equiv 0.$$

Thus the solutions to (4.1) must be a discrete set. ∎

Lemma 4.4 *Let $|\mathcal{D}| = 4$. Suppose that $m_{\mathcal{D}}(x)$ has infinitely many zeros in $\mathbf{T}^2$. Then the digits in $\mathcal{D}$ form the vertices of a trapezoid.*

Proof. Let $\mathcal{D} = \{0, d_1, d_2, d_3\}$. We have

$$4m_{\mathcal{D}}(x) = 1 + \exp(2\pi i\langle d_1, x\rangle) + \exp(2\pi i\langle d_2, x\rangle) + \exp(2\pi i\langle d_3, x\rangle) = 0.$$

Multiply $\exp(2\pi i\langle \frac{d_2+d_3}{2}, x\rangle)$ to the equation we obtain

$$-\sin 2\pi\langle \tfrac{d_2+d_3}{2}, x\rangle + \sin 2\pi\langle \tfrac{2d_1-d_2-d_3}{2}, x\rangle = 0, \qquad (4.2)$$

$$-\cos 2\pi\langle \tfrac{d_2+d_3}{2}, x\rangle + \cos 2\pi\langle \tfrac{2d_1-d_2-d_3}{2}, x\rangle + 2\cos 2\pi\langle \tfrac{d_2-d_3}{2}, x\rangle = 0. \qquad (4.3)$$

(4.2) gives $\langle d_1 - d_2 - d_3, x\rangle \equiv 0 \pmod 1$ or $\langle d_1, x\rangle \equiv \frac{1}{2} \pmod 1$.

For $\langle d_1 - d_2 - d_3, x\rangle \equiv 0 \pmod 1$ the equation (4.3) yields

$$2\cos 2\pi\langle \frac{d_2+d_3}{2}, x\rangle + 2\cos 2\pi\langle \frac{d_2-d_3}{2}, x\rangle = 0.$$

Hence we have

$$\begin{cases} \langle d_1 - d_2 - d_3, x\rangle & \equiv 0 \pmod 1, \\ \langle \frac{d_2+d_3}{2}, x\rangle \pm \langle \frac{d_2-d_3}{2}, x\rangle & \equiv \frac{1}{2} \pmod 1. \end{cases} \qquad (4.4)$$

Since $\langle \frac{d_2+d_3}{2}, x\rangle \pm \langle \frac{d_2-d_3}{2}, x\rangle = \langle d_2, x\rangle$ or $\langle d_3, x\rangle$, equations (4.4) have infinitely many solutions in $\mathbf{T}^2$ only if $d_2 \parallel d_1 - d_2 - d_3$ or $d_3 \parallel d_1 - d_2 - d_3$. In either case, $\{0, d_1, d_2, d_3\}$ must form the vertices of a trapezoid.

For $\langle d_1, x\rangle \equiv \frac{1}{2} \pmod 1$ the equation (4.3) yields

$$2\cos 2\pi\langle \frac{d_2-d_3}{2}, x\rangle = 0.$$

Hence we have

$$\begin{cases} \langle d_1, x\rangle & \equiv \frac{1}{2} \pmod 1, \\ \langle d_2 - d_3, x\rangle & \equiv \frac{1}{2} \pmod 1. \end{cases} \tag{4.5}$$

Equations (4.5) have infinitely many solutions in $\mathbf{T}^2$ only if $d_1 \parallel d_2 - d_3$. This also implies that $\{0, d_1, d_2, d_3\}$ form the vertices of a trapezoid. ∎

Theorem 4.5 *Let $A \in \mathbf{M}_2(\mathbf{Z})$ be expanding and $|\det(A)| \geq 3$. Then there exists a primitive complete residue system $\mathcal{D}$ modulo A such that $m_{\mathcal{D}}(x)$ has finitely many zeros in $\mathbf{T}^2$.*

Proof. We divide the proof into three cases.

Case 1. $q = |\det(A)| > 3$ *and* $\gcd(A) = 1$.

In this case $\mathbf{Z}^2/A\mathbf{Z}^2$ is cyclic according to Lemma 4.2. So there exists a $d_1 \in \mathbf{Z}^2$ such that d_1 generates $\mathbf{Z}^2/A\mathbf{Z}^2$. Let $d^* = d_1/\gcd(d_1)$. Then d^* also generates $\mathbf{Z}^2/A\mathbf{Z}^2$, and $\gcd(d^*) = 1$. Hence there exists a $v \in \mathbf{Z}^2$ such that $\det([d^*, v]) = 1$. Suppose that $v \equiv \lambda d^* \pmod A$ for some $\lambda \in \mathbf{Z}$ and let $v^* = v - \lambda d^*$. Then $v^* \equiv 0 \pmod A$ and $\det([d^*, v^*]) = 1$.

If $q = 4$, then let $\mathcal{D} = \{0, d^*, 2d^*, 3d^* + v^*\}$. Clearly $\mathcal{D}$ is a complete residue system modulo A, and it is primitive because $\det([d^*, 3d^* + v^*]) = 1$. Furthermore, the digits of $\mathcal{D}$ do not form the vertices of a trapezoid. Hence $m_{\mathcal{D}}(x)$ has finitely many zeros in $\mathbf{T}^2$.

Now suppose that $q > 4$. Let $\lambda \in \mathbf{Z}$ be sufficiently large and let

$$\mathcal{D}_1 = \Big\{\lambda v^*, d^*, 2d^*, \ldots, (q-3)d^*, (q-2)d^* - \lambda v^*, v^* - d^*\Big\}.$$

It is clear that $\mathcal{D}_1$ is a complete residue system modulo A. Let $\mathcal{D}_2 = \mathcal{D}_1 - \alpha d^*$ where $\alpha = (q-2)/2$. Then $\mathcal{D}_2 \setminus \{(v^* - d^*) - \alpha d^*\}$ is centrally symmetric. So

$$\mathrm{Im}(m_{\mathcal{D}_2}(x)) = \sin 2\pi \langle v^* - d^* - \alpha d^*, x\rangle.$$

Thus $\mathrm{Im}(m_{\mathcal{D}_2}(x)) = 0$ yields

$$2\langle v^* - d^* - \alpha d^*, x\rangle \equiv 0 \pmod 1. \tag{4.6}$$

Since v^* and $v^* - d^* - \alpha d^*$ are linearly independent, it follows from Lemma 4.3 that (4.6) together with $\mathrm{Re}(m_{\mathcal{D}_2}(x)) = 0$ yield only finitely many solutions in $\mathbf{T}^2$. Hence $m_{\mathcal{D}_2}(x)$ has finitely many zeros in $\mathbf{T}^2$.

Let $\mathcal{D} = \mathcal{D}_1 - d^* = \mathcal{D}_2 + (\alpha - 1)d^*$. Then $\mathcal{D}$ is a complete residue system modulo A with $0 \in \mathcal{D}$. $\mathcal{D}$ is primitive because d^*, $v^* - 2d^* \in \mathcal{D}$ and $\det([d^*, v^* - 2d^*]) = 1$. Finally, $m_{\mathcal{D}}(x) = \exp(2\pi i \langle (\alpha - 1)d^*, x\rangle) m_{\mathcal{D}_2}(x)$. So $m_{\mathcal{D}}(x)$ has finitely many zeros in $\mathbf{T}^2$.

Case 2. $r = \gcd(A) > 1$.

Let $q = |\det(A)|$ and $q_1 = q/r$. Then $\mathbf{Z}^2/A\mathbf{Z}^2 \cong \mathbf{Z}_r \oplus \mathbf{Z}_{q_1}$. We say that the ordered pair $(d_1, d_2) \in \mathbf{Z}^2 \times \mathbf{Z}^2$ generates $\mathbf{Z}^2/A\mathbf{Z}^2$ if $rd_1 \equiv q_1 d_2 \equiv 0 \pmod A$ and $\{id_1 + jd_2 : 0 \leq i < r, 0 \leq j < q_1\}$ forms a complete residue system modulo A. We show that there exists a pair (d_1^*, d_2^*) which generates $\mathbf{Z}^2/A\mathbf{Z}^2$ such that $\det([d_1^*, d_2^*]) = 1$.

To see this, choose an arbitrary pair (d_1, d_2) that generates $\mathbf{Z}^2/A\mathbf{Z}^2$. Set $d_2^* = d_2/\gcd(d_2)$ and let $d_3 \in \mathbf{Z}^2$ such that $\det([d_3, d_2^*]) = 1$. We have $d_3 \equiv \lambda_1 d_1^* + \lambda_2 d_2^* \pmod{A}$. Now let $d_1^* = d_3 - \lambda_2 d_2^*$. Then $\det([d_1^*, d_2^*]) = 1$, which also implies that $\gcd(d_1^*) = 1$. So $rd_1^* \equiv r\lambda_1 d_1 \equiv 0 \pmod{A}$ but $td_1^* \not\equiv 0 \pmod{A}$ for any $0 < t < r$. Hence (d_1^*, d_2^*) generates $\mathbf{Z}^2/A\mathbf{Z}^2$.

If $q = 4$, then we must have $r = q_1 = 2$. In this case let

$$\mathcal{D} = \left\{0, d_1^*, d_2^*, d_1^* + d_2^* + qv\right\}$$

where $v \in \mathbf{Z}^2$ such that the digits of $\mathcal{D}$ do not form the vertices of a trapezoid. So $m_{\mathcal{D}}(x)$ has finitely many zeros in $\mathbf{T}^2$, and clearly $\mathcal{D}$ is a primitive complete residue system modulo A.

Now suppose that $q > 4$. Since $r|q_1$, we must have $q_1 \geq 4$. Let $\lambda \in \mathbf{Z}$ be sufficiently large and let $\mathcal{D} = \{d_{i,j} : \ 0 \leq i < r, 0 \leq j < q_1\}$ where $d_{i,j} = id_1^* + jd_2^*$ except for the following:

$$d_{r-1,q_1-1} = -d_1^* - d_2^*, \quad d_{r-1,0} = (r-1)d_1^* - \lambda q d_2^*, \quad d_{0,q_1-1} = (q_1-1)d_2^* + \lambda q d_2^*.$$

Then $\mathcal{D}$ is a complete residue system modulo A. It is primitive because $d_{0,1} = d_2^*$, $d_{1,1} = d_1^* + d_2^*$ and $\det([d_1^* + d_2^*, d_2^*]) = 1$. Let

$$u^* = \frac{r-1}{2}d_1^* + \frac{q_1-1}{2}d_2^*$$

and $\mathcal{D}_1 = \mathcal{D} - u^*$. Then $\mathcal{D}_1 \setminus \{d_{0,0} - u^*, d_{r-1,q_1-1} - u^*\}$ is centrally symmetric. So

$$0 = \mathrm{Im}(m_{\mathcal{D}_1}(x)) = -\sin 2\pi\langle u^*, x\rangle + \sin 2\pi\langle d_{r-1,q_1-1} - u^*, x\rangle$$

yields

$$\langle d_{r-1,q_1-1} - 2u^*, x\rangle \equiv 0 \pmod{1} \quad \text{or} \quad \langle d_{r-1,q_1-1}, x\rangle \equiv \frac{1}{2} \pmod{1}. \tag{4.7}$$

Notice that d_2^*, $d_{r-1,q_1-1} - 2u^*$ are linearly independent, and so are d_2^*, d_{r-1,q_1-1}. Thus it follows from Lemma 4.3 that (4.7) together with $\mathrm{Re}(m_{\mathcal{D}_1}(x)) = 0$ yield only finitely many solutions in $\mathbf{T}^2$. Hence $m_{\mathcal{D}}(x) = \exp(2\pi i\langle u^*, x\rangle)m_{\mathcal{D}_1}(x)$ has finitely many zeros in $\mathbf{T}^2$.

Case 3. $q = |\det(A)| = 3$.

Let d^*, v^* be as in Case 1 and let $\mathcal{D} = \{0, d^*, v^* - d^*\}$. Then $\mathcal{D}$ is a complete residue system modulo A and it is primitive because $\det([d^*, v^* - d^*]) = 1$. Now $m_{\mathcal{D}}(x) = 0$ gives

$$1 + \cos 2\pi\langle d^*, x\rangle + \cos 2\pi\langle v^* - d^*, x\rangle = 0, \tag{4.8}$$

$$\sin 2\pi\langle d^*, x\rangle + \sin 2\pi\langle v^* - d^*, x\rangle = 0. \tag{4.9}$$

It is easily checked that (4.8) and (4.9) reduce to

$$\langle d_2^*, x\rangle \equiv 0 \pmod{1}, \quad \langle d_1^*, x\rangle \equiv \frac{1}{3}, \frac{2}{3} \pmod{1}.$$

Hence $m_{\mathcal{D}}(x)$ has finitely many zeros in $\mathbf{T}^2$. ∎

Theorem 4.1 follows immediately from Theorem 4.5.

5 Haar Bases in $\mathbf{R}^2$: Case $|\det(A)| = 2$

Our object in this section is to prove:

Theorem 5.1 *Let $A \in \mathbf{M}_2(\mathbf{Z})$ be an expanding matrix with $|\det(A)| = 2$. If $\mathcal{D}$ is a primitive complete residue system modulo A, then $\mu(T(A, \mathcal{D})) = 1$. Moreover, such a $\mathcal{D}$ exists.*

Furthermore we shall completely classify all expanding $A \in \mathbf{M}_2(\mathbf{Z})$ with $|\det(A)| = 2$, and determine all primitive complete residue systems modulo A.

Call two integer matrices A and B *integrally similar*, and write $A \sim B$, if there exists an integer unimodular matrix $P \in \mathbf{GL}_2(\mathbf{Z})$ such that $P^{-1}AP = B$. Now, denote

$$C_1 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}.$$

Lemma 5.2 *Let $A \in \mathbf{M}_2(\mathbf{Z})$ be expanding. If $\det(A) = -2$ then A is integrally similar to C_1. If $\det(A) = 2$ then A is integrally similar to one of the following matrices: C_2, $\pm C_3$, $\pm C_4$.*

Proof. Let $A = [a_{ij}]$ and define the *weight* $p(A)$ of A to be

$$p(A) := -a_{11}a_{22}.$$

Since the two eigenvalues of A satisfy $|\lambda_1\lambda_2| = 2$ and $|\lambda_1|, |\lambda_2| > 1$, we have $|a_{11} + a_{22}| \leq |\lambda_1| + |\lambda_2| < 3$. Hence $p(A) \geq -1$. We prove that $A \sim B$ for some matrix

$$B = \begin{bmatrix} 0 & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

by induction on the weight $p(A)$.

For the base case $p(A) = -1$ we have $|a_{11}| = |a_{22}| = 1$. Since $a_{12}a_{21} = -p(A) - \det(A)$ is either -1 or 3, either $|a_{12}| = 1$ or $|a_{21}| = 1$. Without loss of generality let $|a_{21}| = 1$. Then by taking $\lambda = \operatorname{sign}(a_{11}a_{21})$ we have

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

For $p(A) = 0$ we have either $|a_{11}| = 0$ or $|a_{22}| = 0$. If $a_{11} = 0$ we are done. Suppose $|a_{22}| = 0$. Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & a_{21} \\ a_{12} & a_{11} \end{bmatrix}.$$

Now assume that the hypothesis is true when the weight $p(A) < r$ where $r > 0$. We show it is also true when $p(A) = r$. Since $|a_{11}|, |a_{22}| \geq 1$, it implies that either $|a_{21}| \leq |a_{11}|$

or $|a_{12}| \leq |a_{22}|$, for if otherwise we would have $|\det(A)| = |a_{21}a_{12} - a_{11}a_{22}| \geq 3$. So without loss of generality we assume that $|a_{21}| \leq |a_{11}|$. Let $\lambda = \mathrm{sign}(a_{11}a_{21})$ and

$$A_1 = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a_{11} - \lambda a_{21} & \star \\ \star & a_{22} + \lambda a_{21} \end{bmatrix}.$$

Then

$$\begin{aligned} p(A_1) &= -(a_{11} - \lambda a_{21})(a_{22} + \lambda a_{21}) \\ &= p(A) + \lambda^2 a_{21}^2 + \lambda a_{21}(a_{22} - a_{11}). \end{aligned}$$

Because $a_{11}a_{22} = -p(A) = -r < 0$, it follows that

$$\begin{aligned} p(A_1) &= p(A) + a_{21}^2 - \mathrm{sign}(a_{11}a_{21})a_{21}(a_{11} - a_{22}) \\ &< p(A) + a_{21}^2 - \mathrm{sign}(a_{11}a_{21})a_{21}a_{11} \\ &\leq p(A) \;=\; r. \end{aligned}$$

Hence $A_1 \sim B$ for some

$$B = \begin{bmatrix} 0 & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

This proves the hypothesis because $A \sim A_1$.

Now suppose that $\det(A) = -2$. Then $b_{12}b_{21} = 2$. It follows from $|\lambda_1|, |\lambda_2| > 1$ that $b_{22} = 0$. One can easily check that for whichever combination of b_{12} and b_{21} we always have $B \sim C_1$. For example, if $b_{12} = -1$ and $b_{21} = -2$, then $PBP^{-1} = C_2$ where

$$P = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Suppose that $\det(A) = 2$. Then $b_{12}b_{21} = -2$. Again it follows from $|\lambda_1|, |\lambda_2| > 1$ that $|b_{22}| \leq 2$. Denote

$$\tilde{C}_3 = \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix}.$$

It is easily checked that we will have $PBP^{-1} = C_2,\ \pm\tilde{C}_3$, or $\pm C_4$ by taking P to be one of the following matrices:

$$I,\quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Finally, $C_3 = Q\tilde{C}_3Q^{-1}$ where

$$Q = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

∎

We will make use of the following criterion for $\mu(T(A, \mathcal{D})) > 1$.

Lemma 5.3 (Cohen's Condition) *Let $\mathcal{D}$ be a residue system of A. Then $\mu(T(A,\mathcal{D}))=1$ if and only if there exists a bounded fundamental domain K of the lattice $\mathbf{Z}^2$ which contains a neighborhood of the origin such that $|m_{\mathcal{D}}(B^{-j}x)|>0$ where $B=A^T$ holds for all $x\in\overline{K}$ and $j\geq 1$.*

Proof. This is proved in [9], Theorem 3. (It is a special case of the results of Cohen [5].) ∎

Proof of Theorem 5.1. Because $T(B,\mathcal{D})=PT(A,P\mathcal{D})$ where $B=PAP^{-1}$, it suffices to prove the theorem for A being one of the matrices C_1, C_2, $\pm C_3$, and $\pm C_4$.

Case 1. $A=C_1=\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$

First we observe that $\mathcal{D}_0=\{0,e_1\}$ is a primitive residue system modulo A because $e_2=Ae_1\in\mathbf{Z}[A,\mathcal{D}_0]$. Furthermore, $\mu(T(A,\mathcal{D}_0))=1$ because $T(A,\mathcal{D}_0)$ is simply the unit square $[0,1]\times[0,1]$.

Suppose $\mathcal{D}=\{0,\,d\}$ where $d=[a,b]^T$. Then $d=Qe_1$ where

$$Q=\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}.$$

Since $AQ=QA$, $\mathcal{D}$ is primitive if and only if

$$\det(Q)=a^2-2b^2=\pm 1. \tag{5.1}$$

(5.1) is a classical Pell's equation. Its solutions are $a=\pm p_n$, $b=\pm q_n$ for $n\in\mathbf{N}$, where p_n/q_n is the n-th convergent of the continued fraction of $\sqrt{2}$. For example, the first several (p_n,q_n) pairs are $(1,0)$, $(1,1)$, $(3,2)$, $(7,5)$, $(17,12)$, $(41,27)$, etc. (see [20]). For those $\mathcal{D}$ we have $\mu(T(A,\mathcal{D}))=|\det(Q)|\mu(T(A,\mathcal{D}_0))=1$. So the theorem is true for $A=C_1$.

Case 2. $A=C_2=\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}.$

A proof for this case can be found in [9]. In this case the only primitive residue systems modulo A are $\mathcal{D}=\{0,\,d\}$ where $d=e_1$ and $d=-e_1$, both make $T(A,\mathcal{D})$ a unit square.

Case 3. $A=\pm C_3=\pm\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$

A proof for this case, which employed Cohen's Condition, can also be found in [9]. In this case the only primitive residue systems modulo A are $\mathcal{D}=\{0,\,d\}$ where $d=\pm e_1$ and $d=\pm e_2$. In all four cases the corresponding tile $T(A,\mathcal{D})$ are the well–known "twin dragon" tiles.

Case 4. $A=\pm C_4=\pm\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}.$

First we observe that $\mathcal{D}_0=\{0,e_1\}$ is a primitive residue system modulo A because $e_2=-Ae_1\in\mathbf{Z}[A,\mathcal{D}_0]$. We show that $\mu(T(A,\mathcal{D}_0))=1$ using Cohen's Condition.

Let $K_1 = [-1/2, 1/2)^2$ and denote

$$\begin{aligned} V^+ &:= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in K_1 : \ (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 < \delta^2 \right\}, \\ V^- &:= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in K_1 : \ (x_1 + \frac{1}{2})^2 + (x_2 + \frac{1}{2})^2 < \delta^2 \right\} \end{aligned}$$

for some small $\delta > 0$. Let $K = (K_1 \setminus (V^+ \cup V^-)) \cup (V^+ + [-1, 0]^T) \cup (V^- + [1, 0]^T)$. Then K is a fundamental domain of the lattice $\mathbf{Z}^2$. Now

$$m_{\mathcal{D}_0}(x) = \frac{1}{2}\Big(1 + \exp(2\pi i \langle e_1, x\rangle)\Big) = \exp(\pi i x_1) \cos \pi x_1. \tag{5.2}$$

So $|m_{\mathcal{D}_0}(x)| = 0$ if and only if $x_1 \equiv \frac{1}{2} \pmod 1$. But we have

$$B^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 0 \end{bmatrix}, \quad B^{-2} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

So it is easy to check that $|m_{\mathcal{D}_0}(B^{-j}x)| > 0$ for all $x \in \overline{K}$ and $j = 1, 2$. Also since $B^{-2}K \subset K$, it implies $|m_{\mathcal{D}_0}(B^{-j}x)| > 0$ for all $x \in \overline{K}$ and j. Hence $\mu(T(A, \mathcal{D}_0)) = 1$ from Cohen's Condition.

Now let $\mathcal{D} = \{0, d\}$ where $d = [a, b]^T$. Notice that $d = Qe_1$ where

$$Q = \begin{bmatrix} a & -2b \\ b & a+b \end{bmatrix}$$

and we have $AQ = QA$. So $\mathcal{D}$ is primitive if and only if $|\det(Q)| = |a^2 + ab + 2b^2| = 1$, which implies $d = \pm e_1$. Clearly, $\mu(T(A, \mathcal{D})) = 1$ in either case. ∎

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