INTEGRAL SELF-AFFINE TILES IN $\mathbb{R}^n$ II. LATTICE TILINGS

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Abstract. Let A be an expanding $n \times n$ integer matrix with $|\det(A)| = m$. A *standard digit set* $\mathcal{D}$ for A is any complete set of coset representatives for $\mathbb{Z}^n/A(\mathbb{Z}^n)$. Associated to a given $\mathcal{D}$ is a set $T(A, \mathcal{D})$, which is the attractor of an affine iterated function system, satisfying $T = \cup_{d\in\mathcal{D}}(T+d)$. It is known that $T(A, \mathcal{D})$ tiles $\mathbb{R}^n$ by some subset of $\mathbb{Z}^n$. This paper proves that every standard digit set $\mathcal{D}$ gives a set $T(A, \mathcal{D})$ which tiles $\mathbb{R}^n$ with a lattice tiling.

1. Introduction

Suppose that A is an $n \times n$ real matrix which is *expanding*, i.e. all its eigenvalues λ_i have $|\lambda_i| > 1$, and that $|\det(A)| = m$ is an integer. Associated to any finite set $\mathcal{D} \subset \mathbb{R}^n$ with $|\mathcal{D}| = m$ there is then a unique compact set $T = T(A, \mathcal{D})$ which satisfies the set-valued functional equation

$$A(T) = \bigcup_{d\in\mathcal{D}} (T + d), \tag{1.1}$$

which is given explicitly by

$$T(A, \mathcal{D}) := \Big\{ \sum_{k=1}^{\infty} A^{-k} d_k : \ \text{all } d_k \in \mathcal{D} \Big\}. \tag{1.2}$$

We call the vectors $d \in \mathcal{D}$ *digits*, based on the viewpoint that (1.2) gives a multidimensional generalization of a radix expansion for the members of T. The set $T(A, \mathcal{D})$ is called a *self-affine tile* if it has positive Lebesgue measure. For most pairs $(A, \mathcal{D})$ the set $T(A, \mathcal{D})$ has Lebesgue measure 0, and only special pairs $(A, \mathcal{D})$ yield self-affine tiles.

The name "self-affine tile" refers to a geometric interpretation of the functional equation (1.1): it says that the affinely dilated set $A(T)$ is perfectly tiled by the m translates $T+\mathcal{D}$ of T, and that the overlaps $(T+d)\cap(T+d')$ have measure zero for distinct d, $d' \in \mathcal{D}$. Moreover it can then easily be shown using the functional equation that T tiles $\mathbb{R}^n$ by translation. Many examples of such tiles have fractal boundaries, cf. Falconer [9], Section 8.3.

A *lattice self-affine tile* is a self-affine tile $T = T(A, \mathcal{D})$ produced by a pair $(A, \mathcal{D})$ such that the difference set $\Delta(\mathcal{D}) = \mathcal{D} - \mathcal{D}$ is contained in a lattice Λ which is A-invariant in the sense that

$$A(\Lambda) \subseteq \Lambda \ . \tag{1.3}$$

Such self-affine tilings always give a tiling of $\mathbb{R}^n$ by a set of translations $\mathcal{S}$ contained in Λ. An *integral self-affine tile*[1] is a special case of lattice self-affine tile where $T = T(A, \mathcal{D})$ has

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[1] Strictly speaking the integrality property is associated to the pair $(A, \mathcal{D})$ For any self-affine tile there are infinitely many choices of $(A, \mathcal{D})$ with $T = T(A, \mathcal{D})$. Some of these pairs might be non-integral.

an integer matrix $A \in M_n(\mathbb{Z})$ and an integer digit set $\mathcal{D} \subseteq \mathbb{Z}^n$; in this case one can take $\Lambda = \mathbb{Z}^n$. The study of lattice self-affine tiles can always be reduced to the special case of integral self-affine tiles by an affine transformation, cf. Lemma 2.1 below.

This paper continues a study of integral self-affine tiles, and studies the question: which integral self-affine tiles can tile $\mathbb{R}^n$ with a lattice tiling?

One motivation for studying the structure of tilings concerns the construction of orthonormal wavelet bases in $\mathbb{R}^n$. Gröchenig and Madych [12] (cf. Theorem 1) showed that the characteristic function $\chi_T(x)$ of an integral self-affine tile T is a scaling function of a multiresolution analysis that produces an orthonormal wavelet basis of $L^2(\mathbb{R}^n)$ if and only if T tiles $\mathbb{R}^n$ with the lattice $\mathbb{Z}^n$. This is equivalent to that the Lebesgue measure $\mu(T(A, \mathcal{D})) = 1$.

In studying lattice tilings for integral self-affine tiles, without loss of generality we may restrict consideration to a special subclass of $(A, \mathcal{D})$ which we call *primitive*. Associate to any integral pair $(A, \mathcal{D})$ the A-invariant sublattice $\mathbb{Z}[A, \mathcal{D}]$ of $\mathbb{Z}^n$ that contains the difference set $\mathcal{D} - \mathcal{D}$. When $0 \in \mathcal{D}$ this is:

$$\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}[\mathcal{D}, A(\mathcal{D}), \ldots, A^{n-1}(\mathcal{D})] \ .$$

A pair $(A, \mathcal{D})$ is *primitive* if $\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}^n$, and we then call $\mathcal{D}$ a *primitive digit set* for A. Part I observed that if $T = T(A, \mathcal{D})$ is an integral self-affine tile there is another integral self-affine tile $\tilde{T} = T(\tilde{A}, \tilde{\mathcal{D}})$ with $(\tilde{A}, \tilde{\mathcal{D}})$ primitive and $0 \in \tilde{\mathcal{D}}$, such that

$$T = B(\tilde{T}) + v \ , \tag{1.4}$$

for some[2] $B \in M_n(\mathbb{Z})$ with $|\det(B)| \neq 0$ and some $v \in \mathbb{Z}^n$. This shows that T has a lattice tiling of $\mathbb{R}^n$ if and only if $\tilde{T}$ does. Consequently it suffices to study primitive digit sets.

Part I [18] introduced a distinction between *standard digit sets* and *nonstandard digit sets.* A primitive digit set is called *standard* if it forms a complete residue system (mod A), i.e. a complete set of coset representatives of the group $\mathbb{Z}^n / A(\mathbb{Z}^n)$, otherwise it is *nonstandard.* (The extension of this definition to imprimitive digit sets is given in part I.) All standard digit sets give self-affine tiles, i.e. the measure $\mu(T(A, \mathcal{D})) > 0$. However most nonstandard digit sets have $\mu(T(A, \mathcal{D})) = 0$. Part I showed that if $|\det(A)| = p$ is prime and $A^2\mathbb{Z}^n \not\supseteq p\mathbb{Z}^n$, then all nonstandard digit sets have $\mu(T(A, \mathcal{D})) = 0$. However when $|\det(A)| \neq p$ there exist nonstandard digit sets with $\mu(T(A, \mathcal{D})) > 0$. Part I also proved that the measure condition $\mu(T(A, \mathcal{D})) = 1$ necessary to get a multiresolution analysis giving a wavelet basis can never hold for nonstandard digit sets.

The distinction between standard and nonstandard digit sets is important for tiling questions. This paper considers only standard digit sets and proves:

Theorem 1.1. *Every integral self-affine tile T coming from a standard digit set gives a lattice tiling of $\mathbb{R}^n$ with some lattice $\Gamma \subseteq \mathbb{Z}^n$.*

This result was first conjectured by Gröchenig and Haas [11], who proved that it is true in the one-dimensional case. The hypothesis of a standard digit set cannot be removed from this conjecture, for there are integral self-affine tiles T coming from non-standard digit sets that have no lattice tilings, e.g. $A = [4]$ and $\mathcal{D} = \{0, 1, 8, 9\}$ has $T = [0, 1] \cup [2, 3]$.

To indicate why establishing Theorem 1.1 is a nontrivial problem in higher dimensions, we observe that iterating the functional equation (1.1) does not necessarily find lattice tilings. The functional equation (1.1) can be used to directly produce *self-replicating tilings* of $\mathbb{R}^n$,

[2] The columns of B then form a basis of the lattice $\mathbb{Z}[A, \mathcal{D}]$.

which are translation tilings of $\mathbb{R}^n$ consistent with (1.1) in the sense that for each tile $T+v$ in the tiling, the inflated tile $A(T+v)$ is a finite union of tiles in the tiling. (The concept of self-replicating tiling is due to Kenyon [16].) However the primitive pair $(A,\mathcal{D})$ with

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \mathcal{D} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}, \tag{1.5}$$

has a standard digit set $\mathcal{D}$, and the tile $T(A,\mathcal{D})$ has the property that *all* self-replicating tilings using $T(A,\mathcal{D})$ are non-periodic tilings, hence are not lattice tilings, cf. Lagarias and Wang [18], Example 2.3. Nevertheless, this particular tile does have a lattice tiling,[3] using the lattice $3\mathbb{Z}\oplus\mathbb{Z} = \{\left[\begin{smallmatrix} 3a \\ b\end{smallmatrix}\right] : a,b\in\mathbb{Z}\}$.

To place these results in a more general context, we remark that it remains an open question whether every tile T which tiles $\mathbb{R}^n$ by translation has a periodic tiling. (A *tile* is a compact set of positive measure, which is the closure of its interior, and has a boundary of measure zero.) Venkov [29] proved that every convex set T that tiles $\mathbb{R}^n$ by translation has a lattice tiling, and his result was also found by McMullen [22]. Nonconvex tiles need not have any lattice tilings, e.g. on $\mathbb{R}$, take $T=[0,1]\cup[2,3]$.

The contents of the paper are as follows. §2 describes a Fourier-analytic tiling criterion taken from Gröchenig and Haas [11], which implies that a lattice tiling exists only when a certain scaling operator has a nonconstant eigenfunction of eigenvalue 1. In §3 we suppose that such a nonconstant eigenfunction exists, and introduce a notion of *special eigenfunction* $f(x)$. A key to our approach is a result showing that the zero set Z_f of a special eigenfunction, when projected onto the torus $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$, is invariant under the linear map A^T (Lemma 3.2). The general idea of obtaining information from zero sets of special eigenfunctions goes back to Conze and Raugi [6]. In §4 we introduce the notion of of *stretched tile,* which is a tile whose smallest A-invariant lattice generated by the differenced digit set $\mathcal{D}-\mathcal{D}$ is $\mathbb{Z}^n$, but which has $\mu(T(A,\mathcal{D}))>1$. Stretched tiles $T(A,\mathcal{D})$ essentially correspond to the case where special eigenfunctions exist for $(A,\mathcal{D})$. We use a recent result of Cerveau, Conze and Raugi [4], together with Lemma 3.2, to prove that the zero sets of special eigenfunctions of stretched tiles contain translates of an A^T-invariant vector space of dimension ≥ 1 (Theorem 4.1). In §5 we explicitly construct a class of stretched tiles whose digit sets have a quasi-product form (Theorem 5.1). In §6 we use Theorem 4.1 to prove a structure theorem for those $(A,\mathcal{D})$ giving stretched tiles, which shows that they all essentially arise from the construction of §4 (Theorem 6.1). §6 uses this structure theorem to prove that all stretched tiles $T(A,\mathcal{D})$ give lattice tilings by some sublattice of $\mathbb{Z}^n$. Lattice tiling property of self-affine tiles is also discussed in a recent preprint of Conze, Hervé and Raugi [7].

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2. Fourier-Analytic Tiling Criterion

It is known that $T(A,\mathcal{D})$ tiles $\mathbb{R}^n$ by translation with some tiling set Γ satisfying

$$\Gamma \subseteq \mathbb{Z}[A,\mathcal{D}], \tag{2.1}$$

cf. Gröchenig and Haas [11], or Lagarias and Wang [17]. If $\Gamma=\mathbb{Z}[A,\mathcal{D}]$, then $T(A,\mathcal{D})$ tiles $\mathbb{R}^n$ with a lattice tiling, and this occurs if and only if the Lebesgue measure $\mu(T(A,\mathcal{D})$ of

[3] Kenyon [16] states a result (Theorem 12) which would imply the truth of the Lattice Tiling Conjecture, and which furthermore asserts that there always is a lattice tiling with an A-invariant lattice. However this result is false. The tile $(A,\mathcal{D})$ in (1.5) is a counterexample to it, as is shown in Lagarias and Wang [18], Section 4.

the tile is

$$\mu(T(A,\mathcal{D})) = [\mathbb{Z}^n : \mathbb{Z}[A,\mathcal{D}]] = \det(\mathbb{Z}[A,\mathcal{D}])\ . \tag{2.2}$$

Vince [30] and Gröchenig and Haas [11] give criteria for the equality $\Gamma = \mathbb{Z}[A,\mathcal{D}]$. We follow the latter, see Lemma 2.3 below.

As a preliminary fact we recall that the study of general lattice self-affine tiles can be reduced to the study of integral self-affine tiles that are primitive.

Lemma 2.1. *Let $T = T(A,\mathcal{D})$ be an integral self-affine tile in $\mathbb{R}^n$. Then there is an invertible affine transformation $L(x) = Bx + v$ such that $L(T) = \tilde{T}$, where $\tilde{T} = T(\tilde{A},\tilde{\mathcal{D}})$ is an integral self-affine tile with $0 \in \mathcal{D}$ and $(\tilde{A},\tilde{\mathcal{D}})$ primitive, i.e. $\tilde{A} \in M_n(\mathbb{Z})$, $\tilde{\mathcal{D}} \subset \mathbb{Z}^n$ and $\mathbb{Z}[\tilde{A},\tilde{\mathcal{D}}] = \mathbb{Z}^n$. Furthermore $\tilde{A}$ is similar to A over Q.*

Proof. This is Lemma 2.1 of Lagarias and Wang [18]. □

For a digit set $\mathcal{D}$ we define the *digit function* $g_{\mathcal{D}} : \mathbb{R}^n \to \mathbb{C}$ by

$$g_{\mathcal{D}}(x) := \frac{1}{|\mathcal{D}|}\sum_{d\in\mathcal{D}} \exp(2\pi i\langle d, x\rangle). \tag{2.3}$$

We also define the *correlation function* $u_{\mathcal{D}} : \mathbb{R}^n \to \mathbb{R}$ by

$$u_{\mathcal{D}}(x) := |g_{\mathcal{D}}(x)|^2 = \frac{1}{|\mathcal{D}|^2}\sum_{d,d'\in\mathcal{D}} \exp(2\pi i\langle\ d-d', x\rangle). \tag{2.4}$$

In the rest of this section, we always assume that $\mathcal{D}$ is a complete residue system (mod A). We also assume that $\mathcal{D}^T$ is some complete residue system (mod A^{T}).

Lemma 2.2. *For all $x \in \mathbb{R}^n$, we have*

$$\sum_{l\in\mathcal{D}^T} u_{\mathcal{D}}\big((A^T)^{-1}(x+l)\big) \equiv 1. \tag{2.5}$$

Proof. See Gröchenig and Haas [11], Lemma 5.1. □

We now define a linear operator $\hat{C}_{A,\mathcal{D}}$ on the space $\Omega(\mathbb{R}^n)$ of exponential polynomials, where $\Omega(\mathbb{R}^n)$ consists of all

$$f(x) = \sum_{m\in\mathbb{Z}^n} a_m \exp(2\pi i\langle m, x\rangle), \quad a_m \in \mathbb{R}, \tag{2.6}$$

with only finitely many $a_m \neq 0$. Define the *transfer operator* $\hat{C}_{A,\mathcal{D}} : \Omega(\mathbb{R}^n) \to \Omega(\mathbb{R}^n)$ by

$$\hat{C}_{A,\mathcal{D}} f(x) := \sum_{l\in\mathcal{D}^T} u_{\mathcal{D}}\big((A^T)^{-1}(x+l)\big) f\big((A^T)^{-1}(x+l)\big). \tag{2.7}$$

It is easy to check that $\hat{C}_{A,\mathcal{D}}$ is a linear operator that maps $\Omega(\mathbb{R}^n)$ into itself, by expanding the terms $u_{\mathcal{D}}(\cdot)$ using (2.4), and $\hat{C}_{A,\mathcal{D}}$ is independent of the choice of $\mathcal{D}^T$.

We will be concerned with the action of $\hat{C}_{A,\mathcal{D}}$ on the space $\Omega^+(\mathbb{R}^n)$ of real cosine polynomials

$$f(x) = \sum_{m\in\mathbb{Z}^n} a_m \cos(2\pi\langle m, x\rangle),\ a_m \in \mathbb{R}, \tag{2.8}$$

with only finitely many $a_m \neq 0$. This is exactly the set of functions $f(x)$ in $\Omega(\mathbb{R}^n)$ left fixed by the involution $Jf(x) = f(-x)$. It is easy to check that $\hat{C}_{A,\mathcal{D}}$ commutes with the involution J, hence it has $\Omega^+(\mathbb{R}^n)$ as an invariant subspace.

Lemma 2.2 shows that the constant functions are eigenfunctions of $\hat{C}_{A,\mathcal{D}}$ with eigenvalue 1. Gröchenig and Haas [11] (Proposition 5.3) give the following eigenfunction criterion for $T(A,\mathcal{D})$ to have a $\mathbb{Z}^n$-tiling:

Lemma 2.3 ($\mathbb{Z}^n$-Tiling Criterion). *$T(A,\mathcal{D})$ tiles $\mathbb{R}^n$ with a $\mathbb{Z}^n$-tiling if and only if the only solutions $f(x) \in \Omega^+(\mathbb{R}^n)$ of*

$$\hat{C}_{A,\mathcal{D}} f(x) = f(x) \tag{2.9}$$

are constant functions.

3. Zero Set of Eigenfunctions

Througout this section, A denotes an expanding matrix in $M_n(\mathbb{Z})$ and $\mathcal{D}$ denotes a complete residue system (mod A).

If $\mu(T(A,\mathcal{D})) = 1$ then $T(A,\mathcal{D})$ lattice tiles $\mathbb{R}^n$ with the lattice $\mathbb{Z}^n$, so we need only study the case when $\mu(T(A,\mathcal{D})) > 1$. Our basic approach to finding a lattice tiling is to study the structure of the zero set of a specially chosen nonconstant eigenfunction $f(x)$. This approach was used by Gröchenig and Haas in the one-dimensional case; they attribute the idea to Conze and Raugi [6].

Lemma 3.1. *Suppose that there exists a nonconstant $\tilde{f}(x) \in \Omega^+(\mathbb{R}^n)$ satisfying*

$$\hat{C}_{A,\mathcal{D}} \tilde{f}(x) = \tilde{f}(x). \tag{3.1}$$

Then there exists such an eigenfunction $f(x)$ satisfying

$$f(x) \geq 0 \qquad \text{and} \qquad f(0) > 0, \tag{3.2}$$

which has a nonempty (real) zero set $Z_f = \{x \in \mathbb{R}^n : f(x) = 0\}$.

Proof. Suppose that $\tilde{f}(x) \in \Omega^+(\mathbb{R}^n)$ is nonconstant and satisfies (3.1). Define

$$f_1(x) = \tilde{f}(x) - \min_{y \in \mathbb{R}^n} \tilde{f}(y), \quad f_2(x) = \max_{y \in \mathbb{R}^n} \tilde{f}(y) - \tilde{f}(x).$$

Clearly both $f_i(x) \geq 0$ for all $x \in \mathbb{R}^n$, and

$$f_2(x) + f_1(x) = \max_{y \in \mathbb{R}^n} \tilde{f}(y) - \min_{y \in \mathbb{R}^n} \tilde{f}(y) > 0.$$

Now we define $f(x)$ to be any one of the $f_1(x)$, $f_2(x)$ that satisfies $f_i(0) > 0$. The zero set Z_f is nonempty by construction, and our choice of $f(x)$ guarantees that $f(x) \geq 0$ and $f(0) > 0$, proving the lemma. $\square$

Remark. We call an $f(x)$ having the properties of Lemma 3.1 a *special eigenfunction* of $(A,\mathcal{D})$. The property $f(0) > 0$ and the periodicity of $f(x)$ (mod $\mathbb{Z}^n$) guarantees that $\mathbb{Z}^n \cap Z_f = \emptyset$, a fact that will be important later.

Lemma 3.2. *Let $f(x)$ be a special eigenfunction of $(A,\mathcal{D})$. Then*

$$Z_f \subseteq A^T(Z_f) + \mathbb{Z}^n. \tag{3.3}$$

Proof. If $x \in Z_f$ then $\hat{C}_{A,\mathcal{D}} f(x) = f(x) = 0$. Let $\mathcal{D}^T$ be a complete residue system $\pmod{A^T}$. Then by definition

$$\sum_{l \in \mathcal{D}^T} u_{\mathcal{D}}\big((A^T)^{-1}(x+l)\big) f\big((A^T)^{-1}(x+l)\big) = 0. \tag{3.4}$$

Since $f(z) \geq 0$ everywhere, every term on the right-hand sum must be zero. Now Lemma 2.2 implies that some $z^* = (A^T)^{-1}(x+l)$ gives

$$u_{\mathcal{D}}(z^*) > 0,$$

hence $f(z^*) = 0$. Now $A^T(z^*) = x + l$, so $x \in A^T(Z_f) + \mathbb{Z}^n$. Thus $Z_f \subseteq A^T(Z_f) + \mathbb{Z}^n$. □

Now, for each $l \in \mathbb{Z}^n$ define the map $\tau_l : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\tau_l(x) = (A)^{-1}(x+l). \tag{3.5}$$

Definition 3.1. *Let $\mathcal{D}^T$ be a complete residue system* $\pmod{A^T}$. *We call a subset Y of $\mathbb{R}^n$* τ-invariant with respect to $\mathcal{D}^T$ *if for any $x \in Y$,*

$$l \in \mathcal{D}^T \quad \text{and} \quad u_{\mathcal{D}}(\tau_l(x)) > 0 \quad \Longrightarrow \quad \tau_l(x) \in Y. \tag{3.6}$$

Y is minimal *if it does not contain a proper subset which is also τ-invariant with respect to $\mathcal{D}^T$.*

We shall simply call such a set Y *τ-invariant* when there is no ambiguity. If Y is *periodic*, i.e. $Y = Y + \mathbb{Z}^n$, then Y is τ-invariant with respect to some $\mathcal{D}^T$ implies that Y is τ-invariant with respect to all complete residue systems $\pmod{A^T}$. The zero set Z_f of a special eigenfunction f is always τ-invariant as a result of (3.4). Let $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. We call $\bar{Y} \subseteq \mathbb{T}^n$ *τ-invariant* if $\pi_n^{-1}(\bar{Y}) \subseteq \mathbb{R}^n$ is τ-invariant, where $\pi_n : \mathbb{R}^n \to \mathbb{T}^n$ is the cannonical covering map.

Gröchenig and Haas [11] settled the one-dimensional case of Theorem 1.1 by showing that $\mu(T(A,\mathcal{D})) > 1$ can never occur for primitive pair $(A, \mathcal{D})$ when $n = 1$. The essential part of their proof is contained in the following lemma.

Lemma 3.3. *If there exists a uniformly discrete nonempty invariant subset Y_f with $Y_f = Y_f + \mathbb{Z}^n$, then we have $\mathbb{Z}(A,\mathcal{D}) \subset \Gamma$, where Γ is a proper A-invariant lattice, and $\mathcal{D}$ is not primitive.*

Proof. We argue by contradiction. Suppose not, and let $Y_f \subseteq Z_f$ be a uniformly discrete nonempty τ-invariant set with $Y_f = Y_f + \mathbb{Z}^n$. Let $\bar{Y}_f := \pi_n(Y_f)$. Suppose that $\mathcal{D}^T$ is a complete residue system $\pmod{A^T}$. By Lemma 2.2, for each $y \in Y_f$ there exists at least one $l \in \mathcal{D}^T$ such that $u_{\mathcal{D}}(\tau_l(y)) > 0$, and so $\tau_l(y) \in Y_f$. Because $A^T(\tau_l(y)) \equiv y \pmod{\mathbb{Z}^n}$, we have therefore $A^T(Y_f) + \mathbb{Z}^n \supseteq Y_f$. Hence $A_*^T(\bar{Y}_f) \supseteq \bar{Y}_f$, where $A_*^T : \mathbb{T}^n \to \mathbb{T}^n$ is the induced map from A^T. This implies $A_*^T(\bar{Y}_f) = \bar{Y}_f$ because $\bar{Y}_f$ is finite, and so A_*^T acts as a permutation on $\bar{Y}_f$. Thus for any $y \in Y_f$ we have $(A^T)^k(y) \equiv y \pmod{\mathbb{Z}^n}$ for some finite k. Solving this equation shows that y is rational, i.e. $y \in \mathbb{Q}^n$.

We show that for each $y \in Y_f$ there exists exactly one $l \in \mathcal{D}^T$ such that $\tau_l(y) \in Y_f$. Suppose there were distinct $l_1, l_2 \in \mathcal{D}^T$ such that $\tau_{l_1}(y), \tau_{l_2}(y) \in Y_f$. Then we have $A^T(\tau_{l_1}(y)) \equiv A^T(\tau_{l_2}(y)) \equiv y \pmod{\mathbb{Z}^n}$. But $\tau_{l_1}(y) \not\equiv \tau_{l_2}(y) \pmod{\mathbb{Z}^n}$. This contradicts the fact that A_*^T is a permutation on $\bar{Y}_f$.

So now for any $y \in Y_f$ there exists an $l^* \in \mathcal{D}^T$ such that $u_{\mathcal{D}}(\tau_{l^*}(y)) > 0$ and $u_{\mathcal{D}}(\tau_l(y)) = 0$ for all $l \in \mathcal{D}^T$ and $l \neq l^*$; hence $u_{\mathcal{D}}(\tau_{l^*}(y)) = 1$. Since $y \in Y_f$ is arbitrarily chosen, we have

$$u_{\mathcal{D}}(y) = |g_{\mathcal{D}}(y)|^2 = 1, \qquad \text{all } y \in Y_f. \tag{3.7}$$

Using the definition (2.3) of $g_{\mathcal{D}}(x)$, and that $0 \in \mathcal{D}$, (3.7) holds if and only if

$$\langle d, y\rangle \equiv 0 \pmod 1, \qquad \text{all } d \in \mathcal{D} \text{ and all } y \in Y_f.$$

We use this fact to define a new lattice

$$\Gamma = \{w : w \in \mathbb{Z}^n \text{ and } \langle w, y\rangle \in \mathbb{Z} \text{ for all } y \in Y_f\}. \tag{3.8}$$

Because Y_f lies in finitely many $\mathbb{Z}^n$-equivalence classes, Γ is a full rank sublattice of $\mathbb{Z}^n$. Also $Y_f \cap \mathbb{Z}^n = \emptyset$, because $Z_f \cap \mathbb{Z}^n = \emptyset$, hence we have $\Gamma \neq \mathbb{Z}^n$. We next show that

$$A(\Gamma) \subseteq \Gamma. \tag{3.9}$$

To see this, given $w \in \Gamma$ and $y \in Y_f$, there is a $y_1 \in Y_f$ such that $A^T(y) = y_1 + l$ for some $l \in \mathbb{Z}^n$, and

$$\langle Aw, y\rangle = \langle w, A^T y\rangle = \langle w, y_1 + l\rangle = \langle w, y_1\rangle + \langle w, l\rangle \in \mathbb{Z},$$

since $\langle w, y\rangle \in \mathbb{Z}$ by definition of Γ, and $\langle w, l\rangle \in \mathbb{Z}$ since both $w, l \in \mathbb{Z}^n$.

Now (3.7) implies that $\mathcal{D} \subseteq \Gamma$ hence $\Delta(\mathcal{D}) = \mathcal{D} - \mathcal{D} \subseteq \Gamma$, so (3.9) implies that $\mathbb{Z}[A, \mathcal{D}] \subseteq \Gamma$. But Γ is a proper subset of $\mathbb{Z}^n$, contradicting $\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}^n$. $\square$

We now can settle the one-dimensional case, where $A = [\pm m]$ with $m \geq 2$, and a standard digit set $\mathcal{D} = \{d_1, \ldots, d_m\} \subseteq \mathbb{Z}$ is just a complete residue system (mod m). The primitivity condition $\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}$ is equivalent to

$$\gcd(\mathrm{d} - \mathrm{d}' : \mathrm{d}, \mathrm{d}' \in \mathcal{D}) = 1. \tag{3.10}$$

Theorem 3.4 (Gröchenig and Haas). *Suppose that $A = [\pm m]$ and $\mathcal{D}$ is a complete residue system* (mod m). *Set $d = \gcd(\mathrm{d} - \mathrm{d}' : \mathrm{d}, \mathrm{d}' \in \mathcal{D})$. Then $T(A, \mathcal{D})$ tiles $\mathbb{R}$ by the lattice $d\mathbb{Z}$, and $\mu(T(A, \mathcal{D})) = d$.*

Proof. We reduce to the case that $d = 1$ using Lemma 2.1. Now Lemma 3.3 applies to show that $\mu(T(A, \mathcal{D})) = 1$, because the real zero set Z_f of any nonconstant trigonometric polynomial must be discrete. $\square$

When the dimension $n \geq 2$ the case $\mu(T(A, \mathcal{D})) > 1$ can occur, as in the example (1.5) of §1.

4. Stretched Tiles and Hyperplane Zeros of Special Eigenfunctions

To prove that all standard digit sets give tiles having lattice tilings, it suffices to study the case of $(A, \mathcal{D})$ such that $\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}^n$ and $\mu(T(A, \mathcal{D})) > 1$.

Definition 4.1. *We call $T(A, \mathcal{D})$ a* stretched tile *if*

$$\mu(T(A, \mathcal{D})) > [\mathbb{Z}^n : \mathbb{Z}[A, \mathcal{D}]].$$

Lemma 2.3 and Lemma 3.1 combine to show that a stretched tile has a special eigenfunction. The proof of Theorem 1.1 rests on a special property of the real zero set of a special eigenfunction Z_f of a stretched tile which is that it contains translates of certain linear subspaces of $\mathbb{R}^n$, stated as Theorem 4.1 below.

It appears that the global structure of the set Z_f is a union of translates of rational subspaces of $\mathbb{R}^n$ of various dimensions. A *rational subspace* V of $\mathbb{R}^n$ is a linear space having a basis consisting of rational vectors $v \in \mathbb{Q}^n$. This would follow from:

Hyperplane Zeros Conjecture. *Let $h : \mathbb{R}^n \to \mathbb{C}$ be an analytic function that is periodic* $(\mathrm{mod}\ \mathbb{Z}^n)$*. Suppose that there is an expanding integer matrix A such that*

$$Z_h \subseteq A(Z_h) + \mathbb{Z}^n.$$

Then

$$Z_h = \bigcup_{i=1}^{m} (x_i + V_i) + \mathbb{Z}^n, \tag{4.1}$$

in which each $x_i \in \mathbb{R}^n$ and each V_i is a rational subspace of $\mathbb{R}^n$. (The V_i need not all have the same dimension.)

We derive a weak result in the direction of this conjecture for a special eigenfunction of a stretched tile, which will suffice to prove our main result.

Theorem 4.1. *Let $A \in M_n(\mathbb{Z})$ be expanding and $\mathcal{D}$ be a primitive complete residue system* $(\mathrm{mod}\ \mathrm{A})$*. Let $\mathcal{D}^T$ be a complete residue system* $(\mathrm{mod}\ \mathrm{A}^{\mathrm{T}})$*. Suppose that $\mu(T(A, \mathcal{D})) > 1$, and let $f(x)$ be a special eigenfunction for $(A, \mathcal{D})$. Then the real zero set Z_f contains a finite number of translates $\{y_i + W : 0 \le i \le k-1\}$ of an A^T-invariant proper rational subspace W of $\mathbb{R}^n$ such that:*

(i) $A^T y_{i+1} \equiv y_i \pmod{\mathbb{Z}^n}$ *for all $0 \le i \le k-1$, where $y_k := y_0$.*
(ii) *For every $x \in y_i + W$ we have*

$$\sum_{\substack{l \in \mathcal{D}^T \\ \tau_l(x) \in y_{i+1} + W + \mathbb{Z}^n}} u_{\mathcal{D}}(\tau_l(x)) = 1. \tag{4.2}$$

The main ingredient in the proof of this theorem is a result of Cerveau, Conze and Raugi [4]. First, we prove:

Lemma 4.2. *Let V be a subspace of $\mathbb{R}^n$. Then $\pi_n(V)$ is closed in $\mathbb{T}^n$ if and only if V is a rational subspace of $\mathbb{R}^n$.*

Proof. We first show that if V is a rational subspace of $\mathbb{R}^n$ then $\pi_n(V)$ is closed in $\mathbb{T}^n$. Let $w_1, w_2, \ldots, w_r \in \mathbb{Z}^n$ form a basis of V. Suppose that $z^* \in \mathbb{T}^n$ is in the closure of $\pi_n(V)$. Then we may find a sequence $\{x_j\}$ in V such that $\lim_{j\to\infty} \pi_n(x_j) = z^*$. Write

$$x_j = \sum_{k=1}^{r} b_{j,k} w_k.$$

Since all $w_k \in \mathbb{Z}^n$, we may choose all $b_{j,k} \in [0, 1)$. Therefore we can find a subsequence $\{j_m\}$ of $\{j\}$ such that

$$\lim_{m\to\infty} b_{j_m,k} = b_k^*, \qquad \text{all } 1 \le k \le r.$$

Let $x^* = \sum_{k=1}^{r} b_k^* w_k$. Clearly, $\pi_n(x^*) = z^*$. Hence $z^* \in \pi_n(W)$. Therefore $\pi_n(V)$ is closed in $\mathbb{T}^n$.

We next prove the following hypothesis: If $v \in \mathbb{R}^n$ then the closure of $\pi_n(\mathbb{R}v)$ in $\mathbb{T}^n$ is a rational subspace. To see this, let $v = [\beta_1, \ldots, \beta_n]^T$. Without loss of generality we assume that $\beta_1, \dots, \beta_r$ are linearly independent over $\mathbb{Q}$ while $\beta_k = \sum_{j=1}^{r} a_{k,j}\beta_j$ with $a_{k,j} \in \mathbb{Q}$ for all $1 \le k \le n$. The set

$$\left\{ m \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} \pmod{\mathbb{Z}^r} :\ m \in \mathbb{Z} \right\}$$

is dense in $\mathbb{T}^r$ (see Cassels [3], Theorem I, p. 64). Now let $V_0 = \{Ax : x \in \mathbb{R}^r\}$ where $A = [a_{k,j}]$. Then V_0 is a rational subspace of $\mathbb{R}^n$, and $\pi_n(V_0)$ is contained in the closure of $\pi_n(\mathbb{R}v)$. But $\pi_n(V_0)$ is closed and $V_0 \supseteq \mathbb{R}v$. Hence the closure of $\pi_n(\mathbb{R}v)$ is $\pi_n(V_0)$, proving the hypothesis.

Finally, let $v_1, \dots, v_r$ be a basis of V. Suppose that $\bar{W}_j$ is the closure of $\pi_n(\mathbb{R}v_j)$ in $\mathbb{T}^n$. Then the closure of $\pi_n(V)$ contains $\bar{W}_1 + \cdots + \bar{W}_r$. But $\bar{W}_1 + \cdots + \bar{W}_r$ is closed in $\mathbb{T}^n$ because it is a rational subspace, and it contains $\pi_n(V)$. Hence the closure of $\pi_n(V)$ is $\bar{W}_1 + \cdots + \bar{W}_r$, proving the lemma. ∎

Corollary 4.3. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be continuous and periodic* $\pmod{\mathbb{Z}^n}$. *Suppose that V is a subspace of $\mathbb{R}^n$ such that $v_0 + V \subseteq Z_f$ where $v_0 \in \mathbb{R}^n$. Then $v_0 + W \subseteq Z_f$ where W is the smallest rational subspace of $\mathbb{R}^n$ containing V.*

Proof. First, let $\{V_\alpha\}$ be a set of rational subspaces of $\mathbb{R}^n$. Then $\pi_n(\bigcap_\alpha V_\alpha) = \bigcap_\alpha \pi_n(V_\alpha)$ is closed in $\mathbb{T}^n$; so $\bigcap_\alpha V_\alpha$ must be a rational subspace of $\mathbb{R}^n$. This implies that the minimal rational subspace W containing V exists. Since $f(x)$ is periodic $\pmod{\mathbb{Z}^n}$ we may view it as a continuous function defined on $\mathbb{T}^n$. Now, $\pi_n(v_0) + \pi_n(W)$ is the closure of $\pi_n(v_0) + \pi_n(V)$ in $\mathbb{T}^n$. Hence $\pi_n(v_0) + \pi_n(W)$ is in the zero set of $f : \mathbb{T}^n \to \mathbb{C}$. Thus $v_0 + W \subseteq Z_f$. ∎

Proof of Theorem 4.1. We construct a nonempty minimal compact τ-invariant set Y with respect to $\mathcal{D}^T$ in Z_f as follows, where $f(x)$ is a special eigenfunction of $(A, \mathcal{D})$. Take any point $x_0 \in Z_f$ and set $X_0 = \{x_0\}$ and recursively define the finite sets $\{X_j : j \ge 0\}$ by letting X_j consist of all points x_j such that $x_j = \tau_l(x_{j-1})$ with $x_{j-1} \in X_{j-1}$ and $l \in \mathcal{D}^T$ such that $u_{\mathcal{D}}(x_j) > 0$. Then the τ-invariance of Z_f with respect to $\mathcal{D}^T$ gives $X_j \subseteq Z_f$ for all $j \ge 0$. The set $\bigcup_{j=0}^{\infty} X_j$ lies in a bounded region in $\mathbb{R}^n$ because the mappings τ_l are uniformly contracting with respect to a suitable norm in $\mathbb{R}^n$ (cf. Lagarias and Wang [18], Section 3 or Conze and Raugi [4]). Now let Y_0 be the set of all cluster points of sequences $\{x_j^* : x_j^* \in X_j\}$. Then Y_0 is a compact set, and we show that Y_0 is τ-invariant with respect to $\mathcal{D}^T$. If $y \in Y_0$ and $u_{\mathcal{D}}(\tau_l(y)) > 0$ where $l \in \mathcal{D}^T$, take a subsequence $x_{j_k} \in X_{j_k}$ that converges to y, so that $\tau_l(x_{j_k}) \to \tau_l(y)$. Now $u_{\mathcal{D}}(\tau_l(x_{j_k})) > 0$ for k sufficiently large, hence $\tau_l(x_{j_k}) \in X_{j_k+1}$; so we may construct a sequence having $\tau_l(y)$ as a cluster point, proving $\tau_l(y) \in Y_0$. The existence of a nonempty minimal compact τ-invariant set Y with respect to $\mathcal{D}^T$ contained in Y_0 follows by a Zorn's Lemma argument.

It follows from Theorem 2.8 of Cerveau et al [4] that there exists an A^T-invariant subspace V and $\{y_i \in Y : 0 \le i \le k-1\}$ satisfying (i) such that

$$Y \subseteq \bigcup_{i=0}^{k-1} (y_i + V) \subseteq Z_f, \tag{4.3}$$

with the property that the set $\bigcup_{i=0}^{k-1}(y_i + V)$ is τ-invariant with respect to $\mathcal{D}^T$. Now let W be the smallest rational subspace of $\mathbb{R}^n$ containing V. Since $A^T(W)$ is also a rational subspace containing V and it has the same dimension as W, we must have $A^T(W) = W$.

By Corollary 4.3,

$$Y \subseteq \bigcup_{i=0}^{k-1} (y_i + W) \subseteq Z_f. \tag{4.4}$$

Moreover, since $\pi_n\big(\bigcup_{i=0}^{k-1}(y_i+W)\big)$ is the closure of $\pi_n\big(\bigcup_{i=0}^{k-1}(y_i+V)\big)$ in $\mathbb{T}^n$, we conclude that $\bigcup_{i=0}^{k-1}(y_i+W)$ is τ-invariant with respect to $\mathcal{D}^T$.

We now prove property (ii). Let $x \in y_{i-1}+W$. We show that for any $l \in \mathcal{D}^T$, $u_{\mathcal{D}}(\tau_l(x)) > 0$ only if $\tau_l(x) \in y_i + W + \mathbb{Z}^n$. Suppose this is false, then there exists an $l^* \in \mathcal{D}^T$ with $u_{\mathcal{D}}(\tau_{l^*}(x)) > 0$ such that $\tau_{l^*}(x) \notin y_i + W + \mathbb{Z}^n$. The τ-invariance of $\bigcup_{i=0}^{k-1}(y_i+W)$ with respect to $\mathcal{D}^T$ implies then that $\tau_{\tilde{l}}(x) \in y_j + W$ for some j where $y_i + W \neq y_j + W$. Hence $x \in A^T(y_j + W + \mathbb{Z}^n) \subseteq y_{j-1} + W + \mathbb{Z}^n$. But this could happen only if

$$y_{i-1} + W + \mathbb{Z}^n = y_{j-1} + W + \mathbb{Z}^n.$$

By applying the operator $(A^T)^{k-1}$ to the above equation, we obtain

$$y_i + W + \mathbb{Z}^n = y_j + W + \mathbb{Z}^n,$$

a contradiction. property (2) now follows immediately from Lemma 2.2. $\square$

5. Stretched Tiles and Quasi-Product Form Digit Sets

Our object in this section is to present a large class of pairs $(A,\mathcal{D})$ giving stretched tiles $T(A,\mathcal{D})$. In §6 we shall then prove a structure theorem asserting that all pairs $(A,\mathcal{D})$ with $\mathbb{Z}[A,\mathcal{D}] = \mathbb{Z}^n$ and $\mu(T(A,\mathcal{D})) > 1$ essentially arise from this class.

Suppose now that A is an expanding integer matrix having the block-triangular form

$$A = \begin{bmatrix} A_1 & 0 \\ C & A_2 \end{bmatrix}, \tag{5.1}$$

where A_1 and A_2 are $r \times r$ and $(n-r)\times(n-r)$, respectively, with $1 \leq r \leq n-1$. We say that a digit set $\mathcal{D}$ for A is of *quasi-product form* if it has the form

$$\mathcal{D} := \left\{ \begin{bmatrix} a_i \\ b_i \end{bmatrix} + \begin{bmatrix} 0 \\ Qc_{i,j} \end{bmatrix} : \ 1 \leq i \leq |\det(A_1)|, \ 1 \leq j \leq |\det(A_2)| \right\}, \tag{5.2}$$

with the properties:

i. $\{a_i\} \subset \mathbb{Z}^r$ is a complete residue system (mod A_1), and $\{b_i\} \subset \mathbb{Z}^{n-r}$.
ii. $c_{i,j} \in \mathbb{Z}^{n-r}$ for all i,j and for each i the set $\{Qc_{i,j} : 1 \leq j \leq |\det(A_2)|\}$ is a complete residue system (mod A_2).
iii. $Q \in M_{n-r}(\mathbb{Z})$ has $|\det(Q)| \geq 2$ and $A_2 Q = Q\tilde{A}_2$ for some $\tilde{A}_2 \in M_{n-r}(\mathbb{Z})$.

The conditions (1), (2) imply that $\mathcal{D}$ is necessarily a standard digit set.

Theorem 5.1. *Let A be an expanding integer matrix of block-triangular form*

$$A = \begin{bmatrix} A_1 & 0 \\ C & A_2 \end{bmatrix}, \tag{5.3}$$

where A_1 and A_2 are $r \times r$ and $(n-r)\times(n-r)$, respectively. Suppose that $\mathcal{D}$ is a primitive standard digit set for A which is of quasi-product form (5.2). Then $|\det(Q)|$ divides $\mu(T(A,\mathcal{D}))$, so that $\mu(T(A,\mathcal{D})) > 1$.

Proof. Suppose that $\mathcal{D}$ is of quasi-product form. Consider the block-diagonal matrix

$$\tilde{A} := \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

together with the new digit set

$$\tilde{\mathcal{D}} := \left\{ \begin{bmatrix} a_i \\ Qc_{i,j} \end{bmatrix} : \ 1 \le i \le |\det(A_1)|, \ 1 \le j \le |\det(A_2)| \right\}.$$

We pair the digits of $\mathcal{D}$ and $\tilde{\mathcal{D}}$ by

$$d = \begin{bmatrix} a_i \\ b_i + Qc_{i,j} \end{bmatrix} \in \mathcal{D} \quad \Longleftrightarrow \quad \tilde{d} = \begin{bmatrix} a_i \\ Qc_{i,j} \end{bmatrix} \in \tilde{\mathcal{D}}.$$

There is a simple relationship between $T(A, \mathcal{D})$ and $T(\tilde{A}, \tilde{\mathcal{D}})$, which implies that

$$\mu(T(A, \mathcal{D})) = \mu(T(\tilde{A}, \tilde{\mathcal{D}})). \tag{5.4}$$

Define a map $\phi : T(A, \mathcal{D}) \to T(\tilde{A}, \tilde{\mathcal{D}})$ by $\phi(x) = \tilde{x}$, where if

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \sum_{k=1}^{\infty} A^{-k} d_k, \quad d_k \in \mathcal{D},$$

then

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \sum_{k=1}^{\infty} \tilde{A}^{-k} \tilde{d}_k, \quad \tilde{d}_k \in \tilde{\mathcal{D}},$$

where d_k and $\tilde{d}_k$ are paired digits (cf. (1.2)). Now

$$A^{-k} = \begin{bmatrix} A_1^{-k} & 0 \\ C_k & A_2^{-k} \end{bmatrix}, \quad \tilde{A}^{-k} = \begin{bmatrix} A_1^{-k} & 0 \\ 0 & A_2^{-k} \end{bmatrix},$$

hence

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - \psi(x_1) \end{bmatrix} \tag{5.5}$$

where

$$\psi(x_1) = \sum_{k=1}^{\infty} (C_k a_{i_k} + A_1^{-k} b_{i_k}).$$

The function $\psi : T(A_1, \{a_i\}) \to \mathbb{R}^{n-r}$ is easily checked to be a measurable function, hence (5.4) follows from (5.5) using Fubini's theorem.

Next we define the expanding matrix

$$\widehat{A} = \begin{bmatrix} A_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix},$$

where $\tilde{A}_2 \in M_{n-r}(\mathbb{Z})$ and $A_2 Q = Q\tilde{A}_2$, together with the digit set

$$\widehat{\mathcal{D}} := \left\{ \begin{bmatrix} a_i \\ c_{i,j} \end{bmatrix} : \ 1 \le i \le |\det(A_1)|, \ 1 \le j \le |\det(A_2)| \right\}.$$

Set

$$\tilde{Q} := \begin{bmatrix} I_r & 0 \\ 0 & Q \end{bmatrix},$$

and we then have

$$\tilde{A} = \tilde{Q}\widehat{A}\tilde{Q}^{-1}, \quad \tilde{\mathcal{D}} = \tilde{Q}(\widehat{\mathcal{D}}).$$

So it follows from Lemma 2.1 that

$$T(\tilde{A},\tilde{\mathcal{D}}) = \tilde{Q}T(\widehat{A},\widehat{\mathcal{D}}), \tag{5.6}$$

and hence

$$\mu(T(\tilde{A},\tilde{\mathcal{D}})) = |\det(Q)|\mu(T(\widehat{A},\widehat{\mathcal{D}})),$$

since $|\det(\tilde{Q})| = |\det(Q)|$. Combining this with (5.4) completes the proof. □

Remark. The name *stretched tile* is suggested by (5.6), which shows that in a weak sense the tile $T(A,\mathcal{D})$ is stretched by the matrix $\tilde{Q}$ along the $\mathbb{R}^{n-r}$-coordinate directions. See Theorem 6.1 for the general case.

As an example of Theorem 5.1, consider the pair $(A,\mathcal{D})$ of (1.5). Let $P = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. The the digit set $\mathcal{D}^T = P(\mathcal{D})$ for the matrix $A^T = PAP^{-1}$ is a standard digit set of quasi-product form with $Q = [3]$. Theorem 5.1 asserts that 3 divides $\mu(T(A^T,\mathcal{D}^T))$, and so 3 divides $\mu(T(A,\mathcal{D}))$. In fact, $\mu(T(A,\mathcal{D})) = 3$.

6. Structure Theorem for Stretched Tiles

We now use Theorem 4.1 to prove a structure theorem concerning stretched tiles, which is a converse to Theorem 5.1.

Theorem 6.1. *Let $\mathcal{D}$ be a primitive standard digit set for the expanding matrix $A \in M_n(\mathbb{Z})$, and suppose that $\mu(T(A,\mathcal{D})) > 1$. Then there exists a matrix $P \in GL(n,\mathbb{Z})$ such that the following two conditions hold.*

1. *There is some r with $1 \le r \le n-1$ such that*

$$PAP^{-1} = \begin{bmatrix} B_1 & 0 \\ C & B_2 \end{bmatrix}, \tag{6.1}$$

where B_1, B_2 are $r \times r$ and $(n-r) \times (n-r)$ expanding integer matrices, respectively, and C is an $(n-r) \times r$ integer matrix.

2. *The digit set $P(\mathcal{D})$ of PAP^{-1} is of quasi-product form.*

Before proving this result, we derive a corollary. Write $A_1 \sim_{\mathbb{Z}} A_2$ to mean A_1 is integrally similar to A_2, i.e. there exists some $Q \in GL(n,\mathbb{Z})$ such that $A_2 = QA_1Q^{-1}$. We say that A is *(integrally) reducible* if

$$A \sim_{\mathbb{Z}} \begin{bmatrix} A_1 & 0 \\ C & A_2 \end{bmatrix}, \tag{6.2}$$

where A_1 and A_2 are nonempty. We call A *irreducible* if it is not integrally reducible.

Corollary 6.2. *Suppose that the expanding matrix $A \in M_n(\mathbb{Z})$ is irreducible. Then for all primitive standard digit sets $\mathcal{D}$ the tile $T(A,\mathcal{D})$ lattice tiles $\mathbb{R}^n$ with lattice $\mathbb{Z}^n$.*

Proof. If $\mu(T(A,\mathcal{D})) > 1$ then (6.1) shows that A is integrally reducible, which contradicts the irreducibility of A. Thus $\mu(T(A,\mathcal{D})) = 1$. □

A sufficient condition for irreducibility of A is that the characteristic polynomial of A is irreducible over $\mathbb{Q}$. Using this criterion any expanding matrix A with $|\det(A)| = p$ a prime is irreducible, because if a decomposition (6.2) existed then $|\det(A_1)| > 1$ and $|\det(A_2)| > 1$.

Lemma 6.3. *Let $\mathcal{D}$ be a primitive standard digit set for A. Suppose that $P \in M_n(\mathbb{Z})$ is unimodular. Then $P(\mathcal{D})$ is a primitive standard digit set for PAP^{-1}. Furthermore,*

$$g_{P(\mathcal{D})}(x) = g_{\mathcal{D}}(P^T x), \qquad u_{P(\mathcal{D})}(x) = u_{\mathcal{D}}(P^T x). \tag{6.3}$$

Proof. Since $\mathcal{D}$ is a complete residue system (mod A), for distinct $d_1, d_2 \in \mathcal{D}$ we have $d_1 - d_2 \notin A(\mathbb{Z}^n)$. Hence $P(d_1 - d_2) \notin PAP^{-1}(\mathbb{Z}^n)$, so $P(\mathcal{D})$ is a complete residue system (mod PAP^{-1}). It is primitive because

$$\mathbb{Z}[PAP^{-1}, P(\mathcal{D})] = P\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}^n.$$

Now for any $x \in \mathbb{R}^n$,

$$g_{P(\mathcal{D})}(x) = \frac{1}{|\mathcal{D}|} \sum_{d \in \mathcal{D}} \exp(2\pi i \langle Pd, x\rangle) = \frac{1}{|\mathcal{D}|} \sum_{d \in \mathcal{D}} \exp(2\pi i \langle d, P^T x\rangle) = g_{\mathcal{D}}(P^T x).$$

So $g_{P(\mathcal{D})}(x) = g_{\mathcal{D}}(P^T x)$. Similarly, $u_{P(\mathcal{D})}(x) = u_{\mathcal{D}}(P^T x)$. □

Proof of Theorem 6.1. Since $\mu(T(A, \mathcal{D})) > 1$ there exists a special eigenfunction $f(x)$ for $(A, \mathcal{D})$ by Lemma 3.1. Now Theorem 4.1 states that there exists a rational subspace W of $\mathbb{R}^n$ having $\dim(W) = r$ with $1 \le r \le n-1$, and with $A^T(W) = W$ such that Z_f contains at least one translate of W. It is well-known that one can choose a unimodular matrix $P_1 \in GL(n, \mathbb{Z})$ that maps a given rational subspace W onto the first k-coordinate axes, i.e.

$$P_1(W) = E_r := \left\{ \begin{bmatrix} x_r \\ 0 \end{bmatrix} : x_r \in \mathbb{R}^r \right\}. \tag{6.4}$$

This directly follows from the Hermite normal form decomposition for a rational basis of the vector space W, see Schrijver [25], Theorem 4.1 and Corollary 4.3b. Now W is an invariant subspace of A^T, hence

$$P_1 A^T P_1^{-1} = \begin{bmatrix} B_1^T & C^T \\ 0 & B_2^T \end{bmatrix}, \tag{6.5}$$

for integer matrices B_1^T, B_2^T and C^T. Therefore, taking the transpose yields

$$PAP^{-1} = \begin{bmatrix} B_1 & 0 \\ C & B_2 \end{bmatrix},$$

with $P = (P_1^{-1})^T$. Both B_1 and B_2 are expanding because A is expanding. This proves (1).

We now prove (2). Let

$$B := PAP^{-1} = \begin{bmatrix} B_1 & 0 \\ C & B_2 \end{bmatrix}$$

and $\mathcal{E} := P(\mathcal{D})$. Then $\mathcal{E}$ is a primitive standard digit set for B. We have $u_{\mathcal{E}}(x) = u_{\mathcal{D}}(P^T x)$.

Our object is to analyze the structure of the digit set $\mathcal{E}$, to eventually prove that it is of quasi-product form. For any vector $v \in \mathbb{R}^n$ the notation $v = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$ always means that $v_1 \in \mathbb{R}^r$ and $v_2 \in \mathbb{R}^{n-r}$.

Let $\mathcal{E}_1^T$, $\mathcal{E}_2^T$ be complete residue systems (mod $\mathrm{B}_1^{\mathrm{T}}$) and (mod $\mathrm{B}_2^{\mathrm{T}}$), respectively. Let

$$\mathcal{E}^T := \mathcal{E}_1^T \oplus \mathcal{E}_2^T = \left\{ b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} : b_1 \in \mathcal{E}_1^T,\ b_2 \in B_2^T \right\}. \tag{6.6}$$

Then the fact that B^T is block upper-triangular implies that $\mathcal{E}^T$ a complete residue system (mod B^T). Let $\mathcal{D}^T = (P^T)^{-1}\mathcal{E}^T$. Then $\mathcal{D}^T$ a complete residue system (mod A^T), because $A^T = P^T B^T (P^T)^{-1}$.

Now, let $\{y_j + W : 0 \le j \le k-1\}$ satisfy the properties of Theorem 4.1. We have $A_*^T \circ \pi_n(y_{j+1}) = \pi_n(y_j)$ where $y_k := y_0$, and

$$\sum_{\substack{l \in \mathcal{D}^T \\ (A^T)^{-1}(x+l) \in y_{j+1}+W+\mathbb{Z}^n}} u_{\mathcal{D}}((A^T)^{-1}(x+l)) \equiv 1, \qquad x \in y_j + W. \tag{6.7}$$

Applying the transformations $v_j = (P^T)^{-1}y_j$ and $l = (P^T)^{-1}b$ with $b \in \mathcal{E}^T$ in (6.7), we may rewrite (6.7) as

$$\sum_{\substack{b \in \mathcal{E}^T \\ (B^T)^{-1}(x+b) \in v_{j+1}+E_r+\mathbb{Z}^n}} u_{\mathcal{E}}((B^T)^{-1}(x+b)) \equiv 1, \qquad x \in v_j + E_r, \tag{6.8}$$

using (6.4) and Lemma 6.3.

We proceed to simplify the formula (6.8). Choose $x \in v_j + E_r$ and define

$$\Lambda_j := \{m \in \mathbb{Z}^n : (B^T)^{-1}(x+m) \in v_{j+1} + E_r + \mathbb{Z}^n\}. \tag{6.9}$$

We show that Λ_j is well-defined independent of the choice of $x \in v_j + E_r$. More precisely, denote

$$v_j = \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix}, \qquad 0 \le j \le k$$

where $\alpha_k := \alpha_0$ and $\beta_k := \beta_0$. Then

$$\Lambda_j = \begin{bmatrix} 0 \\ n_j^* \end{bmatrix} + \Lambda \tag{6.10}$$

in which Λ is the lattice $\mathbb{Z}^r \oplus B_2^T(\mathbb{Z}^{n-r})$ and

$$n_j^* := B_2^T \beta_{j+1} - \beta_j. \tag{6.11}$$

To prove these facts, let $x = \left[\begin{smallmatrix} x_1 \\ \beta_j \end{smallmatrix}\right] \in v_j + E_r$. Then $m \in \Lambda_j$ if and only if

$$\begin{bmatrix} B_1^T & C^T \\ 0 & B_2^T \end{bmatrix}^{-1} \left(\begin{bmatrix} x_1 \\ \beta_j \end{bmatrix} + \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \right) = \begin{bmatrix} \alpha_{j+1} \\ \beta_{j+1} \end{bmatrix} + \begin{bmatrix} z \\ k_2 \end{bmatrix}$$

for some $z \in \mathbb{R}^r$ and $k_2 \in \mathbb{Z}^{n-r}$. So the condition for $m \in \Lambda_j$ is

$$(B_2^T)^{-1}(\beta_j + m_2) \equiv \beta_{j+1} \pmod{\mathbb{Z}^{n-r}}.$$

The above is equivalent to

$$\beta_j + m_2 \equiv B_2^T \beta_{j+1} \pmod{B_2^T(\mathbb{Z}^{n-r})},$$

i.e. $m_2 \equiv n_j^* \pmod{B_2^T(\mathbb{Z}^{n-r})}$, which gives (6.10) and (6.11).

Using these formulae, the identity (6.8) becomes

$$\sum_{b_1 \in \mathcal{E}_1^T} \sum_{\substack{b_2 \in \mathcal{E}_2^T \\ b_2 - n_j^* \in B_2^T(\mathbb{Z}^{n-r})}} u_{\mathcal{E}}((B^T)^{-1}(x + \left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right])) \equiv 1 \tag{6.12}$$

for $x \in v_j + E_r$. Note that for each $0 \le j \le k-1$ there is exactly one $b_2 \in \mathcal{E}_2^T$ such that $b_2 \equiv n_j^* \pmod{\mathrm{B}_2^{\mathrm{T}}(\mathbb{Z}^{\mathrm{n-r}})}$; denote this $b_2 \in \mathcal{E}_2^T$ by $b_{2,j}^*$. Then (6.12) is reduced further to

$$\sum_{b_1 \in \mathcal{E}_1^T} u_{\mathcal{E}}(w + (B^T)^{-1} \begin{bmatrix} b_1 \\ 0 \end{bmatrix}) \equiv 1, \tag{6.13}$$

where $w := (B^T)^{-1}(x + \begin{bmatrix} 0 \\ b_{2,j}^* \end{bmatrix})$ and $x \in v_j + E_r$.

We now use (6.12) to establish a series of claims.

Claim 1. *Suppose that $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ and $d' = \begin{bmatrix} d_1' \\ d_2' \end{bmatrix}$ are two digits in $\mathcal{E}$ such that $d_1 - d_1' \in B_1(\mathbb{Z}^r)$. Then $d_1 = d_1'$ and*

$$\langle d_2 - d_2', \beta_j \rangle \equiv 0 \pmod 1, \qquad 0 \le j \le k-1. \tag{6.14}$$

Proof of Claim 1. We make use of the *orthogonality relations* [4] on the abelian group $\mathbb{Z}^r/B_1(\mathbb{Z}^r)$: For all $m = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \in \mathbb{Z}^n$, we have

$$\sum_{b_1 \in \mathcal{E}_1^T} \exp(2\pi i \langle \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \begin{bmatrix} (B_1^T)^{-1} b_1 \\ 0 \end{bmatrix} \rangle) = \begin{cases} |\det(B_1)| & \text{if } m_1 \in B_1(\mathbb{Z}^r), \\ 0 & \text{if } m_1 \notin B_1(\mathbb{Z}^r). \end{cases}$$

Define

$$\mathcal{F} := \{(d, d') \in \mathcal{E} \times \mathcal{E} : d - d' \in B_1(\mathbb{Z}^r) \oplus \mathbb{Z}^{n-r}\}.$$

Using the orthogonality relation above and the definition

$$u_{\mathcal{E}}(x) = \frac{1}{|\det(B)|^2} \sum_{d, d' \in \mathcal{E}} \exp(2\pi i \langle d - d', x \rangle),$$

we obtain

$$\begin{aligned} & \sum_{b_1 \in \mathcal{E}_1^T} \sum_{d, d' \in \mathcal{E}} u_{\mathcal{E}} \left(w + \begin{bmatrix} (B_1^T)^{-1} b_1 \\ 0 \end{bmatrix} \right) \\ = \; & \frac{1}{|\det(B)|^2} \sum_{b_1 \in \mathcal{E}_1^T} \sum_{d, d' \in \mathcal{E}} \exp\left(2\pi i \langle d - d', w + \begin{bmatrix} (B_1^T)^{-1} b_1 \\ 0 \end{bmatrix} \rangle \right) \\ = \; & \frac{1}{|\det(B)|^2} \sum_{d, d' \in \mathcal{E}} \sum_{b_1 \in \mathcal{E}_1^T} \exp\left(2\pi i \langle d - d', w + \begin{bmatrix} (B_1^T)^{-1} b_1 \\ 0 \end{bmatrix} \rangle \right) \\ = \; & \frac{1}{|\det(B_1)| |\det(B_2)|^2} \sum_{(d, d') \in \mathcal{F}} \exp(2\pi i \langle d - d', w \rangle). \end{aligned}$$

Now, the above equation combines with (6.12) to give

$$\sum_{(d, d') \in \mathcal{F}} \exp(2\pi i \langle d - d', w \rangle) \equiv |\det(B_1)| |\det(B_2)|^2. \tag{6.15}$$

We next show that

$$|\mathcal{F}| = |\det(B_1)| \, |\det(B_2)|^2, \tag{6.16}$$

which will force all the exponentials on the left side of (6.15) to be 1. To prove (6.16) note that $\mathcal{E}$ is a complete residue system (mod B); hence the set $\mathcal{F}$ viewed as a subset of

[4] The functions $\chi_{b_1}(m_1) := \exp(2\pi i \langle m_1, (B_1^T)^{-1} b_1 \rangle)$ for $b_1 \in \mathcal{E}_1^T$ form a complete set of characters on $\mathbb{Z}^r/B_1(\mathbb{Z}^r)$.

$\mathbb{Z}^n/B(\mathbb{Z}^n) \times \mathbb{Z}^n/B(\mathbb{Z}^n)$ is a subgroup. Its quotient group is isomorphic to $\mathbb{Z}^r/B_1(\mathbb{Z}^r)$, which has size $|\det(B_1)|$, so (6.16) follows.

Now we know that $\exp(2\pi i\langle d-d', w\rangle) = 1$ for all pairs $(d, d') \in \mathcal{F}$, so

$$\langle d-d', w\rangle \equiv 0 \pmod 1. \tag{6.17}$$

For $x \in v_j + E_r$, write $x = \left[\begin{smallmatrix} x_1 \\ \beta_j \end{smallmatrix}\right]$. Then

$$w = (B^T)^{-1}\Big(\begin{bmatrix} x_1 \\ \beta_j \end{bmatrix} + \begin{bmatrix} 0 \\ b^*_{2,j} \end{bmatrix}\Big) = \begin{bmatrix} z_1 \\ (B_2^T)^{-1}(\beta_j + b^*_{2,j}) \end{bmatrix}$$

for some $z_1 \in \mathbb{R}^r$. Notice that $b^*_{2,j} - n^*_j \in B_2^T(\mathbb{Z}^{n-r})$. By (6.11),

$$(B_2^T)^{-1}(\beta_j + b^*_{2,j}) = (B_2^T)^{-1}(\beta_j + n^*_j + B_2^T m_2) = \beta_{j+1} + m_2$$

for some $m_2 \in \mathbb{Z}^{n-r}$. Hence

$$w = \begin{bmatrix} z_1 \\ \beta_{j+1} + m_2 \end{bmatrix}.$$

(6.17) now becomes

$$\langle d_1 - d_1', z_1\rangle + \langle d_2 - d_2', \beta_{j+1}\rangle \equiv 1 \pmod 1.$$

But $\langle d_1 - d_1', z_1\rangle$ is a continuous function of $z_1 \in \mathbb{R}^r$, and as x_1 runs through $\mathbb{R}^r$ so does z_1. Hence we must have $d_1 - d_1' = 0$, and

$$\langle d_2 - d_2', \beta_{j+1}\rangle \equiv 1 \pmod 1,$$

proving Claim 1. □

Claim 2. *There exists a B_2-invariant proper sublattice Γ of $\mathbb{Z}^{n-r}$ such that for all $d = \left[\begin{smallmatrix} d_1 \\ d_2 \end{smallmatrix}\right]$ and $d' = \left[\begin{smallmatrix} d_1' \\ d_2' \end{smallmatrix}\right]$ in $\mathcal{E}$, if $(d, d') \in \mathcal{F}$ then*

$$d_2 - d_2' \in \Gamma. \tag{6.18}$$

Proof of Claim 2. Define the lattice Γ in $\mathbb{Z}^{n-r}$ by

$$\Gamma := \{m_2 \in \mathbb{Z}^{n-r} : \langle m_2, \beta_j\rangle \equiv 0 \ (\mathrm{mod}\ 1),\ 0 \le \mathrm{j} \le \mathrm{k}-1\}.$$

Then Γ is a sublattice of $\mathbb{Z}^{n-r}$, and it is full rank because all $\beta_j \in \mathbb{Q}^{n-r}$. Claim 1 gives

$$\langle d_2 - d_2', \beta_j\rangle \equiv 0 \pmod 1, \quad 0 \le j \le k-1.$$

Hence $d_2 - d_2' \in \Gamma$.

It remains to check that Γ is a proper sublattice of $\mathbb{Z}^{n-r}$ and $B_2(\Gamma) \subseteq \Gamma$. First, all $y_j + W$ are contained in Z_f for some special eignefunction $f(x)$ of $(A, \mathcal{D})$, so $(y_j + W) \cap \mathbb{Z}^n = \emptyset$. Since $v_j + E_r$ are the images of $y_j + W$ under a unimodular linear map, $(v_j + E_r) \cap \mathbb{Z}^n = \emptyset$. But $v_j = \left[\begin{smallmatrix} \alpha_j \\ \beta_j \end{smallmatrix}\right]$. Hence $\beta_j \notin \mathbb{Z}^{n-r}$; so Γ must be a proper sublattice of $\mathbb{Z}^{n-r}$. Next, we show that Γ is B_2-invariant. (6.11) states that

$$B_2^T \beta_{j+1} \equiv \beta_j \pmod{\mathbb{Z}^{n-r}},$$

hence for any $m_2 \in \Gamma$,

$$\begin{aligned} \langle B_2 m_2, \beta_{j+1}\rangle &= \langle m_2, B_2^T \beta_{j+1}\rangle \\ &\equiv \langle m_2, \beta_j\rangle \pmod 1 \\ &\equiv 0 \pmod 1, \end{aligned}$$

proving $B_2(\Gamma) \subseteq \Gamma$. $\square$

Claim 3. *The digit set $\mathcal{E} = P(\mathcal{D})$ is of quasi-product form.*

Proof of Claim 3. For a given residue class $m_1 + B_1(\mathbb{Z}^r)$ we pick a digit $\left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right] \in \mathcal{E}$ with $a_i \equiv m_1 \pmod{\mathrm{B}_1(\mathbb{Z}^r)}$, which exists since $\mathcal{E}$ is a complete residue system (mod B). Consider all other digits $\left[\begin{smallmatrix} a_i' \\ b_i' \end{smallmatrix}\right] \in \mathcal{E}$ having

$$a_i' \equiv a_i \pmod{B_1(\mathbb{Z}^r)}.$$

It follows from Claim 1 and Claim 2 that $a_i' = a_i$ and $b_i - b_i' \in \Gamma$. Taking a basis matrix $Q \in M_{n-r}(\mathbb{Z})$ for Γ, we can write

$$b_i - b_i' = Qc_{i,j}$$

for some $c_{i,j} \in \mathbb{Z}^{n-r}$. Since $\mathcal{E}$ is a complete residue system (mod B), the set of such $\left[\begin{smallmatrix} a_i' \\ b_i' \end{smallmatrix}\right] \in \mathcal{E}$ has cardinality $|\det(B_2)|$, and $\{Qc_{i,j} : 1 \le j \le |\det(B_2)|\}$ forms a complete residue system (mod B_2). Now because Γ is B_2-invariant, there exists a $\tilde{B}_2 \in M_{n-r}(\mathbb{Z})$ such that

$$B_2 Q = Q\tilde{B}_2.$$

Finally, $|\det(Q)| > 1$ because Γ is a proper sublattice of $\mathbb{Z}^{n-r}$. $\square$

Finally, Theorem 6.1 follows from Lemma 6.3 and Claim 3. $\square$

7. Lattice Tilings

We now use Theorem 6.1 to prove Theorem 1.1.

Proof of Theorem 1.1. We prove the theorem by induction on the Lebesgue measure $\mu(T(A,\mathcal{D}))$ of the tile $T(A,\mathcal{D})$, where $\mathcal{D}$ is a standard digit set for $A \in M_n(\mathbb{Z})$. This measure is an integer, by Theorem 1.1 of part I. The base case is therefore $\mu(T(A,\mathcal{D})) = 1$, in which case $T(A,\mathcal{D})$ tiles by $\mathbb{Z}^n$.

For the induction step, suppose that it is true for all tiles of measure less than k, with $k \ge 2$, and that $\mu(T(A,\mathcal{D})) = k$. We consider first the case that $\mathbb{Z}[A,\mathcal{D}] \neq \mathbb{Z}^n$. The proof of Lemma 2.1 shows that

$$T(A,\mathcal{D}) = Q(T(\tilde{A},\tilde{\mathcal{D}})) + v, \tag{7.1}$$

where $\tilde{A} \in M_n(\mathbb{Z})$ is similar to A over $\mathbb{Q}$, and $Q \in M_n(\mathbb{Z})$ with $|\det(Q)| \ge 2$, hence

$$\mu(T(\tilde{A},\tilde{\mathcal{D}})) = \frac{\mu(T(A,\mathcal{D}))}{|\det(Q)|} < k.$$

The induction hypothesis applies to $(\tilde{A},\tilde{\mathcal{D}})$, so $T(\tilde{A},\tilde{\mathcal{D}})$ tiles with a lattice $\tilde{\Gamma} \subseteq \mathbb{Z}_n$, and (7.1) then shows that $T(A,\mathcal{D})$ tiles $\mathbb{R}^n$ using the lattice $\Gamma = Q(\tilde{\Gamma}) \subseteq \mathbb{Z}^n$.

Next suppose that $\mathbb{Z}[A,\mathcal{D}] = \mathbb{Z}^n$. Since $\mu(T(A,\mathcal{D})) = k \ge 2$, the tile $T(A,\mathcal{D})$ is a stretched tile. Theorem 6.1 shows that there exists a $P \in GL(n,\mathbb{Z})$ with

$$B := PAP^{-1} = \begin{bmatrix} B_1 & 0 \\ C & B_2 \end{bmatrix}, \tag{7.2}$$

where $B_1 \in M_r(\mathbb{Z})$ and $B_2 \in M_{n-r}(\mathbb{Z})$ with $1 \le r \le n-1$, and $\mathcal{E} := P(\mathcal{D})$ has the quasi-product form

$$\mathcal{E} = \left\{ \begin{bmatrix} a_i \\ b_i + Qc_{i,j} \end{bmatrix} : 1 \le i \le |\det(B_1)|,\ 1 \le j \le |\det(B_2)| \right\} \tag{7.3}$$

with $|\det(Q)| \ge 2$. Since

$$T(A, \mathcal{D}) = P^{-1} T(PAP^{-1}, P(\mathcal{D})) = P^{-1} T(B, \mathcal{E}),$$

we only need to show that $T(B, \mathcal{E})$ lattice tiles $\mathbb{R}^n$.

Consider the new pair $(B_1, \mathcal{E}_1)$ where

$$\mathcal{E}_1 = \{a_i : 1 \le i \le |\det(B_1)|\}.$$

$\mathcal{E}_1$ is a complete residue system (mod B$_1$). Furthermore, $\mathbb{Z}[B, \mathcal{E}] = \mathbb{Z}^n$ implies that $\mathbb{Z}[B_1, \mathcal{E}_1] = \mathbb{Z}^r$. Hence $\mathcal{E}_1$ is a primitive standard digit set for B_1.

We claim that we can always find a factorization (7.2), (7.3) with the additional property that

$$\mu(T(B_1, \mathcal{E}_1)) = 1. \tag{7.4}$$

To see this, assume that r is the smallest positive integer with which the factorization (7.2), (7.3) exists. If $r = 1$ then we already have $\mu(T(B_1, \mathcal{E}_1)) = 1$ by Theorem 3.4. Suppose that $\mu(T(B_1, \mathcal{E}_1)) > 1$. Then $r > 1$ and by Theorem 6.1 there exists a $P_1 \in GL(r, \mathbb{Z})$ such that

$$P_1 B_1 P_1^{-1} = \begin{bmatrix} \tilde{B}_1 & 0 \\ C_1 & \tilde{B}_2 \end{bmatrix},$$

where $B_1 \in M_{r_1}(\mathbb{Z})$, $B_2 \in M_{r-r_1}(\mathbb{Z})$ with $1 \le r_1 < r$, and $P_1(\mathcal{E})$ has the quasi-product form. Now if we let

$$\hat{P} = \begin{bmatrix} P_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} P,$$

then

$$\hat{P} A \hat{P}^{-1} = \begin{bmatrix} \tilde{B}_1 & 0 \\ C' & B_2' \end{bmatrix}$$

for some integer matrices B_2' and C', with $\hat{P}(\mathcal{D})$ having the quasi-product form. This is a contradiction because $r_1 < r$. Hence we have $\mu(T(B_1, \mathcal{E}_1)) = 1$, proving the claim.

We next associate to the pair $(B, \mathcal{E})$ a new pair $(\widehat{B}, \widehat{\mathcal{E}})$ given by

$$\widehat{B} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \tag{7.5}$$

$$\widehat{\mathcal{E}} = \left\{ \begin{bmatrix} a_i \\ Qc_{ij} \end{bmatrix} :\ 1 \le i \le |\det(B_1)|,\ 1 \le j \le |\det(B_2)| \right\}. \tag{7.6}$$

The proof of Theorem 5.1 already shows that

$$\mu(T(\widehat{B}, \widehat{\mathcal{E}})) = \mu(T(B, \mathcal{E})) = k,$$

and it also shows that the new pair $(B^\dagger, \mathcal{E}^\dagger)$ given by

$$B^\dagger = \begin{bmatrix} B_1 & 0 \\ 0 & B_2^\dagger \end{bmatrix}, \tag{7.7}$$

$$\mathcal{E}^\dagger = \left\{ \begin{bmatrix} a_i \\ c_{i,j} \end{bmatrix} :\ 1 \le |\det(B_1)|,\ 1 \le j \le |\det(B_2)| \right\} \tag{7.8}$$

where $B_2 Q = Q B_2^\dagger$ has

$$T(\widehat{B}, \widehat{\mathcal{E}}) = \begin{bmatrix} I_r & 0 \\ 0 & Q \end{bmatrix} T(B^\dagger, \mathcal{E}^\dagger). \tag{7.9}$$

Note that

$$\mu(T(B^\dagger, \mathcal{E}^\dagger)) = \frac{\mu(T(\widehat{B}, \widehat{\mathcal{E}}))}{|\det(Q)|} < k,$$

and it is easy to check that $\mathcal{E}^\dagger$ is a complete residue system (mod $\mathrm{B}^\dagger$). So the induction hypothesis applies to show that $T(B^\dagger, \mathcal{E}^\dagger)$ lattice tiles $\mathbb{R}^n$, and hence $T(\widehat{B}, \widehat{\mathcal{E}})$ also lattice tiles $\mathbb{R}^n$ as a result of (7.9).

Assume that $T(\widehat{B}, \widehat{\mathcal{E}})$ tiles $\mathbb{R}^n$ with the lattice Γ. We now come to the main point of the proof: we show that $T(\widehat{B}, \widehat{\mathcal{E}})$ also tiles $\mathbb{R}^n$ with a (possibly different) lattice Γ^* which is a direct sum $\mathbb{Z}^r \oplus \Gamma_1$ where $\Gamma_1 \subseteq \mathbb{Z}^{n-r}$. We start by observing that the orthogonal projection of $T(\widehat{B}, \widehat{\mathcal{E}})$ to its first r-coordinate plane is $T(B_1, \mathcal{E}_1)$. So since $\Gamma \subseteq \mathbb{Z}_n$, every tile

$$T(\widehat{B}, \widehat{\mathcal{E}}) + \gamma, \quad \gamma \in \Gamma,$$

in the tiling by Γ orthogonally projects to

$$T(B_1, \mathcal{E}_1) + \gamma_1, \quad \gamma_1 \in \mathbb{Z}^r$$

where $\gamma := \left[\begin{smallmatrix} \gamma_1 \\ \gamma_2 \end{smallmatrix}\right]$. These projections are measure-disjoint for different γ_1's. Thus the tiling $T(\widehat{B}, \widehat{\mathcal{E}}) + \Gamma$ of $\mathbb{R}^n$ using Γ naturally divides up into cylinders

$$U(\gamma_1) := (T(B_1, \mathcal{E}_1) + \gamma_1) \oplus \mathbb{R}^{n-r}. \tag{7.10}$$

Look at the tiling of the particular cylinder $U(0)$, which is given by $T(\widehat{B}, \widehat{\mathcal{E}}) + \Gamma'$ where

$$\Gamma' = \Gamma \cap \left(\{0\} \oplus \mathbb{Z}^{n-r}\right). \tag{7.11}$$

Clearly Γ' is a sublattice of $\mathbb{Z}^n$, Write $\Gamma' = \{0\} \oplus \Gamma_1$ where $\Gamma_1 \subseteq \mathbb{Z}^{n-r}$. Now Γ_1 is a sublattice of $\mathbb{Z}^{n-r}$, and $T(\widehat{B}, \widehat{\mathcal{E}})$ tiles $U(\gamma_1)$ by $\{\gamma_1\} \oplus \Gamma_1$. Hence $T(\widehat{B}, \widehat{\mathcal{E}})$ tiles $\mathbb{R}^n$ by $\Gamma^* := \mathbb{Z}^r \oplus \Gamma_1$.

Next we claim that the tile $T(B, \mathcal{E})$ also tiles $\mathbb{R}^n$ using the lattice $\Gamma^* = \mathbb{Z}^r \oplus \Gamma_1$. To prove this claim we note that the orthogonal projection of $T(B, \mathcal{E})$ onto its first r-coordinate plane is also $T(B_1, \mathcal{E}_1)$ as a result of the triangular form of B and the quasi-product form of $\mathcal{E}$. Hence $T(B, \mathcal{E})$ also tiles the cylinder $U(\gamma_1)$ for each $\gamma_1 \in \mathbb{Z}^r$. It thus suffices to prove that $T(B, \mathcal{E}) + \Gamma'$ tiles the cylinder $U(0)$, where Γ' is defined in (7.11). Recall that the proof of Theorem 5.1 shows that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in T(B, \mathcal{E}) \quad \Longleftrightarrow \quad \widehat{x} = \begin{bmatrix} x_1 \\ x_2 - \psi(x_1) \end{bmatrix} \in T(\widehat{B}, \widehat{\mathcal{E}}),$$

where $\psi : T(B_1, \mathcal{E}_1) \to \mathbb{R}^{n-r}$ is a certain measurable function, see (5.5). This relation shows that translates of $T(B, \mathcal{E})$ by Γ' inherit the measure-disjointness property from that of translates of $T(\widehat{B}, \widehat{\mathcal{E}})$ by Γ'. It also yields the covering property for the cylinder $U(0)$, since the map $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \to \left[\begin{smallmatrix} x_1 \\ x_2 - \psi(x_1) \end{smallmatrix}\right]$ maps $U(0)$ one-to-one onto itself. Thus $T(B, \mathcal{E})$ tiles $\mathbb{R}^n$ using the lattice $\mathbb{Z}^r \oplus \Gamma_1$. This proves the Theorem. $\square$

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