



NORTH-HOLLAND

The Finiteness Conjecture for the Generalized Spectral Radius of a Set of Matrices

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ABSTRACT

The *generalized spectral radius* $\bar{\rho}(\Sigma)$ of a set Σ of $n \times n$ matrices is $\bar{\rho}(\Sigma) = \limsup_{k \rightarrow \infty} \bar{\rho}_k(\Sigma)^{1/k}$, where $\bar{\rho}_k(\Sigma) = \sup\{\rho(A_1 A_2 \cdots A_k) : \text{each } A_i \in \Sigma\}$. The *joint spectral radius* $\hat{\rho}(\Sigma)$ is $\hat{\rho}(\Sigma) = \limsup_{k \rightarrow \infty} \hat{\rho}_k(\Sigma)^{1/k}$, where $\hat{\rho}_k(\Sigma) = \sup\{\|A_1 \cdots A_k\| : \text{each } A_i \in \Sigma\}$. It is known that $\hat{\rho}(\Sigma) = \bar{\rho}(\Sigma)$ holds for any finite set Σ of $n \times n$ matrices. The *finiteness conjecture* asserts that for any finite set Σ of real $n \times n$ matrices there exists a finite k such that $\hat{\rho}(\Sigma) = \bar{\rho}(\Sigma) = \bar{\rho}_k(\Sigma)^{1/k}$. The *normed finiteness conjecture* for a given operator norm asserts that for any finite set $\Sigma = \{A_1, \dots, A_m\}$ having all $\|A_i\|_{\text{op}} \leq 1$, either $\hat{\rho}(\Sigma) < 1$ or $\hat{\rho}(\Sigma) = \bar{\rho}(\Sigma) = \bar{\rho}_k(\Sigma)^{1/k} = 1$ for some finite k . It is shown that the finiteness conjecture is true if and only if the normed finiteness conjecture is true for all operator norms. The normed finiteness conjecture is proved for a large class of operator norms, extending results of Gurvits. In particular, for polytope norms and for the Euclidean norm, explicit upper bounds are given for the least k having $\bar{\rho}(\Sigma) = \bar{\rho}_k(\Sigma)^{1/k}$. These results imply upper bounds for generalized critical exponents for these norms.

1. INTRODUCTION

The spectral radius $\rho(\mathbf{A})$ of a single matrix \mathbf{A} is the absolute value of the largest eigenvalue of \mathbf{A} , and thus satisfies $\rho(\mathbf{A}) = \rho(\mathbf{A}^k)^{1/k}$ for all $k \geq 1$; hence

$$\rho(\mathbf{A}) = \lim_{k \rightarrow \infty} \rho(\mathbf{A}^k)^{1/k}. \quad (1.1)$$

It can also be computed using matrix norms. A *consistent matrix norm* is a matrix norm satisfying the submultiplicativity property

$$\|\mathbf{M}_1 \mathbf{M}_2\| \leq \|\mathbf{M}_1\| \|\mathbf{M}_2\|.$$

It is well known that for any consistent matrix norm the spectral radius satisfies

$$\rho(\mathbf{A}) \leq \|\mathbf{A}\|$$

and that

$$\rho(\mathbf{A}) = \lim_{k \rightarrow \infty} \|\mathbf{A}^k\|^{1/k}, \quad (1.2)$$

cf. Belitskii and Lyubich (1988), Stewart and Sun (1990).

This paper studies questions concerning the spectral radius of a set Σ of $n \times n$ real matrices. There are two natural notions for the spectral radius of such a set Σ , which generalize the properties (1.1) and (1.2), respectively. The first of these is the *generalized spectral radius* $\bar{\rho}(\Sigma)$, defined by

$$\bar{\rho}(\Sigma) := \limsup_{k \rightarrow \infty} \bar{\rho}_k(\Sigma)^{1/k}, \quad (1.3)$$

where

$$\bar{\rho}_k(\Sigma) = \sup\{\rho(\mathbf{A}_{i_1} \cdots \mathbf{A}_{i_k}) : \text{each } \mathbf{A}_{i_j} \in \Sigma\}. \quad (1.4)$$

The second notion is the *joint spectral radius* $\hat{\rho}(\Sigma)$, defined by

$$\hat{\rho}(\Sigma) := \limsup_{k \rightarrow \infty} \hat{\rho}_k(\Sigma, \|\cdot\|)^{1/k}. \quad (1.5)$$

where $\|\cdot\|$ is a consistent matrix norm and

$$\hat{\rho}_k(\Sigma, \|\cdot\|) = \sup\{\|\mathbf{A}_{i_1} \cdots \mathbf{A}_{i_k}\| : \text{each } \mathbf{A}_{i_j} \in \Sigma\}. \quad (1.6)$$

The quantity $\hat{\rho}(\Sigma)$ is well defined independently of the consistent matrix norm used; however, the quantities $\hat{\rho}_k(\Sigma, \|\cdot\|)$ do depend on the matrix norm $\|\cdot\|$. The notion of joint spectral radius appears in Rota and Strang (1960), and that of generalized spectral radius in Daubechies and Lagarias (1992a).

These notions of spectral radius of a set Σ are closely related. The generalized spectral radius and joint spectral radius satisfy the inequalities

$$\bar{\rho}_k(\Sigma)^{1/k} \leq \bar{\rho}(\Sigma) \leq \hat{\rho}(\Sigma) \leq \hat{\rho}_k(\Sigma, \|\cdot\|)^{1/k} \quad (1.7)$$

for any $k \geq 1$ and any consistent matrix norm $\|\cdot\|$. In particular the right-hand inequality implies that

$$\hat{\rho}(\Sigma) = \liminf_{k \rightarrow \infty} \hat{\rho}_k(\Sigma, \|\cdot\|)^{1/k} = \lim_{k \rightarrow \infty} \hat{\rho}_k(\Sigma, \|\cdot\|)^{1/k}. \quad (1.8)$$

Daubechies and Lagarias (1992a) conjectured, and Berger and Wang (1992) proved, that for finite sets Σ the equality

$$\bar{\rho}(\Sigma) = \hat{\rho}(\Sigma) \quad (1.9)$$

always holds. More generally, Berger and Wang (1992) show that the equality (1.9) holds whenever Σ is a bounded set. In Appendix A of this paper we show further that if $\bar{\rho}(\Sigma) = 0$ then, even when Σ is infinite,

$$\bar{\rho}(\Sigma) = \hat{\rho}(\Sigma) = 0$$

always holds. Daubechies and Lagarias (1992a) observe that there do exist infinite sets Σ for which

$$\bar{\rho}(\Sigma) < \hat{\rho}(\Sigma),$$

e.g.,

$$\Sigma = \left\{ \begin{bmatrix} \frac{1}{2} & 2^n \\ 0 & \frac{1}{2} \end{bmatrix} : n = 1, 2, 3, \dots \right\}.$$

The main object of this paper is to study the following problem.

FINITENESS CONJECTURE. For each finite set Σ of $n \times n$ real matrices there is some finite k such that

$$\hat{\rho}(\Sigma) = \bar{\rho}(\Sigma) = \bar{\rho}_k(\Sigma)^{1/k}. \quad (1.10)$$

This conjecture arose from work of Daubechies and Lagarias (1992a), in connection with the problem of whether there is an effectively computable procedure for deciding whether or not a finite set of matrices Σ with rational entries has joint spectral radius $\hat{\rho}(\Sigma) < 1$. If the finiteness conjecture is true, then such an algorithm exists, namely, for $k = 1, 2, 3, \dots$ compute $\bar{\rho}_k(\Sigma)^{1/k}$ and $\hat{\rho}_k(\Sigma, \|\cdot\|)^{1/k}$, where $\|\cdot\|$ is a fixed consistent matrix norm (e.g., the Frobenius norm), and check whether either of $\hat{\rho}_k(\Sigma, \|\cdot\|)^{1/k} < 1$ or $\bar{\rho}_k(\Sigma) \geq 1$ holds. If so, then $\hat{\rho}(\Sigma) < 1$

or $\widehat{\rho}(\Sigma) \geq 1$, respectively. If $\widehat{\rho}(\Sigma) < 1$, then some $\widehat{\rho}_k(\Sigma, \|\cdot\|)^{1/k} < 1$ by (1.8), while if $\widehat{\rho}(\Sigma) \geq 1$, then assuming the finiteness conjecture would guarantee that, for some k ,

$$\overline{\rho}_k(\Sigma)^{1/k} = \overline{\rho}(\Sigma) = \widehat{\rho}(\Sigma) \geq 1.$$

Hence this algorithm would eventually halt.

We shall prove the finiteness conjecture holds for various special classes of Σ and indicate why it may well be true in general. One indication of its subtlety is that there exist two 2×2 matrices such that the smallest k for which equality occurs in (1.10) is arbitrarily large; see Example 2.1 in Section 2.

The finiteness conjecture does not use matrix norms at all in its statement. However, in Section 2 we show that it is equivalent to the truth (for *all* norms) of a conjecture about matrix norms, the *normed finiteness conjecture*, stated below. The normed finiteness conjecture was apparently first proposed in the former Soviet Union, where it was raised in studying stability questions for certain control problems. Gurvits (1991, 1993, 1994) gives the first published results on it, and attributes it to E. S. Pyatnicky.

Given a norm $\|\cdot\|$ on \mathbb{R}^n , the *operator norm* $\|\cdot\|_{\text{op}}$ on the set $M(n, \mathbb{R})$ of $n \times n$ real matrices induced from it is

$$\|A\|_{\text{op}} := \sup_{\|x\|=1} \{\|Ax\| : x \in \mathbb{R}^n\}.$$

All operator norms are consistent matrix norms.

NORMED FINITENESS CONJECTURE. Let $\|\cdot\|$ be a given norm on \mathbb{R}^n . Suppose that $\Sigma = \{A_i : 1 \leq i \leq m\}$ is a finite set of $n \times n$ real matrices with joint spectral radius $\widehat{\rho}(\Sigma) = 1$, for which

$$\|A_i\|_{\text{op}} \leq 1, \quad 1 \leq i \leq m, \quad (1.11)$$

in the operator norm induced from $\|\cdot\|$. Then there exists a finite k such that

$$\widehat{\rho}(\Sigma) = \overline{\rho}_k(\Sigma)^{1/k} = 1. \quad (1.12)$$

Note that the normed finiteness conjecture has a stronger hypothesis than the finiteness conjecture, namely, it assumes the boundedness of the semigroup $\mathcal{S}(\Sigma)$ generated by Σ . [This follows from (1.11).] In contrast, the set Σ consisting of the single matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

has $\rho(A) = \widehat{\rho}(\Sigma) = 1$, but for any operator norm and any $k \geq 1$

$$\rho(A^k) < \|A^k\|_{\text{op}}.$$

In Sections 3–6 we consider the normed finiteness conjecture for various norms $\|\cdot\|$ on \mathbb{R}^n . Gurvits (1991, 1992, 1994) proved that it is true for norms whose unit ball is a polytope. In Section 3 we prove more generally that the normed finiteness conjecture is true for all piecewise analytic norms in \mathbb{R}^n (Theorem 3.1). A *piecewise analytic norm* is one whose unit ball B has a boundary which is contained in the zero set of a holomorphic function f defined on an open set Ω in \mathbb{C}^n containing B , which has $f(0) \neq 0$. The main innovation in the proof over the methods of Gurvits is a result in symbolic dynamics—Lemma 3.2.

The normed finiteness conjecture differs from the finiteness conjecture in that for certain norms, but not all norms, there exists a finite universal upper bound $\alpha(m, \|\cdot\|)$ for the smallest k in (1.12) for which equality occurs, i.e.

$$\alpha(m, \|\cdot\|) := \sup_{\Sigma} \{ \min(k : \rho_k(\Sigma)^{1/k} = 1) : |\Sigma| = k, \text{ all } \|A_i\| \leq 1 \}.$$

We prove that such a bound $\alpha(m, \|\cdot\|)$ exists for piecewise algebraic norms (Theorem 3.2). A *piecewise algebraic norm* is one whose boundary is contained in the zero set of a polynomial $p(\mathbf{z}) \in \mathbb{R}[z_1, \dots, z_n]$, which has $p(0) \neq 0$. This is the case when the unit ball of $\|\cdot\|$ is a polytope or an ellipsoid, or the l^p norm for rational p , with $1 \leq p \leq \infty$. In Sections 4 and 5 we obtain explicit bounds for $\alpha(m, \|\cdot\|)$ in the polytope and ellipsoid cases, respectively. The bound in the polytope case depends only on the norm $\|\cdot\|$ and not on $m = |\Sigma|$. For the ellipsoid case the bound depends on both m and n , and seems unreasonably large (Theorem 5.1), but we do show that any bound must depend on m (Theorem 5.2).

The results of Sections 3–5 also serve to bound generalized critical exponents. Given an integer $m \geq 1$ and a norm $\|\cdot\|$ on \mathbb{R}^n , the *generalized critical exponent* $\beta(m, \|\cdot\|)$ is the smallest integer k such that for all sets $\Sigma = \{A_i : 1 \leq i \leq m\}$ for which all $\|A_i\|_{\text{op}} \leq 1$, but for which $\widehat{\rho}(\Sigma) < 1$, any product of length k has

$$\|A_{i_k} \cdots A_{i_1}\|_{\text{op}} < 1.$$

The value $\beta(m, \|\cdot\|) = +\infty$ if no finite k exists. The notion of generalized critical exponent is due to Gurvits (1991). The *generalized-critical-exponent problem* is to determine all such constants $\beta(m, \|\cdot\|)$. It is immediate that

$$\beta(1, \|\cdot\|) \leq \beta(2, \|\cdot\|) \leq \beta(3, \|\cdot\|) \leq \cdots.$$

The quantity $\beta(1, \|\cdot\|)$ is called the *critical exponent*. It was defined in Pták (1962) and has been extensively studied; see Belitskii and Lyubich (1988, Section 2.6)

and the survey article of Pták (1993). For example, for the Euclidean norm on \mathbb{R}^n , $\beta(1, \|\cdot\|) = n$. It is immediate from the definitions that

$$\beta(m, \|\cdot\|) \leq \alpha(m, \|\cdot\|) \quad (1.13)$$

for all $m \geq 1$. Thus finite bounds for $\alpha(m, \|\cdot\|)$ automatically bound the corresponding generalized critical exponents. Conversely, since there exists a norm in \mathbb{R}^2 with $\beta(1, \|\cdot\|) = +\infty$, we have $\alpha(m, \|\cdot\|) = +\infty$ for all $m \geq 1$ in this case; see Section 6.

The results of this paper carry over to sets of complex $n \times n$ matrices in $M(n, \mathbb{C})$, by regarding them as real $2n \times 2n$ matrices using the correspondence

$$x + iy \leftrightarrow \begin{bmatrix} x & y \\ -y & x \end{bmatrix},$$

and using the correspondence on column vectors

$$x + iy \leftrightarrow x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

to define their action on column vectors. The natural notions of generalized spectral radius and joint spectral radius for complex matrices are preserved under this correspondence. This correspondence is needed because the definition of piecewise analytic norm (given in Section 3) requires a norm on \mathbb{R}^n , and does not work on \mathbb{C}^n .

We note that the definitions of joint spectral radius and generalized spectral radius make sense for sets Σ in an arbitrary Banach algebra. We expect that the finiteness conjecture fails for finite sets Σ in arbitrary Banach algebras.

Finally we remark that the notions of generalized spectral radius and joint spectral radius naturally arise in studying the smoothness properties of compactly supported wavelets and solutions of two-scale dilation equations [see Daubechies and Lagarias (1991, 1992b), Colella and Heil (1992a, 1992b)] and also arise in studying the dynamical complementarity problem in the theory of stochastic networks [see Kozyakin et al. (1993)].

2. THE NORMED FINITENESS CONJECTURE

In this section we reduce the finiteness conjecture to the normed finiteness conjecture, and then give an example showing that arbitrarily long finite products may be needed to attain the generalized spectral radius.

THEOREM 2.1. *The following are equivalent, for each integer n :*

(i) *The finiteness conjecture is true for all finite sets of real $n \times n$ matrices.*

(ii) *The normed finiteness conjecture is true for all operator norms on all real $n \times n$ matrices.*

Proof. (i) \Rightarrow (ii): This follows using the Berger-Wang equality (1.9).

(ii) \Rightarrow (i): First note that (ii) implies the truth of the normed finiteness conjecture for all operator norms on $l \times l$ matrices, for $1 \leq l \leq n$. Set $\Sigma = \{A_i : 1 \leq i \leq m\}$. The finiteness conjecture is always true when $\bar{\rho}(\Sigma) = 0$ with $k = 1$ in (1.11); hence we may suppose that $\bar{\rho}(\Sigma) > 0$. Since $\bar{\rho}(\lambda\Sigma) = \lambda\bar{\rho}(\Sigma)$ for scalars $\lambda > 0$, we may suppose without loss of generality that $\bar{\rho}(\Sigma) = 1$. Following Berger and Wang (1992, Proposition III and Theorem IV), there exists a similarity transformation $P \in GL(n, \mathbb{C})$ such that all matrices in $P^{-1}\Sigma P$ have the block factorization

$$P^{-1}A_iP = \begin{bmatrix} A_i^{(1)} & & \\ & A_i^{(2)} & * \\ \mathbf{0} & & A_i^{(r)} \end{bmatrix}, \quad (2.1)$$

where $A_i^{(j)}$ is $k_j \times k_j$, such that each set

$$\Sigma_j := \{A_i^{(j)} : 1 \leq i \leq m\} \quad (2.2)$$

generates a bounded semigroup $\mathcal{S}(\Sigma_j)$. Furthermore

$$\bar{\rho}(\Sigma) = \bar{\rho}(P^{-1}\Sigma P) = \max_{i \leq j \leq r} \bar{\rho}(\Sigma_j); \quad (2.3)$$

hence some $\bar{\rho}(\Sigma_j) = 1$. Now the block-triangular form (2.1) implies that

$$\bar{\rho}_k(\Sigma) = \bar{\rho}_k(P^{-1}\Sigma P) = \max_{i \leq j \leq r} \bar{\rho}_k(\Sigma_j), \quad (2.4)$$

It therefore suffices to prove the finiteness conjecture for any Σ_j having $\bar{\rho}(\Sigma_j) = 1$, since it then follows for $P^{-1}\Sigma P$ and Σ by (2.4). Taking such a Σ_j , recall that by Berger and Wang (1992, Lemma II), for any (finite or infinite) Σ generating a bounded semigroup $\mathcal{S}(\Sigma)$ there exists an operator norm $\|\cdot\|_v$ such that

$$\|A\|_v \leq 1, \quad \text{all } A \in \Sigma.$$

Since $\bar{\rho}(\Sigma_j) = 1$, the truth of the normed finiteness conjecture for $\|\cdot\|_v$ implies that there is a finite k and a product in Σ_j , of length k , having an eigenvalue of modulus one. Thus $\bar{\rho}(\Sigma) = \bar{\rho}_k(\Sigma) = 1$ by (2.4). \blacksquare

EXAMPLE 2.1. The set $\Sigma = \{A_1, A_2\}$ defined by

$$A_1 = \alpha^k \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \alpha^{-1} \begin{bmatrix} \cos \frac{\pi}{2k} & \sin \frac{\pi}{2k} \\ -\sin \frac{\pi}{2k} & \cos \frac{\pi}{2k} \end{bmatrix},$$

where

$$1 < \alpha < \left(\cos \frac{\pi}{2k} \right)^{-1}, \quad (2.5)$$

has $\bar{\rho}(\Sigma) = \hat{\rho}(\Sigma) = 1$. In addition, $\bar{\rho}_j(\Sigma) < 1$ for $j \leq k$, and

$$\bar{\rho}_{k+1}(\Sigma) = 1. \quad (2.6)$$

Proof. First, note that

$$A_2^m A_1 = \alpha^{k-m} \begin{bmatrix} \sin \frac{\pi m}{2k} & 0 \\ \cos \frac{\pi m}{2k} & 0 \end{bmatrix}$$

is lower-triangular and has rank at most one; hence

$$\rho(A_2^m A_1) = |\text{tr}(A_2^m A_1)| = \alpha^{k-m} \left| \sin \frac{\pi m}{2k} \right|.$$

If $m > k$, then $\rho(A_2^m A_1) < 1$, and if $m = k$, then $\rho(A_2^m A_1) = 1$. For $1 \leq m < k$, set $\theta = \pi/2k$ and $l = m - k$ and obtain, using (2.5),

$$\begin{aligned} \rho(A_2^m A_1) &= \alpha^{k-m} \cos \frac{\pi(m-k)}{2k} \\ &= \frac{\cos l\theta}{(\cos \theta)^l} = \frac{\cos[(l-1)\theta] \cos \theta - \sin[(l-1)\theta] \sin \theta}{(\cos \theta)^l} \\ &< \frac{\cos[(l-1)\theta]}{(\cos \theta)^{l-1}} \leq \cdots \leq \frac{\cos \theta}{\cos \theta} = 1. \end{aligned}$$

Hence $\rho(A_2^m A_1) < 1$ in this case, so $\rho(A_2^m A_1) = 1$ if and only if $m = k$.

For the general case, since $A_1^2 = \mathbf{0}$ and $\rho(A_2) < 1$, we need only consider finite products $M = A_2^{j_0} A_1 A_2^{j_1} \cdots A_1 A_2^{j_r}$ where each $j_i \geq 0$. Since M has rank at

most one,

$$\begin{aligned}\rho(\mathbf{M}) &= |\operatorname{tr} \mathbf{M}| = |\operatorname{tr}(\mathbf{A}_2^{j_r} \mathbf{M} \mathbf{A}_2^{-j_r})| \\ &= \left| \operatorname{tr}(\mathbf{A}_2^{j_0+j_r} \mathbf{A}_1) \right| \prod_{i=1}^r \left| \operatorname{tr}(\mathbf{A}_2^{j_i} \mathbf{A}_1) \right|.\end{aligned}$$

Hence $\rho(\mathbf{M}) \leq 1$, and $\rho(\mathbf{M}) = 1$ if and only if $j_0 + j_r = k$ and all other $j_i = k$. Thus $\bar{\rho}(\Sigma) = \bar{\rho}_{k+1}(\Sigma) = 1$, while $\bar{\rho}_m(\Sigma) < 1$ if $m < k$. Finally, $\hat{\rho}(\Sigma) = 1$ by (1.9). \blacksquare

This example generates a bounded semigroup, so, by the results of Berger and Wang (1992), there exists a norm on \mathbb{R}^n giving $\|\mathbf{A}_i\|_{\text{op}} \leq 1$, $i = 1, 2$, for this example. Its unit ball can be chosen to be a polygon having vertices at

$$\alpha^{k-j} \begin{bmatrix} \pm \sin \frac{\pi j}{2k} \\ \pm \cos \frac{\pi j}{k} \end{bmatrix}, \quad 0 \leq j \leq k.$$

3. PIECEWISE ANALYTIC NORMS

A *piecewise analytic norm* is any norm $\|\cdot\|$ on \mathbb{R}^n whose unit ball \mathbf{B} has a boundary $\partial\mathbf{B}$ contained in the zero set of a holomorphic function $f(\mathbf{z})$ defined on a connected open domain Ω in \mathbb{C}^n containing $\mathbf{0}$, which has $f(\mathbf{0}) \neq 0$.

More generally, given a collection \mathcal{F} of holomorphic functions $f : \Omega \rightarrow \mathbb{C}$, let

$$V_{\Omega}(\mathcal{F}) = \{\mathbf{z} \in \Omega : f(\mathbf{z}) = 0 \text{ for all } f \in \mathcal{F}\}$$

denote their common zero set in \mathbb{C}^n . We also call $\|\cdot\|$ a *piecewise analytic norm* if

$$\partial\mathbf{B} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\} \subseteq V_{\Omega}(\mathcal{F}) \quad (3.1)$$

for some nonempty collection \mathcal{F} of holomorphic functions defined on Ω , and $\mathbf{0} \notin V_{\Omega}(\mathcal{F})$. This second definition has no extra generality, because if an \mathcal{F} satisfies (3.1) then there is also a single function in \mathcal{F} with $f(\mathbf{0}) \neq 0$, and $\mathcal{F}' = \{f\}$ also has the property (3.1).

A piecewise analytic norm $\|\cdot\|$ has a unit ball whose boundary $\partial\mathbf{B}$ consists of a finite number of real analytic pieces; this motivates our terminology. To prove this fact, observe that the set $V_{\Omega}(f)$ is an *analytic set* as defined in Hervé (1963, p. 27). The regular points $V_{\Omega}^*(f)$ of $V_{\Omega}(f)$ are dense in $V_{\Omega}(f)$, and form a union of connected components each of which is associated to an irreducible analytic set; cf. Hervé (1963, pp. 84, 97). By compactness only finitely many of these irreducible components intersect $V_{\Omega}(f) \cap \mathbf{B}$. Because \mathbb{R}^n is a Lagrangian

submanifold of \mathbb{C}^n , $\partial\mathbf{B}$ must be covered by a finite number of these irreducible analytic sets of complex codimension one. Each of these intersected with $\partial\mathbf{B}$ gives a real-analytic piece of the boundary of real codimension one in \mathbb{R}^n , which proves the fact. *Warning:* not all norms whose unit balls have a boundary that is a finite union of real-analytic pieces are piecewise analytic norms.

Our object is to prove the following result.

THEOREM 3.1. *The normed finiteness conjecture is true for all piecewise analytic norms $\|\cdot\|$ on \mathbb{R}^n .*

Our proof of Theorem 3.1 does not provide any explicit bound for k in (1.11). It is based on two auxiliary results. The first of these is the following Noetherian property of zero sets of holomorphic maps.

LEMMA 3.1. *Let $\|\cdot\|$ be a norm on \mathbb{R}^n with unit ball \mathbf{B} . Given any collection $\mathcal{F} = \{f_\alpha : \alpha \in I\}$ of functions holomorphic on an open connected domain Ω on \mathbb{C}^n containing \mathbf{B} , there exists a finite subset $\mathcal{F}' = \{f_i : 1 \leq i \leq m\}$ of \mathcal{F} such that*

$$V_\Omega(\mathcal{F}') \cap \mathbf{B} = V_\Omega(\mathcal{F}) \cap \mathbf{B}. \quad (3.2)$$

Proof. We use the basic fact that the ring $\mathcal{H}_{\mathbf{z}}^n$ of germs of holomorphic functions at a point $\mathbf{z} \in \mathbb{C}^n$ is Noetherian; cf. Hörmander (1973, Theorem 6.3.3). In consequence, using Hervé (1963, Corollary 3, p. 37), for each $\mathbf{z} \in \Omega$ there exists a finite subset $\mathcal{F}_{\mathbf{z}}$ of elements of \mathcal{F} and an open neighborhood $U(\mathbf{z})$ of \mathbf{z} such that $\mathcal{F}_{\mathbf{z}}$ and \mathcal{F} have the same zero sets on $U(\mathbf{z})$, i.e.,

$$V_{U(\mathbf{z})}(\mathcal{F}_{\mathbf{z}}) = V_{U(\mathbf{z})}(\mathcal{F}). \quad (3.3)$$

Now let \mathbf{z} run over all elements of \mathbf{B} . The sets $U(\mathbf{z})$ form an open cover of the compact set \mathbf{B} , and hence have a finite subcover $\{U(\mathbf{z}_i) : 1 \leq i \leq l\}$. Take

$$\mathcal{F}' = \bigcup_{i=1}^l \mathcal{F}_{\mathbf{z}_i}.$$

Since

$$V_\Omega(\mathcal{F} \cup \mathcal{G}) \subseteq V_\Omega(\mathcal{F}) \cap V_\Omega(\mathcal{G}),$$

one has

$$V_\Omega(\mathcal{F}') \cap \mathbf{B} \subseteq V_\Omega(\mathcal{F}) \cap \mathbf{B},$$

using (3.3). Since $V_\Omega(\mathcal{F}') \supseteq V_\Omega(\mathcal{F})$, (3.2) follows. ■

The second auxiliary result concerns symbol sequences.

LEMMA 3.2. *Let ω' be an infinite word in an alphabet \mathcal{A} of cardinality $|\mathcal{A}| = m$. Then it contains a set of nonempty finite subwords $\{\gamma_i : i \geq 1\}$ such that $\gamma_{i+1} = \gamma_i \beta_i \gamma_i$ for some word β_i , and which satisfy $|\gamma_i| \leq f(i, m)$, where $f(i, m)$ are explicitly computable universal bounds given by (3.5) below.*

This lemma is really a recurrence result in symbolic dynamics. Given a finite alphabet \mathcal{A} and a (possibly infinite) set Δ of finite words from \mathcal{A} , the *one-sided subshift* ζ_Δ determined by Δ is the set of all one-sided infinite words

$$\omega = \alpha_1 \alpha_2 \alpha_3 \dots, \quad \alpha_i \in \mathcal{A},$$

that contain no block in Δ , i.e., $\alpha_i \alpha_{i+1} \dots \alpha_j \notin \Delta$ whenever $i \leq j$. A word ω is *recurrent* if any subword occurring in ω occurs at least twice; see Furstenberg (1981, Proposition 1.10). Now Lemma 3.2 immediately implies the following result, by taking the word $\omega = \lim_{i \rightarrow \infty} \gamma_i$.

COROLLARY 3.1. *Let ζ_Δ be a one-sided subshift of a finite alphabet \mathcal{A} of cardinality $|\mathcal{A}| = m$. There exist universal bounds $\{f(i, m) : i = 1, 2, \dots\}$ such that the following holds: If $\zeta_\Delta \neq \emptyset$, then ζ_Δ contains a recurrent word $\omega = \alpha_1 \alpha_2 \dots \alpha_m \dots$ such that each initial word $\alpha_1 \alpha_2 \dots \alpha_r$ occurs twice without overlap in the first $f(i, m)$ symbols of ω .*

It is well known that any nonempty one-sided subshift contains a recurrent word and, more generally, a uniformly recurrent word; see Furstenberg (1981, Theorem 1.15). The interesting feature of Corollary 3.1 lies in the explicit bounds $f(i, m)$.

Proof of Lemma 3.2. Write

$$\omega' = \alpha'_1 \alpha'_2 \alpha'_3 \dots, \quad \text{each } \alpha'_i \in \mathcal{A}.$$

Let \mathcal{A}^* denote the set of finite words from \mathcal{A} . Given a word $\sigma \in \mathcal{A}^*$, let $|\sigma|$ denote the number of symbols in σ . The (upper asymptotic) *density* of σ in ω' is

$$\bar{d}(\sigma) := \limsup_{k \rightarrow \infty} \frac{1}{k} \#\{i \leq k : \sigma = \alpha'_i \dots \alpha'_{i+|\sigma|-1}\}.$$

There certainly exist $\sigma \in \mathcal{A}^*$ with $\bar{d}(\sigma) > 0$, since if $\mathcal{A} = \{\beta^{(j)} : 1 \leq j \leq |\mathcal{A}|\}$ then

$$\sum_{j=1}^l \bar{d}(\beta^{(j)}) \geq 1.$$

The basic ingredient of the proof is:

Claim. If $\bar{d}(\sigma) > 0$, then there exists a word $\tau \in \mathcal{A}^*$ with $\bar{d}(\sigma\tau\sigma) > 0$. In fact, if $s := 2|\sigma|\{1 + [\bar{d}(\sigma)]^{-1}\}$ then there is such a word τ with $|\tau| \leq s$ and $\bar{d}(\sigma\tau\sigma) \geq s^{-1}|\mathcal{A}|^{-s-1}$.

To prove the claim, suppose that $\bar{d}(\sigma) \geq 1/k$. Consider the set $\mathcal{E} = \{i : \sigma = \alpha'_i \alpha'_{i+1} \cdots \alpha'_{i+|\sigma|-1} \text{ and } \sigma = \alpha'_j \alpha'_{j+1} \cdots \alpha'_{j+|\sigma|-1} \text{ for some } j \text{ with } i + |\sigma| \leq j \leq i + (2k + 1)|\sigma|\}$. Suppose that $\bar{d}(\mathcal{E}) < 1/2k$. Now the occurrences of σ consist of all indices $i \in \mathcal{E}$ (and these have upper density less than $1/2k$) and also of some indices $i \notin \mathcal{E}$, which occur in clumps of at most $|\sigma|$ overlapping copies of σ , all lying in the interval $i \leq j < i + 2|\sigma|$, and which are followed by an interval of length $2k|\sigma|$ free of any occurrence of σ ; hence these contribute upper density at most $1/2k$. But this implies $\bar{d}(\sigma) < 1/k$, a contradiction. Thus $\bar{d}(\mathcal{E}) \geq 1/2k$, so that

$$\sum_{|\tau| \leq 2k|\sigma|} \bar{d}(\sigma\tau\sigma) \geq \bar{d}(\mathcal{E}) \geq \frac{1}{2k}.$$

Consequently some $\tau \in \mathcal{A}^*$ with $|\tau| \leq 2k|\sigma|$ has

$$\bar{d}(\sigma\tau\sigma) \geq \frac{1}{2k}|\mathcal{A}|^{-2k|\sigma|-1} \geq s^{-1}|\mathcal{A}|^{-s-1},$$

proving the claim.

Now choose σ_1 with $\bar{d}(\sigma_1) \geq |\mathcal{A}|^{-1}$, as may be done. Using the claim, there exists an infinite sequence $\{\sigma_i : i \geq 1\}$ with

$$\sigma_{i+1} = \sigma_i \tau_i \sigma_i \tag{3.4}$$

and $\bar{d}(\sigma_i) > 0$ for all i .

Finally, one can easily derive bounds $f(i, m)$ inductively, using the claim to obtain a suitable recursion. The initial conditions are

$$f(1, m) = 1, \tag{3.5a}$$

$$d(1, m) = m, \tag{3.5b}$$

and, for $i \geq 2$, the recursion is

$$s(i, m) = 2f(i-1, m)[1 + d(i-1, m)], \tag{3.5c}$$

$$d(i, m) = s(i, m)m^{s(i, m)+1} \tag{3.5d}$$

$$f(i, m) = 2f(i-1, m) + s(i, m). \tag{3.5e}$$

The resulting $f(i, m)$ grows like a tower of exponentials of height i . ■

Proof of Theorem 3.1. Let $\Sigma = \{\mathbf{A}_i : 1 \leq i \leq m\}$ have $\bar{\rho}(\Sigma) = 1$ and

$\|A_i\|_{\text{op}} \leq 1$ for the given operator norm on $M(n, \mathbb{R})$ induced from the piecewise analytic norm $\|\cdot\|$.

There is a function $f(\mathbf{z})$ with $f(\mathbf{0}) \neq \mathbf{0}$, which is holomorphic on an open connected domain Ω in \mathbb{C}^n which contains the unit ball \mathbf{B} of $\|\cdot\|$, such that

$$\partial\mathbf{B} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\} \subseteq V_{\Omega}(f') = \{\mathbf{z} \in \Omega : f(\mathbf{z}) = \mathbf{0}\}.$$

View elements $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ as column vectors with $z_j = x_j + iy_j$, and note that

$$\text{Int } \mathbf{B} = \{\mathbf{x} : \|\mathbf{x}\| < 1\} = \mathbf{B} - \partial\mathbf{B}.$$

For small enough $\varepsilon > 0$, the domain Ω contains

$$\begin{aligned} \Omega' &= \{z = (z_1, \dots, z_n)^T : (x_1, \dots, x_n) \in (1 + \varepsilon) \text{Int } \mathbf{B} \\ &\quad \text{and } (y_1, \dots, y_n) \in \varepsilon \text{Int } \mathbf{B}\}, \end{aligned}$$

which is a connected open set. The matrices A_i in Σ are real with $\|A_i\|_{\text{op}} \leq 1$; hence each A_i maps \mathbf{B} into \mathbf{B} and also maps Ω' into Ω' .

Given a sequence of matrices $(A_{d_1}, A_{d_2}, \dots)$ in Σ , we construct a sequence $\mathcal{F} = \{f_k : k \geq 0\}$ of functions $f_0(\mathbf{z}) = f(\mathbf{z})$ and

$$f_k(\mathbf{z}) := f(A_{d_k} A_{d_{k-1}} \cdots A_{d_1} \mathbf{z}).$$

All f_i are holomorphic on Ω' , since $A_i(\Omega') \subseteq \Omega'$.

We study the zero sets

$$Z_m := \mathbf{B} \cap V_{\Omega'}(f_0, f_1, \dots, f_m). \quad (3.6)$$

These are compact sets with $Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \cdots$, and they don't contain $\mathbf{0}$, because $\mathbf{0} \notin Z_0$. By Lemma 3.1 this sequence has the Noetherian property that there is a finite r with

$$Z_r = Z_{r+1} = Z_{r+2} = \cdots. \quad (3.7)$$

We call $Z := Z_r$ the *limit set* of $\omega = (d_1, d_2, \dots)$.

Our object is to produce a sequence

$$\omega = (d_1, d_2, \dots) \quad (3.8)$$

such that the construction above yields a nonempty limit set Z , and also a finite product $\tilde{\mathbf{A}} := A_{d_k} A_{d_{k-1}} \cdots A_{d_1}$ for which

$$\tilde{\mathbf{A}}(Z) \subseteq Z. \quad (3.9)$$

If so, then the spectral radius $\rho(\tilde{\mathbf{A}}) = 1$. For Z is compact and doesn't contain $\mathbf{0}$; hence there are constants r_0 and r_1 with

$$0 < r_0 < \|\mathbf{x}\| < r_1 \quad \text{for all } \mathbf{x} \in Z.$$

Given any $\mathbf{x}_0 \in Z$, (3.9) yields

$$\|\tilde{\mathbf{A}}^j\|_{\text{op}} \geq \frac{\|\tilde{\mathbf{A}}^j \mathbf{x}_0\|}{\|\mathbf{x}_0\|} \geq \frac{r_0}{r_1}.$$

Thus

$$\rho(\tilde{\mathbf{A}}) = \lim_{j \rightarrow \infty} \|\tilde{\mathbf{A}}^j\|_{\text{op}}^{1/j} \geq 1.$$

Since

$$\|\tilde{\mathbf{A}}\| = \|\mathbf{A}_{d_k} \cdots \mathbf{A}_{d_1}\|_{\text{op}} \leq 1,$$

we conclude $\rho(\tilde{\mathbf{A}}) = 1$. This shows that $\bar{\rho}_k(\Sigma) = 1$, proving the normed finiteness conjecture in this case.

We will construct a suitable ω using Lemma 3.2. First we consider the set

$$\mathcal{T} = \{(d_1, d_2, \dots, d_k) : \|\mathbf{A}_{d_k} \cdots \mathbf{A}_{d_1}\|_{\text{op}} = 1\}.$$

This set is infinite, for if it were finite, then $\rho_k(\Sigma) < 1$ for large enough k and, by (1.7),

$$\bar{\rho}(\Sigma) \leq \hat{\rho}(\Sigma) \leq \rho_k(\Sigma)^{1/k} < 1,$$

contradicting $\bar{\rho}(\Sigma) = 1$. By Konig's infinity lemma there exists an infinite sequence

$$\omega' = (d_1, d_2, d_3, \dots), \quad \text{each } d_i \in \{1, 2, \dots, m\}, \quad (3.10)$$

with all products

$$\|\mathbf{A}_{d_k} \mathbf{A}_{d_{k-1}} \cdots \mathbf{A}_{d_1}\|_{\text{op}} = 1. \quad (3.11)$$

Now on the alphabet $\mathcal{A} = \{1, 2, \dots, m\}$ consider the set

$$\Delta = \{(e_1, \dots, e_k) : \|\mathbf{A}_{e_k} \mathbf{A}_{e_{k-1}} \cdots \mathbf{A}_{e_1}\|_{\text{op}} = 1\}. \quad (3.12)$$

The one-sided subshift \mathcal{S}_Δ on \mathcal{A} contains ω' by (3.11), so it is nonempty. By Lemma 3.2 it contains an infinite word

$$\omega = (d_1, d_2, d_3, \dots)$$

such that every initial block (d_1, d_2, \dots, d_k) occurs infinitely many times as a block in ω . The definition of \mathcal{S}_Δ gives

$$\|\mathbf{A}_{d_j} \mathbf{A}_{d_{j-1}} \cdots \mathbf{A}_{d_1}\|_{\text{op}} = 1 \quad (3.13)$$

for all $j \geq 1$.

It remains to show that this is the desired ω . First we check that its associated limit set Z is nonempty. This follows from (3.13). By definition there exists some

\mathbf{x}_j with $\|\mathbf{x}_j\| = 1$ and $\|\mathbf{A}_{d_j} \cdots \mathbf{A}_{d_1} \mathbf{x}_j\| = 1$. Then by submultiplicativity of the operator norm,

$$\|\mathbf{A}_{d_i} \cdots \mathbf{A}_{d_1} \mathbf{x}_j\| = 1 \quad \text{for } 1 \leq i \leq j,$$

whence

$$\mathbf{A}_{d_i} \cdots \mathbf{A}_{d_1} \mathbf{x}_j \in \partial \mathbf{B} \subseteq V_{\Omega'}(f_0). \quad (3.14)$$

Thus $f_i(\mathbf{x}_j) = 0$ for $0 \leq i \leq j$, and so $\mathbf{x}_j \in Z_j$. Since all Z_j are nonempty, the Noetherian property (3.7) gives $Z \neq \emptyset$. Second, we check that (3.9) holds. Given $Z = Z_r$, take the smallest $k \geq r$ such that the block $(d_{k+1}, \dots, d_{k+r})$ is identical with the initial block (d_1, d_2, \dots, d_r) . Then

$$\begin{aligned} & \mathbf{B} \cap V_{\Omega'}(f_k, f_{k+1}, \dots, f_r) \\ &= \{\mathbf{x} \in \mathbf{B} : \mathbf{w} = \mathbf{A}_{d_k} \cdots \mathbf{A}_{d_1} \mathbf{x} \text{ has } f_j(\mathbf{w}) = 0 \text{ for } 0 \leq j \leq r\}. \\ &= \{\mathbf{x} \in \mathbf{B} : \mathbf{A}_{d_k} \cdots \mathbf{A}_{d_1} \mathbf{x} \in Z_r\}, \end{aligned} \quad (3.15)$$

because $\mathbf{w} \in \mathbf{B}$, since \mathbf{A}_i maps \mathbf{B} into \mathbf{B} . Now

$$\begin{aligned} \mathbf{B} \cap V_{\Omega'}(f_k, f_{k+1}, \dots, f_{k+r}) &\supseteq \mathbf{B} \cap V(f_0, f_1, \dots, f_{k+r}) \\ &= Z_{k+r} = Z_r. \end{aligned}$$

Together with (3.15) this yields

$$\mathbf{A}_{d_k} \cdots \mathbf{A}_{d_1}(Z_r) \subseteq Z_r,$$

which is (3.9). ■

Theorem 3.1 unfortunately does not apply to the l^p norm. For irrational p , the holomorphic function $\sum_{i=1}^n z_i^p$ whose zero set defines the boundary of the unit ball has singularities where a coordinate vanishes, and it also has a singularity at $\mathbf{z} = \mathbf{0}$. (The case of l^p norms with p rational is handled by Theorem 3.2 following.)

The bound for the constant k in (1.12) produced by Theorem 3.1 may depend on the set Σ . This is due to the ineffectiveness of the constant r occurring in the Noetherian bound in Lemma 3.1. Note that Lemma 3.2 supplies an effective bound $f(r, m)$ if the bound for r is known.

For piecewise algebraic norms we can obtain an explicit bound for r , and hence a finite bound $\alpha(m, \|\cdot\|)$. A *piecewise algebraic norm* is one whose unit ball \mathbf{B} has boundary $\partial \mathbf{B}$ contained in the zero set of a real polynomial $p(\mathbf{z}) \in \mathbb{R}[z_1, \dots, z_n]$ which has $p(\mathbf{0}) \neq 0$. One can show that the l^p norm for rational p is a piecewise algebraic norm.

THEOREM 3.2. *If $\|\cdot\|$ is a piecewise algebraic norm on \mathbb{R}^n , then for each $m \geq 1$ there is a finite bound $\alpha(m, \|\cdot\|)$ such that if $|\Sigma| = m$, then*

$$\widehat{\rho}(\Sigma) = \overline{\rho}_k(\Sigma)^{1/k}$$

holds with $k = \alpha(m, \|\cdot\|)$. Consequently the generalized critical exponents $\beta(m, \|\cdot\|)$ are finite for all $m \geq 1$.

The special case $m = 1$ of this theorem is a known result concerning critical exponents; cf. Belitskii and Lyubich (1988, Theorem 2.6.1). To handle the case $m \geq 2$, however, we need the following lemma.

LEMMA 3.3. *Let $p(\mathbf{z}) \in \mathbb{C}[z_1, \dots, z_n]$ be a polynomial in n variables of total degree at most d . Let $\{\mathbf{A}_i : i = 1, 2, \dots\}$ be any sequence of $n \times n$ complex matrices, and consider the family $\{V_m : m \geq 0\}$ of zero sets*

$$V_m = \{\mathbf{z} \in \mathbb{C}^n : p(\mathbf{A}_j \mathbf{A}_{j-1} \cdots \mathbf{A}_1 \mathbf{z}) = 0 \text{ for } 0 \leq j \leq m\}.$$

This family contains at most $\binom{n+d}{d}$ different sets.

Proof. Certainly

$$V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots.$$

Let W_m denote the span of

$$\{p(\mathbf{A}_j \cdots \mathbf{A}_1 \mathbf{z}) : 0 \leq j \leq m\}$$

regarded as a \mathbb{C} -vector space, which is contained in the vector space of all polynomials of total degree d . The latter has dimension $\binom{n+d}{d}$. Now $V_{m_1} = V_{m_2}$ if $W_{m_1} = W_{m_2}$, but not necessarily vice versa. Since

$$W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots,$$

there are at most $\binom{n+d}{d}$ distinct W_k . ■

Proof of Theorem 3.2. Follow the proof of Theorem 3.1 to obtain an infinite word $\omega = (d_1, d_2, \dots)$ with all

$$\|\mathbf{A}_{d_k} \mathbf{A}_{d_{k-1}} \cdots \mathbf{A}_1\| = 1,$$

which satisfies all the conclusions of Lemma 3.2. Set $p_0(\mathbf{z}) = p(\mathbf{z})$ and $p_k(\mathbf{z}) = p(\mathbf{A}_{d_k} \cdots \mathbf{A}_{d_1} \mathbf{z})$ for $k \geq 1$, and let

$$Z_k := \mathbf{B} \cap \{\mathbf{z} \in \mathbb{R}^n : p(\mathbf{z}) = p_1(\mathbf{z}) = \cdots = p_k(\mathbf{z}) = 0\}.$$

Then $Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \cdots$, and each Z_i is nonempty, as in (3.14). Let W_k denote the vector space of polynomials spanned by $\{p_0(\mathbf{z}), \dots, p_k(\mathbf{z})\}$. Now $Z_k \neq Z_{k+1}$ implies that $W_k \subsetneq W_{k+1}$, while Lemma 3.3 says that $w_k := \dim W_k$ takes at most $\binom{n+d}{d}$ different values.

It now suffices to produce a bound $\alpha(m, \|\cdot\|)$ such that there exists some Z_j with

$$A_{d_k} \cdots A_{d_1}(Z_j) \subseteq Z_j \quad (3.16)$$

for $k \leq \alpha(m, \|\cdot\|)$, because then $\rho(A_{d_k} \cdots A_{d_1}) = 1$ as in the proof of Theorem 3.1. The idea is that, although we have no bound on how many Z_j will occur before the sequence stabilizes, if the W_j remain constant for too long, the bound $f(r, m)$ in Lemma 3.2 will apply to give a Z_j satisfying (3.16), where r is the smallest value of j such that W_j assumed its current value w . If (3.16) doesn't hold for any smaller value of w , we can inductively bound this j by

$$j \leq r(w),$$

where $r(w)$ is given by the recursion

$$r(w) = r(w-1) + f(r(w-1), m). \quad (3.17)$$

Since $w \leq \binom{n+d}{d}$ by Lemma 3.3, we obtain the bound

$$\alpha(m, \|\cdot\|) \leq 1 + \sum_{w=1}^{\binom{n+d}{d}} r(w),$$

which completes the proof. ■

4. POLYTOPE NORMS

A norm on \mathbb{R}^n is a *polytope norm* if its unit ball is a polytope. Gurvits (1992) proves that the normed finiteness conjecture holds for polytope norms; his proof is by contradiction and does not give a bound for $\alpha(m, \|\cdot\|)$. Here we show that the normed finiteness conjecture is true for polytope norms, with a universal bound depending only on the polytope P and not on $|\Sigma|$. Given a convex polytope P in \mathbb{R}^n , let $f_j(P)$ be the number of j -dimensional faces of P .

THEOREM 4.1. *Let $\|\cdot\|$ be a polytope norm on \mathbb{R}^n with unit ball P , and $\|\cdot\|_{\text{op}}$ its associated operator norm on $n \times n$ matrices. If $\Sigma = \{A_1, \dots, A_m\}$ has joint spectral radius $\widehat{\sigma}(\Sigma) = 1$ and all $\|A_i\|_{\text{op}} \leq 1$, then there exists some finite product $A_{d_1} \cdots A_{d_k}$ with*

$$k \leq \frac{1}{2} \sum_{j=0}^{n-1} f_j(P), \quad (4.1)$$

which has spectral radius $\sigma(A_{d_1} \cdots A_{d_k}) = 1$.

This result immediately yields for polytope norms the generalized critical exponent bounds

$$\beta(m, \|\cdot\|) \leq \alpha(m, \|\cdot\|) \leq \frac{1}{2} \sum_{j=0}^{n-1} f_j(P),$$

which are independent of m .

Proof of Theorem 4.1. The polytope \mathbf{P} uniquely decomposes into a disjoint union of open faces of various dimensions, with $\mathbf{P}^\circ := \text{int } \mathbf{P}$ being its unique n -dimensional face. In algebraic terms, each $(n-1)$ -face of \mathbf{P} is the intersection of \mathbf{P} with a hyperplane $L_i(\mathbf{x}) = 0$, where the linear form

$$L_i(\mathbf{x}) = \sum_{j=1}^n c_j x_j + c_0,$$

has $c_0 > 0$, so that the half space $L_i(\mathbf{x}) > 0$ contains \mathbf{P}° . Each k -dimensional open face is determined by the conditions that exactly $n-k$ of the $L_i(\mathbf{x})$ are identically zero and the remainder are strict inequalities of the form $L_i(\mathbf{x}) > 0$. Let \mathcal{X} denote the collection of the $\sum_{j=0}^n f_j(\mathbf{P})$ open faces of \mathbf{P} . Note that since \mathbf{P} is centrally symmetric around $\mathbf{0}$, if γ is an open face then so is $-\gamma$, and $\gamma \neq -\gamma$ except for the open n -face \mathbf{P}° . Thus $f_j(\mathbf{P})$ is even for $0 \leq j \leq n-1$.

Claim. *Given $\mathbf{A}_i \in \Sigma$. For each open face $\gamma \in \mathcal{X}$ there is a unique $\gamma' \in \mathcal{X}$ such that*

$$\mathbf{A}_i(\gamma) \subseteq \gamma'. \quad (4.2)$$

To prove the claim, note first that $\|\mathbf{A}_i\|_{\text{op}} \leq 1$ means $\mathbf{A}_i(\mathbf{P}) \subseteq \mathbf{P}$. Thus it suffices to show that the image $\mathbf{A}_i(\gamma)$ is in at most one open face. We argue by contradiction. Suppose $\mathbf{A}_i(\gamma)$ were in two open faces. These two faces differ in at least one facet constraint $L_j(\mathbf{x})$, with one having $L_j(\mathbf{x}) = 0$ and the other $L_j(\mathbf{x}) > 0$. Then there are points $\mathbf{x}_1, \mathbf{x}_2 \in \gamma$ with $L_j(\mathbf{A}_i \mathbf{x}_1) > 0$ and $L_j(\mathbf{A}_i \mathbf{x}_2) = 0$. Since γ is (relatively) open, there exists $\varepsilon > 0$ with

$$\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2 \in \gamma$$

for $-\varepsilon \leq \lambda \leq 1 + \varepsilon$. In particular $\mathbf{x}' = -\varepsilon \mathbf{x}_1 + (1 + \varepsilon) \mathbf{x}_2 \in \gamma$ and $L_j(\mathbf{A}_i \mathbf{x}') < 0$, which implies $\mathbf{A}_i \mathbf{x}' \notin \mathbf{P}$, contradicting $\mathbf{A}_i(\mathbf{P}) \subseteq \mathbf{P}$, so the claim follows.

The claim shows that each $\mathbf{A}_i \in \Sigma$ defines a mapping $\phi_i : \mathcal{X} \rightarrow \mathcal{X}$. In particular $\phi_i(\mathbf{P}^\circ) = \mathbf{P}^\circ$ for all i , because $\mathbf{0} \in \mathbf{P}^\circ$ and \mathbf{A}_i maps $\mathbf{0}$ to $\mathbf{0}$.

The hypothesis $\widehat{\sigma}(\Sigma) = 1$ guarantees that there exist arbitrarily long products $\mathbf{A}_{d_1} \mathbf{A}_{d_2} \cdots \mathbf{A}_{d_k}$ with

$$\|\mathbf{A}_{d_k} \mathbf{A}_{d_{k-1}} \cdots \mathbf{A}_{d_1}\|_{\text{op}} = 1.$$

Take such a product with $k = \frac{1}{2} \sum_{j=0}^{n-1} f_j(\mathbf{P})$, and set $\widehat{\mathbf{A}}_j := \mathbf{A}_{d_j} \mathbf{A}_{d_{j-1}} \cdots \mathbf{A}_{d_1}$. Since $\|\widehat{\mathbf{A}}_k\|_{\text{op}} = 1$, there exists $\mathbf{x}_0 \in \partial \mathbf{P}$ with $\widehat{\mathbf{A}}_k \mathbf{x}_0 \in \partial \mathbf{P}$. Now let $\mathbf{x}_j := \mathbf{A}_j \mathbf{x}_0$ and let γ_j be the unique open face of \mathbf{P} containing \mathbf{x}_j . The claim now implies that

$$\mathbf{A}_{d_j}(\gamma_{j-1}) \subseteq \gamma_j, \quad 1 \leq j \leq k.$$

Since $\widehat{\mathbf{A}}_k(\mathbf{x}_0) \in \partial \mathbf{P}$, one has $\gamma_k \neq \mathbf{P}^\circ$. Then all $\gamma_j \neq \mathbf{P}^\circ$, because the remark above shows that if $\gamma_j = \mathbf{P}^\circ$ then $\gamma_{j+1} = \mathbf{P}^\circ$, whence $\gamma_k = \mathbf{P}^\circ$, a contradiction.

Now we have $k+1$ faces $\{\gamma_j : 0 \leq j \leq k\}$, and since there are exactly $2k$ faces of dimension $\leq n-1$ in \mathbf{P} , there must occur either $\gamma_i = \gamma_j$ or $\gamma_i = -\gamma_j$ for some $i > j$. We assert that $\mathbf{C} = \mathbf{A}_{d_i} \mathbf{A}_{d_{i-1}} \cdots \mathbf{A}_j$ has the desired properties. Certainly (4.2) holds, and $\mathbf{C}(\gamma_j) \subseteq \pm \gamma_j$. Consequently

$$\mathbf{C}^2(\overline{\gamma_j}) \subseteq \overline{\gamma_j},$$

where $\overline{\gamma_j}$ is the closure of γ_j . Since $\overline{\gamma_j}$ is compact and doesn't contain $\mathbf{0}$, one concludes as in the proof of Theorem 3.1 that

$$\rho(\mathbf{C})^2 = \rho(\mathbf{C}^2) \geq 1.$$

Since $\rho(\mathbf{C}) \leq \|\mathbf{C}\|_{\text{op}} = 1$, we conclude that $\rho(\mathbf{C}) = 1$. ■

We remark that Gurvits (1991, 1992, 1994) proves the following result:

THEOREM 4.2 ((Gurvits)). *Let $\|\cdot\|$ be a polytope norm on \mathbb{R}^n with associated operator norm $\|\cdot\|_{\text{op}}$ on $n \times n$ matrices. Suppose $\Sigma = \{\mathbf{A}_1, \dots, \mathbf{A}_m\}$ has all $\|\mathbf{A}_i\|_{\text{op}} \leq 1$. Let*

$$\mathcal{L}_\Sigma = \{(d_1, \dots, d_j) : \|\mathbf{A}_{d_j} \cdots \mathbf{A}_{d_1}\| = 1\}.$$

Then \mathcal{L}_Σ is a regular language in the alphabet $\mathcal{A} = \{1, 2, \dots, m\}$.

Regular languages are those languages recognizable by a finite-state automaton. A finite-state automaton recognizing the language \mathcal{L}_Σ is implicit in our proof of Theorem 4.1. The states of the machine are the set $\mathcal{X} - \{\mathbf{P}^\circ\}$. A transition labelled i goes from state γ_j to γ'_j if $\phi_i(\gamma_j) = \gamma'_j$ and $\gamma'_j \neq \mathbf{P}^\circ$.

5. EUCLIDEAN NORM

We prove the normed finiteness conjecture for the Euclidean norm on \mathbb{R}^n , with a universal bound $\alpha(m, \|\cdot\|)$ for k that depends only on the dimension n and the cardinality m of Σ . This result applies to ellipsoid norms, because any ellipsoid

norm can be transformed to the Euclidean norm case by a similarity transformation. In this section $\|\mathbf{x}\| = (\sum_{i=1}^n x_i^2)^{1/2}$ is the Euclidean norm.

THEOREM 5.1. *Let $\|\cdot\|$ be the Euclidean norm on \mathbb{R}^n , and $\|\cdot\|_{\text{op}}$ its associated operator norm on $n \times n$ matrices. Define a function $g(d, m)$ recursively by $g(0, m) = 1$ and*

$$g(d+1, m) = m^{g(d, m)} + g(d, m). \quad (5.1)$$

If $\Sigma = \{\mathbf{A}_1, \dots, \mathbf{A}_m\}$ has generalized spectral radius $\bar{\rho}(\Sigma) = 1$ and all $\|\mathbf{A}_i\|_{\text{op}} \leq 1$, then there exists some finite product $\mathbf{A}_{d_k} \cdots \mathbf{A}_{d_1}$ with

$$k \leq g(n-1, m), \quad (5.2)$$

which has spectral radius $\rho(\mathbf{A}_{d_k} \cdots \mathbf{A}_{d_1}) = 1$.

In particular we obtain the bounds

$$\beta(m, \|\cdot\|) \leq \alpha(m, \|\cdot\|) \leq g(n-1, m)$$

for the Euclidean norm in \mathbb{R}^n . Note that $g(n-1, 1) = n$, which with the known result $\beta(1, \|\cdot\|) = n$ yields

$$\beta(1, \|\cdot\|) = \alpha(1, \|\cdot\|) = n.$$

The bound $g(n, m)$ grows extremely rapidly; e.g. $g(5, 2) \geq 2^{2059}$. It is presumably far from the truth. Theorem 5.2 below shows that any bound for $\alpha(m, \|\cdot\|)$ must depend on $m = |\Sigma|$.

The proof of Theorem 5.1 is analogous to the proofs in Sections 3 and 4 in that it studies how products of matrices in Σ map the boundary $\partial\mathbf{B} = \{\mathbf{x} : \|\mathbf{x}\| = 1\}$ into itself.

LEMMA 5.1. *Suppose $\|\mathbf{A}\|_{\text{op}} \leq 1$. Then*

$$V(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{A}\mathbf{x}\| = \|\mathbf{x}\|\}$$

and

$$V^*(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{A}^T \mathbf{x}\| = \|\mathbf{x}\|\}$$

are both vector spaces, and have equal dimension.

Proof. One has

$$\begin{aligned} \|\mathbf{x}\|^2 &= \|\mathbf{A}\mathbf{x}\|^2 = \langle \mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{A}^T \mathbf{A}\mathbf{x} \rangle \\ &\leq \|\mathbf{x}\| \|\mathbf{A}^T \mathbf{A}\mathbf{x}\| \leq \|\mathbf{x}\|^2, \end{aligned}$$

using the Cauchy-Schwartz inequality and $\|\mathbf{A}^T\|_{\text{op}} \leq 1$. Equality can hold only when $\mathbf{A}^T \mathbf{A} \mathbf{x} = \mathbf{x}$. Thus $V(\mathbf{A}) = \ker(I - \mathbf{A}^T \mathbf{A})$ is a vector space. Similarly $V^*(\mathbf{A}) = \ker(I - \mathbf{A} \mathbf{A}^T)$, and they have the same dimension. ■

Note that without the hypothesis $\|\mathbf{A}\|_{\text{op}} \leq 1$ the set $V = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{A}\mathbf{x}\| = \|\mathbf{x}\|\}$ need not be a vector space.

Proof of Theorem 5.1 Let \mathcal{X} denote the set of vector subspaces of \mathbb{R}^n . For any matrix \mathbf{A} with $\|\mathbf{A}\|_{\text{op}} \leq 1$ and a vector space W , define

$$\psi_{\mathbf{A}}(W) = \{\mathbf{y} : \mathbf{x} \in W, \mathbf{y} = \mathbf{A}\mathbf{x}, \text{ and } \|\mathbf{x}\| = \|\mathbf{y}\|\}.$$

By Lemma 5.1 this is a vector space, and $\psi_{\mathbf{A}} : \mathcal{X} \rightarrow \mathcal{X}$. In fact

$$\psi_{\mathbf{A}}(W) = \mathbf{A}(W) \cap V^*(\mathbf{A}),$$

so that $\dim \psi_{\mathbf{A}}(W) \leq \dim W$.

It is also clear that $\psi_{\mathbf{AB}}(W) = \psi_{\mathbf{A}} \circ \psi_{\mathbf{B}}(W)$.

Claim. Suppose $0 \leq d \leq n - 1$. Given any sequence $\mathbf{C}_1, \dots, \mathbf{C}_r$ drawn from Σ such that

$$\dim[\psi_{\mathbf{C}_r} \circ \psi_{\mathbf{C}_{r-1}} \circ \dots \circ \psi_{\mathbf{C}_1}(\mathbb{R}^n)] \geq n - d,$$

with $r \geq g(d, m)$, then there exists $r \geq j > i \geq 1$ with $\rho(\mathbf{C}_j \mathbf{C}_{j-1} \dots \mathbf{C}_i) = 1$.

The claim is proved by induction on d . It's true for $d = 0$ because if $\psi_{\mathbf{C}_1}(\mathbb{R}^n) = \mathbb{R}^n$ then \mathbf{C}_1 is an isometry so $\rho(\mathbf{C}_1) = 1$.

For the induction step, suppose it is true for $d - 1$ and that

$$r = g(d, m) = m^{g(d-1, m)} + g(d - 1, m).$$

We may suppose that

$$\dim[\psi_{\mathbf{C}_j} \circ \psi_{\mathbf{C}_{j-1}} \circ \dots \circ \psi_{\mathbf{C}_1}(\mathbb{R}^n)] = n - d \quad (5.3)$$

for $r \geq j \geq g(d - 1, m)$, for otherwise the induction hypothesis for $d - 1$ produces a product $\rho(\mathbf{C}_j \mathbf{C}_{j-1} \dots \mathbf{C}_i) = 1$. Now there exist $r - g(d - 1, m) + 1 \geq m^{g(d-1, m)} + 1$ blocks $\{\mathbf{C}_j, \mathbf{C}_{j-1}, \dots, \mathbf{C}_i\}$ of length $j - i + 1 = g(d - 1, m)$, so by the pigeonhole principle two such blocks are identical, say from j_1 to i_1 and j_2 to i_2 , where $j_2 > j_1$ and $\mathbf{C}_{j_2-l} = \mathbf{C}_{j_1-l}$ for $0 \leq l \leq g(d - 1, m) - 1$. Set

$$W = \psi_{\mathbf{C}_{j_1}} \circ \psi_{\mathbf{C}_{j_1-1}} \circ \dots \circ \psi_{\mathbf{C}_{i_1}}(\mathbb{R}^n)$$

and

$$W' = \psi_{C_{j_1}} \circ \psi_{C_{j_1-1}} \circ \cdots \circ \psi_{C_{i_1}}(\mathbb{R}^n).$$

Certainly $W \subseteq W'$. If $W \neq W'$ then $\dim W' \geq n-d+1$, and the induction hypothesis applied to the sequence $C_{j_1}, C_{j_1-1}, \dots, C_{i_1}$ then produces $\rho(C_j \cdots C_i) = 1$. Thus we may suppose $W' = W$ and $\dim(W) = n-d$, so that

$$\begin{aligned} W &= \psi_{C_{j_2}} \circ \cdots \circ \psi_{C_{i_2}}(\mathbb{R}^n) \\ &= \psi_{C_{j_2}} \circ \cdots \circ \psi_{C_1}(\mathbb{R}^n) \\ &= \psi_{C_{j_2}} \circ \cdots \circ \psi_{C_{j_1+1}}(W). \end{aligned}$$

Since $\dim W \geq 1$, $W \cap \partial B$ is nonempty and

$$W \cap \partial B = \psi_{C_{j_2}} \circ \cdots \circ \psi_{C_{j_1+1}}(W \cap \partial B).$$

This implies $\rho(C_{j_2} \cdots C_{j_1+1}) = 1$ as in the proof of Theorem 3.1 (starting from (3.9)), and the induction step is completed.

Theorem 5.1 follows immediately from the Claim. For, given an infinite sequence A_{i_1}, A_{i_2}, \dots with

$$\|A_{i_k} \cdots A_{i_1}\|_{\text{op}} = 1, \quad k = 1, 2, \dots,$$

one has

$$\dim(\psi_{A_{i_k}} \circ \cdots \circ \psi_{A_{i_1}}(\mathbb{R}^n)) \geq 1$$

for all $k \geq 1$; hence choosing $r = g(n-1, m)$ gives the desired result. \blacksquare

It is possible to prove, for the Euclidean norm, and for any finite set Σ with all $\|A_i\|_{\text{op}} \leq 1$, that the language

$$\mathcal{L}_\Sigma = \{(d_1, \dots, d_j) : \|A_{d_j} \cdots A_{d_1}\|_{\text{op}} = 1\}$$

is a regular language. Further proof is required, because the set $\{V(A_{i_k} \cdots A_{i_1}) : k \geq 0 \text{ and } 1 \leq i_j \leq m \text{ for all } j\}$ may contain an infinite number of distinct vector spaces. We omit the details.

Now we give a lower bound for the quantity $\alpha(m, \|\cdot\|)$.

THEOREM 5.2. *For the Euclidean norm on \mathbb{R}^n ,*

$$\alpha(m, \|\cdot\|) \geq m.$$

Proof. The rank one matrix

$$A = \begin{bmatrix} 0 & 0 & \cos \theta \\ 0 & 0 & \sin \theta \\ 0 & 0 & 0 \end{bmatrix}$$

has $\|A\|_{\text{op}} \leq 1$ with

$$V(A) = \mathbb{R} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad V^*(A) = \mathbb{R} \left(\begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix} \right).$$

By suitable rotation of such an A we can find a similar matrix A' mapping any one-dimensional space $V_1 = V(A')$ to any other one-dimensional space $V_2 = V^*(A')$ of \mathbb{R}^3 , provided only that V_1 and V_2 are orthogonal. It is then easy to construct a set $\Sigma = \{A_i : 1 \leq i \leq m\}$ with all $V(A_i)$ distinct one-dimensional spaces, with $V(A_{i+1}) = V^*(A_i)$ for $1 \leq i \leq m-1$ and $V(A_1) = V^*(A_m)$. Then $\rho(A_m A_{m-1} \cdots A_1) = 1$, while all shorter products are nilpotent. ■

6. CONCLUDING REMARKS

Consider the normed finiteness conjecture for an arbitrary norm $\|\cdot\|$. As the proofs in Sections 3–5 illustrate, the key problem is to understand how those products $A_{i_k} \cdots A_{i_1}$ with $\|A_{i_k} \cdots A_{i_1}\|_{\text{op}} = 1$ map the boundary ∂B of the unit ball of $\|\cdot\|$ into itself. One can assign to a product $A_{i_k} \cdots A_{i_1}$ the set

$$S^*(i_1, \dots, i_k) = \{A_{i_k} \cdots A_{i_1} \mathbf{x} : \|A_{i_k} \cdots A_{i_1} \mathbf{x}\| = \|\mathbf{x}\|\}.$$

Any infinite product with

$$\|A_{i_k} \cdots A_{i_1}\|_{\text{op}} = 1, \quad k = 1, 2, 3, \dots,$$

produces a sequence of such sets, which must have unusual structure to avoid having a Noetherian inclusion property. It seems likely to us that the convexity of the unit ball ∂B together with the convexity of all the maps $A_{i_k} \cdots A_{i_1}$ prevents pathology. For example, the condition $A_i(B) \subseteq B$ forces a kind of “curvature-increasing” property on the image $A_i(\partial B)$ where it touches ∂B . This leads us to speculate that the normed finiteness conjecture is true for all norms.

Another possibility is that Theorem 3.1 actually covers all the norms that matter in the finiteness conjecture. Call a norm on \mathbb{R}^n *extremal* for a set $\Sigma = \{A_i : 1 \leq i \leq m\}$ if $\widehat{\rho}(\Sigma) = 1$ and all $\|A_i\|_{\text{op}} \leq 1$. Call a set Σ of matrices *product-bounded* if the semigroup $\mathcal{S}(\Sigma)$ of all finite products of elements of Σ is bounded. Berger

and Wang (1992) show that Σ is product-bounded if and only if there exists an operator norm $\|\cdot\|_v$ with $\|A\|_v \leq 1$ for all $A \in \Sigma$. That is, any product-bounded set Σ in \mathbb{R}^n with $\widehat{\rho}(\Sigma) = 1$ has at least one extremal norm.

EXTREMALITY CONJECTURE. Any finite set of product-bounded matrices in \mathbb{R}^n with joint spectral radius 1 has a piecewise analytic extremal norm.

In view of Theorem 3.1 the truth of the extremality conjecture would imply that of the finiteness conjecture.

Which norms are extremal norms? We note that the Euclidean norm on \mathbb{R}^2 is the only extremal norm for any rotation matrix

$$A_1 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

such that θ/π is irrational. Similarly the Euclidean norm in \mathbb{R}^n is extremal for any finite set of rotations which generate a dense subgroup of the orthogonal group $O(n, \mathbb{R})$. It is also easy to construct Σ which have an extremal norm with a unit ball that is a polytope. Direct sums then give extremal norms having unit balls that are products of such unit balls.

There remains the possibility that the finiteness conjecture is false, i.e. that the normed finiteness conjecture is false for some operator norm. Belitskii and Lyubich (1988, Section 2.6) give an example of a norm in \mathbb{R}^2 having critical exponent $+\infty$. This norm has a piecewise analytic boundary, but is not a piecewise analytic norm, because it has $f(\mathbf{0}) = 0$ for all homomorphic functions vanishing on ∂B . This suggests a class of norms to study for possible counterexamples to the normed finiteness conjecture.

Finally, Corollary 3.1 suggests the following problem.

REPEATED-BLOCK PROBLEM. Find best possible bounds $f(i, m)$ such that every one-sided subshift S_Δ on a finite alphabet \mathcal{A} of m letters with $S_\Delta \neq \emptyset$ contains a word ω such that for all $i \geq 1$ its initial block $\alpha_1 \cdots \alpha_i$ occurs at least twice without overlap in the first $f(i, m)$ symbols of ω .

APPENDIX A. SETS WITH GENERALIZED SPECTRAL RADIUS ZERO

Here we show that the generalized spectral radius and joint spectral radius coincide for all sets Σ of $n \times n$ matrices, of any cardinality, having $\widehat{\rho}(\Sigma) = 0$.

THEOREM A.1. If Σ is any set of $n \times n$ matrices whose generalized spectral radius $\rho(\Sigma)$ is zero, then

$$\mathbf{A}_{i_1} \mathbf{A}_{i_2} \cdots \mathbf{A}_{i_n} = \mathbf{0} \quad (\text{A.1})$$

for any n matrices in Σ . Consequently

$$\widehat{\rho}(\Sigma) = \overline{\rho}(\Sigma) = 0. \quad (\text{A.2})$$

Proof. If $\rho(\Sigma) = 0$, then $\rho(\mathbf{A}) = 0$ for any finite product $\mathbf{A} = \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_k}$ from Σ . Hence all elements of the semigroup $\mathcal{S}(\Sigma)$ generated by Σ are nilpotent. The \mathbb{C} -vector space $\mathcal{A}(\Sigma)$ spanned by $\mathcal{S}(\Sigma)$ is closed under multiplication; hence it is a matrix algebra. It is then a nilpotent ring by Jacobson (1964, Theorem VIII.5.1). Alternatively, it is easy to see that $\text{tr } \mathbf{M} = 0$ for all elements of $\mathcal{A}(\Sigma)$. Now $\text{tr } \mathbf{M}^j = 0$ for $1 \leq j \leq n$ implies that \mathbf{M} is nilpotent. Hence $\mathcal{A}(\Sigma)$ is a nil \mathbb{C} -algebra and hence nilpotent by Herstein (1968, pp. 19–20). Jacobson shows (1964, p. 202) that there is a similarity transformation taking $\mathcal{A}(\Sigma)$ to a ring of strictly upper-triangular matrices, whence (A.1) holds. Then (A.2) is immediate. ■

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