# The Uniformity of Non-Uniform Gabor Bases 

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#### Abstract

There have been extensive studies on non-uniform Gabor bases and frames in recent years. But interestingly there have not been a single example of a compactly supported orthonormal Gabor basis in which either the frequency set or the translation set is non-uniform. Nor has there been an example in which the modulus of the generating function is not a characteristic function of a set. In this paper, we prove that in the one dimension and if we assume that the generating function $g(x)$ of an orthonormal Gabor basis is supported on an interval, then both the frequency and the translation sets of the Gabor basis must be lattices. In fact, the Gabor basis must be the "trivial" one in the sense that $|g(x)|=c \chi_{\Omega}(x)$ for some fundamental interval of the translation set. We also give examples showing that compactly supported non-uniform orthonormal Gabor bases exist in higher dimensions.


## 1 Introduction

Let $\mathcal{F}$ and $\mathcal{T}$ be two discrete subsets $\mathbb{R}^{d}$, and let $g(x) \in L^{2}\left(\mathbb{R}^{d}\right)$. The Gabor system (also known as the Weyl-Heisenberg system) with respect to $\mathcal{F}, \mathcal{T}$ and $g$ is the following family of functions in $L^{2}\left(\mathbb{R}^{d}\right)$ :

$$
\begin{equation*}
\mathbf{G}(\mathcal{F}, \mathcal{T}, g):=\left\{e^{2 \pi i \lambda \cdot x} g(x-p) \mid \lambda \in \mathcal{F}, p \in \mathcal{T}\right\} \tag{1.1}
\end{equation*}
$$

[^0]Such a family was first introduced by Gabor [12] in 1946 for signal processing, and is still widely used today. We call $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ an (orthonormal) Gabor basis if it is an orthonormal basis for $L^{2}\left(\mathbb{R}^{d}\right)$, and a Gabor frame if it is a frame for $L^{2}\left(\mathbb{R}^{d}\right)$. Gabor bases and frames have been extensively studied. Apart from their important applications in digital signal processing, they are significant mathematical entities on their own. They are an integral part of time-frequency analysis, and are closely related to the study of wavelets and spectral sets. Some of the recent developments on Gabor systems can be found in the bibliography of this paper.

Most of the study of Gabor bases have focused on "uniform" sets $\mathcal{F}$ and $\mathcal{T}$, i.e. they are taken to be lattices. For full rank lattices $\mathcal{F}=A\left(\mathbb{Z}^{d}\right)$ and $\mathcal{T}=B\left(\mathbb{Z}^{d}\right)$, a Gabor basis $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ must satisfy $|\operatorname{det}(A B)|=1$, see Rieffel [27] for $d=1$ and Ramanathan and Steger [26] for arbitrary $d$. Conversely, if $|\operatorname{det}(A B)|=1$ then there exists a function $g(x) \in L^{2}\left(\mathbb{R}^{d}\right)$ such that $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ is an orthonormal Gabor basis, see Han and Wang [15]. The function $g$ is not necessarily compactly supported. A compactly supported $g$ can be found if $\mathcal{F}$ and $\mathcal{T}^{*}$ (the dual lattice of $\mathcal{T})$ are commensurable. Also known is the Balian-Low Theorem, which states that if $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ is an orthonormal Gabor basis and $g$ is compactly supported then $g$ cannot be very smooth, see [3].

The study of non-uniform or irregular Gabor bases and frames, i.e. those without the lattice condition on $\mathcal{F}$ or $\mathcal{T}$, has gained considerable interest (see e.g. Casazza and Christensen [5]). It is known that if $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ is a Riesz basis then both $\mathcal{F}$ and $\mathcal{T}$ must be uniformly discrete, i.e. there exists an $\epsilon>0$ such that they are $\epsilon$-separated. The density result by Ramanathan and Steger [26] was actually established in a much more general setting. In a Gabor basis $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ the sets $\mathcal{F}$ and $\mathcal{T}$ satisfy the density condition $D(\mathcal{F}) D(\mathcal{T})=1$ where $D(\cdot)$ is the Beurling density. For a set $\mathcal{J}$ in $\mathbb{R}^{d}$ the upper and lower Beurling density of $\mathcal{J}$ respectively are defined as

$$
\begin{aligned}
D^{+}(\mathcal{J}) & =\limsup _{r \rightarrow \infty} \sup _{x \in \mathbb{R}^{d}} \frac{\left|\mathcal{J} \cap\left(x+[0, r]^{d}\right)\right|}{r^{d}} \\
D^{-}(\mathcal{J}) & =\liminf _{r \rightarrow \infty} \inf _{x \in \mathbb{R}^{d}} \frac{\left|\mathcal{J} \cap\left(x+[0, r]^{d}\right)\right|}{r^{d}}
\end{aligned}
$$

If $D^{+}(\mathcal{J})=D^{-}(\mathcal{J})$ then $D(\mathcal{J})=D^{+}(\mathcal{J})=D^{-}(\mathcal{J})$ is the Beurling density of $\mathcal{J}$. But oddly, despite the many studies on non-uniform Gabor bases none of the papers contained a single example of an orthonormal Gabor basis that is non-uniform in the sense that either $\mathcal{F}$ or $\mathcal{T}$ is nonperiodic. In fact, we have not seen an example given in the Gabor literature in which an orthonormal Gabor basis has non-lattice $\mathcal{F}$ and $\mathcal{T}$. Another observation is that there is not a single example of a compactly supported orthonormal Gabor basis in which the generating function $g(x)$ does not satisfy $|g(x)|=\frac{1}{\sqrt{\mu(\Omega)}} \chi_{\Omega}(x)$ for some bounded set $\Omega$. These observations lead to the following questions: Are there any non-uniform orthonormal Gabor bases, and are there compactly supported orthonormal Gabor bases $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ in which $|g(x)|$ is
not a characteristic function?
As we shall demonstrate, there are indeed non-uniform orthonormal Gabor bases in dimension $d \geq 2$. In the one dimension there exist orthonormal Gabor bases $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ in which neither $\mathcal{F}$ nor $\mathcal{T}$ is a lattice. These results follow rather easily from the work on spectral sets. Nevertheless, if $g$ is compactly supported we establish:

Theorem 1.1 Let $g(x) \in L^{2}(\mathbb{R})$ be compactly supported and let $\mathcal{F}, \mathcal{T}$ be subsets of $\mathbb{R}$. Suppose that $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ is an orthonormal Gabor basis and $\mathcal{F}$ is periodic. Then $\mathcal{T}$ must be periodic.

If in addition we assume that $\operatorname{supp}(g)$ is an interval, then the main theorem of ours below states that the only such orthonormal Gabor bases are the "trivial" bases:

Theorem 1.2 Let $g(x) \in L^{2}(\mathbb{R})$ such that $\operatorname{supp}(g)$ is an interval, and let $\mathcal{F}, \mathcal{T}$ be subsets of $\mathbb{R}$. Suppose that $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ is an orthonormal Gabor basis. Then both $\mathcal{F}$ and $\mathcal{T}$ must be (possibly translated) lattices. In other words there exist real numbers $a \geq 0$ and $b_{1}, b_{2}$ such that $\mathcal{F}=a \mathbb{Z}+b_{1}$ and $\mathcal{T}=a^{-1} \mathbb{Z}+b_{2}$. Furthermore, $|g(x)|=$ $\sqrt{a} \chi_{\Omega}(x)$ where $\Omega$ is an interval of length $a^{-1}$.

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## 2 Proof of Theorems

Throughout of this paper we shall use $e(x)$ to denote $e^{2 \pi i x}$.

Lemma 2.1 Let $f(x)$ be a compactly supported function in $L^{1}(\mathbb{R})$, and $\mathcal{T} \subset \mathbb{R}$ be a discrete set with $D^{+}(\mathcal{T})<\infty$. Suppose that $\sum_{p \in \mathcal{T}} f(x-p)=c$ for all $x \in \mathbb{R}$. Then $\mathcal{T}$ is a union of (possibly translated) lattices,

$$
\mathcal{T}=\bigcup_{j=1}^{N}\left(L_{j} \mathbb{Z}+b_{j}\right)
$$

for some real $L_{j} \neq 0$ and $b_{j}, 1 \leq j \leq N$.

Proof. See Kolountzakis and Lagarias [18].
Proof of Theorem 1.1. $\mathcal{F}$ is periodic so we may write $\mathcal{F}=L \mathbb{Z}+\mathcal{A}$ for some $L \neq 0$ and finite set $\mathcal{A} \subset \mathbb{R}$. Without loss of generality we assume that $L=1$ and $|\mathcal{A}|=m$.

Since we have an orthonormal Gabor basis $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$, applying Parseval's identity to the function $\phi_{t}(x)=\chi_{[t, t+1]}(x)$ yields

$$
\begin{aligned}
\left\|\phi_{t}\right\|^{2} & =\sum_{p \in \mathcal{T}} \sum_{\lambda \in \mathcal{F}}\left|\left\langle\phi_{t}(x), e(\lambda x) g(x-p)\right\rangle\right|^{2} \\
& =\sum_{p \in \mathcal{T}} \sum_{\lambda \in \mathcal{F}}\left|\int_{t}^{t+1} e(-\lambda x) \overline{g(x-p)} d x\right|^{2} \\
& =\sum_{p \in \mathcal{T}} \sum_{a \in \mathcal{A}} \sum_{n \in \mathbb{Z}}\left|\int_{t}^{t+1} e(-(n+a) x) \overline{g(x-p)} d x\right|^{2} .
\end{aligned}
$$

Observe that $\{e((n+a) x): n \in \mathbb{Z}\}$ is an orthonormal basis for $L^{2}([t, t+1])$ for any $t$. Therefore another application of Parseval's identity yields

$$
\sum_{n \in \mathbb{Z}}\left|\int_{t}^{t+1} e(-(n+a) x) \overline{g(x-p)} d x\right|^{2}=\int_{t}^{t+1}|g(x-p)|^{2} d x
$$

Thus

$$
\begin{aligned}
\left\|\phi_{t}\right\|^{2} & =\sum_{p \in \mathcal{T}} \sum_{a \in \mathcal{A}} \sum_{n \in \mathbb{Z}}\left|\int_{t}^{t+1} e(-(n+a) x) \overline{g(x-p)} d x\right|^{2} \\
& =\sum_{p \in \mathcal{T}} \sum_{a \in \mathcal{A}} \int_{t}^{t+1}|g(x-p)|^{2} d x \\
& =|\mathcal{A}| \sum_{p \in \mathcal{T}} \int_{t}^{t+1}|g(x-p)|^{2} d x .
\end{aligned}
$$

Now set $f(t)=\int_{t}^{t+1}|g(x)|^{2} d x$. Then $\int_{t}^{t+1}|g(x-p)|^{2} d x=f(t-p)$. We have $\left\|\phi_{t}\right\|^{2}=1$ for all $t$. So

$$
\begin{equation*}
\sum_{p \in \mathcal{T}} \int_{t}^{t+1}|g(x-p)|^{2} d x=\sum_{p \in \mathcal{T}} f(t-p)=|\mathcal{A}|^{-1} \tag{2.1}
\end{equation*}
$$

As $g$ is compactly supported, so must be $f(t)$, and $\int_{\mathbb{R}} f(t) d t<\infty$. By Lemma 2.1 $T$ must be a union of (possibly translated) lattices,

$$
\mathcal{T}=\bigcup_{j=1}^{k}\left(L_{j} \mathbb{Z}+b_{j}\right)
$$

for some real $L_{j} \neq 0$ and $b_{j}$. We claim that all $L_{i} / L_{j} \in \mathbb{Q}$. If not, say $L_{1} / L_{2}$ is irrational, then a theorem of Kronecker (see [6]) states that $L_{1} \mathbb{Z}-L_{2} \mathbb{Z}$ is dense
in $\mathbb{R}$. Hence there exist $p_{1} \in L_{1} \mathbb{Z}+b_{1}$ and $p_{2} \in L_{2} \mathbb{Z}+b_{2}$ such that $p_{1}-p_{2}$ can become arbitrarily small. This contradicts the fact that $g\left(x-p_{1}\right)$ and $g\left(x-p_{2}\right)$ are orthogonal in $L^{2}(\mathbb{R})$. Hence all $L_{j}$ must be commensurable. So $T$ is periodic, proving the theorem.

We next prove our main theorem, Theorem 1.2. We will break it down into several lemmas.

Lemma 2.2 Let $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ be an orthonormal Gabor basis. Then $D(\mathcal{F})$ and $D(\mathcal{T})$ both exist and $D(\mathcal{F}) D(\mathcal{T})=1$.

Proof. It was shown in [26] that $D(\mathcal{F} \times \mathcal{T})$ exists and is equal to 1 . But it is easy to show that $D^{-}(\mathcal{F} \times \mathcal{T})=D^{-}(\mathcal{F}) D^{-}(\mathcal{T})$ and $D^{+}(\mathcal{F} \times \mathcal{T})=D^{+}(\mathcal{F}) D^{+}(\mathcal{T})$. The lemma follows immediately.

Lemma 2.3 Let $\mathcal{F}$ be a uniformly discrete subset of $\mathbb{R}$ with $D^{+}(\mathcal{F}) \leq 1$. Suppose that for some sequence $\left\{c_{\lambda}: \lambda \in \mathcal{F}\right\} \in \ell^{2}(\mathcal{F})$ we have

$$
\sum_{\lambda \in \mathcal{F}} c_{\lambda} e(\lambda x)=0
$$

in $L^{2}([a, b])$ with $b-a>1$. Then $c_{\lambda}=0$ for all $\lambda \in \mathcal{F}$.

Proof. Assume $c_{\lambda_{0}} \neq 0$ for some $\lambda_{0} \in \mathcal{F}$. Let $\mathcal{F}_{0}=\mathcal{F} \backslash\left\{\lambda_{0}\right\}$. Then

$$
e\left(\lambda_{0} x\right)=\sum_{\lambda \in \mathcal{F}_{0}} b_{\lambda} e(\lambda x)
$$

where $b_{\lambda}=-c_{\lambda} / c_{\lambda_{0}}$ in $L^{2}([a, b])$. A theorem of Young (see [30], page 129) states that $\left\{e(\lambda x): \lambda \in \mathcal{F}_{0}\right\}$ is complete in $L^{2}([a, b])$.

Now Theorem 2.4 of Seip [29] states that $\mathcal{F}$ can be extended to $\mathcal{F}^{\prime}$ so that $\{e(\lambda x)$ : $\left.\lambda \in \mathcal{F}^{\prime}\right\}$ is a Riesz basis for $L^{2}([a, b])$. Therefore

$$
\left\|\sum_{\lambda \in \mathcal{F}} c_{\lambda} e(\lambda x)\right\|^{2} \geq B \sum_{\lambda \in \mathcal{F}}\left|c_{\lambda}\right|^{2}
$$

for some $B>0$. This is a contradiction.

Lemma 2.4 Let $\mathcal{F}$ be a uniformly discrete subset of $\mathbb{R}$ such that $D^{-}(\mathcal{F})>0$. Let $\left\{c_{\lambda}: \lambda \in \mathcal{F}\right\}$ be a sequence in $\ell^{2}(\mathcal{F})$. Then $f(x):=\sum_{\lambda \in \mathcal{F}} c_{\lambda} e(\lambda x) \in L^{2}([a, b])$ for any interval $[a, b]$, and $\|f\|^{2} \leq C \sum_{\lambda \in \mathcal{F}}\left|c_{\lambda}\right|^{2}$ where $C$ depends only on $\mathcal{F}$ and $b-a$.

Proof. The fact that $\mathcal{F}$ is uniformly discrete and $D^{-}(\mathcal{F})>0$ implies that there exists a sufficiently small $\delta>0$ such that $\{e(\lambda x): \lambda \in \mathcal{F}\}$ is a frame for $L^{2}([s, s+\delta])$ for any $s$. Let $\|\cdot\|_{\Omega}$ denote the $L^{2}$-norm for $L^{2}(\Omega)$. It follows from Duffin and Schaeffer [9] that the sum converges in $L^{2}([s, s+\delta])$ with

$$
\begin{equation*}
\|f\|_{[s, s+\delta]}^{2} \leq B \sum_{\lambda \in \mathcal{F}}\left|c_{\lambda}\right|^{2} \tag{2.2}
\end{equation*}
$$

where $B$ is the upper frame bound for the frame. Subdividing $[a, b]$ into $K \leq(b-a) / \delta$ intervals of length $\delta$ or less yields

$$
\begin{equation*}
\|f\|_{[a, b]}^{2} \leq K B \sum_{\lambda \in \mathcal{F}}\left|c_{\lambda}\right|^{2} \tag{2.3}
\end{equation*}
$$

We remark that if the condition $D^{-}(\mathcal{F})>0$ is dropped in the above lemma then $f(x):=\sum_{\lambda \in \mathcal{F}} c_{\lambda} e(\lambda x) \in L^{2}([a, b])$ still holds, see [30], section 4.3. It is unclear whether (2.3) also holds. Nevertheless the weaker result is sufficient for our purpose.

Lemma 2.5 Let $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ be an orthonormal Gabor basis for $L^{2}(\mathbb{R})$ with $\operatorname{supp}(g)=$ $[0, a]$. Suppose that $D(\mathcal{F})=1$ and let $\mathcal{T}=\left\{y_{n}: n \in \mathbb{Z}\right\}$ with $y_{n}<y_{n+1}$. Then $y_{n+1}-y_{n} \leq 1$ for all $n \in \mathbb{Z}$.

Proof. Assume the lemma is false. Then without loss of generality we may assume $y_{1}-y_{0}>1$ and $y_{0}=0$. We shall derive a contradiction.

Clearly $a \geq y_{1}$, for if not then any function $h(x) \in L^{2}(\mathbb{R})$ with $\operatorname{supp}(h) \subseteq\left[a, y_{1}\right]$ will be orthogonal to all functions in $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$, a contradiction. We now choose $\varepsilon>0$ such that $y_{1}-\varepsilon>1$ and $y_{-1}+\varepsilon<0$. Let $f(x)$ be any $L^{2}$ function supported in $[a-\varepsilon, a]$. Then

$$
\begin{equation*}
f(x)=\sum_{n \in \mathbb{Z}} \sum_{\lambda \in \mathcal{F}} c_{n, \lambda} e(\lambda x) g\left(x-y_{n}\right), \tag{2.4}
\end{equation*}
$$

where $c_{n, \lambda}=\int_{a-\varepsilon}^{a} f(t) e(-\lambda t) \overline{g\left(t-y_{n}\right)} d t$. By the choice of $\varepsilon$ the coefficients $c_{n, \lambda}=0$ for all but finitely many $n$, in particular $c_{n, \lambda}=0$ for all $n<0$. So

$$
\begin{align*}
f(x) & =\sum_{n=0}^{N} \sum_{\lambda \in \mathcal{F}} c_{n, \lambda} e(\lambda x) g\left(x-y_{n}\right) \\
& =\sum_{\lambda \in \mathcal{F}} c_{0, \lambda} e(\lambda x) g(x)+\sum_{n=1}^{N} \sum_{\lambda \in \mathcal{F}} c_{n, \lambda} e(\lambda x) g\left(x-y_{n}\right) . \tag{2.5}
\end{align*}
$$

For $n>1$ we have $y_{n} \geq y_{1}>1$, so $g\left(x-y_{n}\right)=0$ for $x \in\left[0, y_{1}-\varepsilon\right]$.

We now restrict $f(x)$ to $x \in\left[0, y_{1}-\varepsilon\right]$ as a function in $L^{2}\left(\left[0, y_{1}-\varepsilon\right]\right)$, which is 0 . Note that $\mathcal{F}$ is uniformly discrete and $D^{-}(\mathcal{F})>0$ because $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ is an orthonormal basis. By Lemma 2.4 each sum $\sum_{\lambda \in \mathcal{F}} c_{n, \lambda} e(\lambda x)$ converges in $L^{2}\left(\left[0, y_{1}-\varepsilon\right]\right)$ for each $n$. Note that $g\left(x-y_{n}\right)=0$ on $\left[0, y_{1}-\varepsilon\right]$ for all $n>0$, so (2.5) yields

$$
0=f(x)=\sum_{\lambda \in \mathcal{F}} c_{0, \lambda} e(\lambda x) g(x)
$$

on $\left[0, y_{1}-\varepsilon\right]$. But $g(x) \neq 0$ on $\left[0, y_{1}-\varepsilon\right]$. Thus

$$
\sum_{\lambda \in \mathcal{F}} c_{0, \lambda} e(\lambda x)=0
$$

on $\left[0, y_{1}-\varepsilon\right]$. It follows from Lemma 2.3 that $c_{0, \lambda}=0$ for all $\lambda \in \mathcal{F}$. However, $c_{0, \lambda}=\langle f(x), e(\lambda x) g(x)\rangle=0$ implies that $f(x) \overline{g(x)}$ is orthogonal to the set of functions $\{e(\lambda x): \lambda \in \mathcal{F}\}$ for all functions $f(x)$ with $\operatorname{supp}(f) \subseteq[a-\varepsilon, a]$. This is a contradiction.

For any $\mathcal{F} \in \mathbb{R}$ the upper asymptotic density $D_{u}(\mathcal{F})$ is defined as

$$
D_{u}(\mathcal{F}):=\limsup _{r \rightarrow \infty} \frac{|\mathcal{F} \cap[-r, r]|}{2 r}
$$

It is easy to see that $D^{-}(\mathcal{F}) \leq D_{u}(\mathcal{F}) \leq D^{+}(\mathcal{F})$. In particular if $D(\mathcal{F})$ exists then $D_{u}(\mathcal{F})=D(\mathcal{F})$. The following lemma is a key to proving Theorem 1.2. It was proved in Laba and Wang [20] in a much stronger form. We include the weak form here for completeness.

Lemma 2.6 Let $\mathcal{F}$ be a subset of $\mathbb{R}$ with $D(\mathcal{F})=1$. Suppose that $D_{u}(\mathcal{F}-\mathcal{F}) \leq 1$. Then $\mathcal{F}-\mathcal{F}=\mathbb{Z}$.

Proof. Without loss of generality we assume that $0 \in \mathcal{F}$. Hence $\mathcal{F}-\mathcal{F} \supseteq \mathcal{F}$. Clearly this means $D_{u}(\mathcal{F}-\mathcal{F}) \geq D(\mathcal{F})=1$. This yields $D_{u}(\mathcal{F}-\mathcal{F})=1$. We prove $\mathcal{F}-\mathcal{F}$ is a group.

Denote $\mathcal{G}=\mathcal{F}-\mathcal{F}$. For any $a \in \mathcal{F}$ observe that $\mathcal{F}-a$ has Beurling density $D(\mathcal{F}-a)=1$. But $\mathcal{F}-a \subseteq \mathcal{G}$ and $D_{u}(\mathcal{G})=1$. This implies that $\mathcal{G}=(\mathcal{F}-a) \cup \mathcal{E}_{a}$ with $D_{u}\left(\mathcal{E}_{a}\right)=0$. Similarly $\mathcal{G}=(\mathcal{F}-b) \cup \mathcal{E}_{b}$ for any $b \in \mathcal{F}$ with $D_{u}\left(\mathcal{E}_{b}\right)=0$. Denote $\mathcal{F}_{a, b}=(\mathcal{F}-a) \cap(\mathcal{F}-b)$. We therefore must have $D_{u}\left(\mathcal{F}_{a, b}\right)=1$ since $\mathcal{G} \backslash \mathcal{F}_{a, b}=\mathcal{E}_{a} \cup \mathcal{E}_{b}$ has Beurling density 0 . It follows that $D_{u}\left(\mathcal{F}_{a, b}+b\right)=1$. In other words, $\mathcal{F} \cap(\mathcal{F}-a+b)$ has upper asymptotic density 1 .

Now take any $a_{1}, a_{2}, b_{1}, b_{2} \in \mathcal{F}$. The above yields that both $\mathcal{F} \cap\left(\mathcal{F}-a_{1}+b_{1}\right)$ and $\mathcal{F} \cap\left(\mathcal{F}-a_{2}+b_{2}\right)$ have upper asymptotic density 1 . But both are subsets of $\mathcal{F}$, which itself has upper asymptotic density 1 . Therefore the two sets must
intersect, or the upper asymptotic density of $\mathcal{F}$ would have to be at least 2 . This means there exist $c_{1}, c_{2} \in \mathcal{F}$ such that $c_{1}-a_{1}+b_{1}=c_{2}-a_{2}+b_{2}$, which gives us $\left(b_{2}-a_{2}\right)-\left(b_{1}-a_{1}\right)=c_{1}-c_{2} \in \mathcal{G}$. Since $a_{1}, a_{2}, b_{1}, b_{2} \in \mathcal{F}$ are arbitrary, we conclude that $\mathcal{G}$ is closed under subtraction. Hence $\mathcal{G}$ is a group.

The only discrete subgroups of $\mathbb{R}$ with bounded densities are cyclic groups. So $\mathcal{G}=L \mathbb{Z}$ for some $L \in \mathbb{R}$. But the density of $\mathcal{G}$ is 1 . Hence $\mathcal{G}=\mathcal{F}-\mathcal{F}=\mathbb{Z}$.

Proof of Theorem 1.2. Without loss of generality we may assume that $D(\mathcal{F})=$ $D(\mathcal{T})=1$ by Lemma $2.2,0 \in \mathcal{F}, 0 \in \mathcal{T}$ and that $\operatorname{supp}(g)=[0, a]$. Since $\mathcal{T}$ is uniformly discrete we may write it as $\mathcal{T}=\left\{y_{n}: n \in \mathbb{Z}\right\}$ in ascending order.

Claim: $\mathcal{F}=\mathbb{Z}$ and if $0<a-\left(y_{k}-y_{n}\right) \leq 1$ then $a-\left(y_{k}-y_{n}\right)=1$.
For $0<b:=a-\left(y_{k}-y_{n}\right) \leq 1$ let $h(x)=g\left(x-y_{n}\right) \overline{g\left(x-y_{k}\right)}$. Then $h(x)$ is supported on an interval of length $b \leq 1$. The orthogonality of the Gabor basis yields $\widehat{h}(\xi)=0$ for all $\xi \in \mathcal{F}-\mathcal{F}, \xi \neq 0$. (We would even have $\widehat{h}(0)=0$ if $k \neq n$.)

But note that by the Paley-Wiener Theorem $\widehat{h}$ is an entire function of exponential type restricted to the reals, and such functions cannot have "too many" zeros. In fact it follows from Theorem 8.4.16 of Boas [2] that the set of zeros of $\widehat{h}$ has an upper asymptotic density at most $b$, i.e. $D^{+}(\mathcal{F}-\mathcal{F}) \leq b \leq 1$. It follows from $D(\mathcal{F})=1$ and Lemma 2.6 that $b=1$ and $\mathcal{F}-\mathcal{F}=\mathbb{Z}$. Since $0 \in \mathcal{F}$ we get $\mathcal{F} \subseteq \mathbb{Z}$.

Now suppose that $\mathcal{F} \neq \mathbb{Z}$. Then we may replace $\mathcal{F}$ by $\mathbb{Z}$ and the orthonormality is again satisfied with the new Gabor system $\mathcal{G}(\mathbb{Z}, \mathcal{T}, g)$. This contradicts the fact that $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ is a basis. Therefore $\mathcal{F}=\mathbb{Z}$. The Claim is proved.

We next prove $\mathcal{T}=\mathbb{Z}$ by proving that for any $n$ we have $y_{n}-y_{n-1}=1$. Assume that this is not true, then there exists an $n$ such that $y_{n}-y_{n-1}<1$. We choose $k$ to be the largest index such that $0 \leq y_{k}-y_{n}<a$. Since each $y_{j+1}-y_{j} \leq 1$ by Lemma 2.5, $0<a-\left(y_{k}-y_{n}\right) \leq 1$. It follows from the Claim that $a-\left(y_{k}-y_{n}\right)=1$. But we now have $0<a-\left(y_{k}-y_{n-1}\right)<1$ because $y_{n}-y_{n-1}<1$. This contradicts Claim 1. Hence $y_{n}-y_{n-1}=1$ for all $n$, and $\mathcal{T}=\mathbb{Z}$.

Finally we prove $a=1$ and $|g(x)|=\chi_{[0,1]}(x)$. It is easy to see that $a \geq 1$, which in fact follows from the Claim. Assume that $a>1$. Then there exists an $n \neq 0$ such that $0<b:=a-n \leq 1$. (The Claim actually gives $b=1$, but we don't need it.) Now the function $h(x)=g(x) \overline{g(x-n)}$ is supported on $[n, a]$ and is orthogonal to $e(\lambda x)$ for all $\lambda \in \mathcal{F}=\mathbb{Z}$. This is impossible unless $h(x)=0$, which is not the case since $g$ is supported on $[0, a]$. Hence $a=1$. So $|g|^{2}(x)$ is orthogonal to $e(\lambda x)$ for all $\lambda \in \mathbb{Z} \backslash\{0\}$. This forces $|g|^{2}(x)=c$. The orthonormality of the Gabor basis now yields $c=1$ and $|g|=\chi_{[0,1]}$.

## 3 Examples and Questions

The study of orthonormal Gabor bases is actually closely related to the study of spectral sets and their spectra, a link that has not been exploited in the Gabor literature. A measurable set $\Omega$ in $\mathbb{R}^{d}$ with positive and finite measure is called a spectral set if there exists an $\mathcal{F} \subset \mathbb{R}^{d}$ such that the set of exponentials $\{e(\lambda \cdot x)$ : $\lambda \in \mathcal{F}\}$ is an orthogonal basis for $L^{2}(\Omega)$. In this case $\mathcal{F}$ is called a spectrum of $\Omega$. A spectral set $\Omega$ may have more than one spectrum. Spectral sets have been studied rather extensively, particularly in recent years. We list some of these studies in the bibliography of this paper. The major unsolved problem concerning spectral sets is the following conjecture of Fuglede [11]:

Fuglede's Spectral Set Conjecture: Let $\Omega$ be a set in $\mathbb{R}^{d}$ with positive and finite Lebesgue measure. Then $\Omega$ is a spectral set if and only if $\Omega$ tiles $\mathbb{R}^{d}$ by translation.

Here by $\Omega$ tiles we mean there exists a $\mathcal{T} \subset \mathbb{R}^{d}$ such that $\Omega+\mathcal{T}$ is a measuredisjoint covering of $\mathbb{R}^{d}$, i.e. $\sum_{p \in \mathcal{T}} \chi_{\Omega}(x-p)=1$ for almost all $x \in \mathbb{R}^{d}$. The set $\mathcal{T}$ is called a tiling set for $\Omega$. The Spectral Set Conjecture remains open in either direction, even in dimension one and for sets that are unions of unit intervals. Furthermore, there appears to be a one-to-one correspondence between spectra of a spectral set and its tilings. In this section we give several examples, based on the study of spectral sets. First we establish:

Lemma 3.1 Let $\Omega \subset \mathbb{R}^{d}$ with $0<\mu(\Omega)<\infty$. Suppose that $\Omega$ is a spectral set with a spectrum $\mathcal{F}$ and it tiles $\mathbb{R}^{d}$ by the tiling set $\mathcal{T}$. Let $g(x)$ be any function with $|g|=\frac{1}{\sqrt{\mu(\Omega)}} \chi_{\Omega}$. Then $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ is an orthonormal Gabor basis for $L^{2}\left(\mathbb{R}^{d}\right)$.

Proof. The proof is rather standard, and we shall give a quick sketch here. The orthonormality is clear. Take any $e\left(\lambda_{1} \cdot x\right) g\left(x-p_{1}\right)$ and $e\left(\lambda_{2} \cdot x\right) g\left(x-p_{2}\right)$ in $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$. If $p_{1} \neq p_{2}$ then $g\left(x-p_{1}\right)$ and $g\left(x-p_{2}\right)$ have disjoint support as a result of the tiling property. So the two functions are orthogonal. If $p_{1}=p_{2}$ then $\lambda_{1} \neq \lambda_{2}$. Hence $\left\langle e\left(\lambda_{1} \cdot x\right), e\left(\lambda_{2} \cdot x\right)\right\rangle=0$ by the spectral set property. Therefore

$$
\left\langle e\left(\lambda_{1} \cdot x\right) g\left(x-p_{1}\right), e\left(\lambda_{2} \cdot x\right) g\left(x-p_{1}\right)\right\rangle=\left\langle e\left(\lambda_{1} \cdot x\right), e\left(\lambda_{2} \cdot x\right)\right\rangle=0 .
$$

To see the completeness observe that the set of functions $\{e(\lambda \cdot x) g(x-p): \lambda \in \mathcal{F}\}$ is complete in $L^{2}(\Omega+p)$ because $\mathcal{F}$ is a spectrum for $\Omega+p$ and $|g(x-p)|$ is a nonzero constant on $\Omega+p$. Now every $f(x) \in L^{2}\left(\mathbb{R}^{d}\right)$ can be expressed as $f(x)=\sum_{p \in \mathcal{T}} f_{p}(x)$ where $f_{p}(x):=f(x) \chi_{\Omega}(x-p)$ as a result of the tiling property. But $f_{p}(x) \in L^{2}(\Omega+p)$. Standard argument now implies $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ is complete in $L^{2}(\mathbb{R})$, proving the lemma.

We shall refer to an orthonormal Gabor basis obtained in such a way as a standard orthonormal Gabor basis. Standard Gabor bases nevertheless yield nontrivial examples of non-uniform Gabor bases.

Example 3.1 In dimension $d \geq 2$ there exist compactly supported orthonormal Gabor bases $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ in which both $\mathcal{F}$ and $\mathcal{T}$ are nonperiodic.

Let $\Omega=[0,1]^{d}$ be the unit $d$-cube. It is well known that there are nonperiodic tilings using the unit cube. In fact, for $d \geq 3$ there are completely aperiodic cube tilings, see Lagarias and Shor [22]. (For $d=2$ the tilings must be half periodic in the sense that it must be periodic either in the horizontal or in the vertical direction.) One simple nonperiodic tiling set for the cube $[0,1]^{2}$ in the two dimension is

$$
\mathcal{T}=\left\{\left(n, m+e^{n}\right): n, m \in \mathbb{Z}\right\},
$$

which is obtained from the standard lattice tiling by shifting the $n$-th column by $e^{n}$. Let $\mathcal{T}$ be any nonperiodic tiling of $\Omega$ and set $\mathcal{F}=\mathcal{T}$. Now a theorem of Lagarias, Reeds and Wang [21] (and independently by Iosevich and Pedersen [16]) states that $\mathcal{F}$ must also be a spectrum for $\Omega$. Therefore for $g(x)=\chi_{\Omega}(x)$ the Gabor system $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ is an orthonormal Gabor basis. By assumption neither $\mathcal{F}$ nor $\mathcal{T}$ is uniform.

Example 3.2 In the one dimension there exist compactly supported orthonormal Gabor bases $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ in which neither $\mathcal{F}$ nor $\mathcal{T}$ is a lattice.

Let $\Omega=[0,1] \cup[2,3]$. We know that this is a spectral set with spectrum $\mathcal{F}=\mathbb{Z}+\left\{0, \frac{1}{4}\right\}$, see e.g. Lagarias and Wang [24]. $\Omega$ tiles with the tiling set $\mathcal{T}=4 \mathbb{Z}+\{0,1\}$. Now for $g(x)=\frac{1}{\sqrt{2}} \chi_{\Omega}(x)$ the Gabor system $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ is an orthonormal Gabor basis. However, neither $\mathcal{F}$ nor $\mathcal{T}$ is a lattice.

Example 3.3 In an orthonormal Gabor basis $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ having one of $\mathcal{F}$ or $\mathcal{T}$ being a lattice does not imply the other must be, even in the one dimension.

Let $\Omega=[0,1] \cup[3,4]$. Again we know that this is a spectral set with two distinct spectra: $\mathcal{F}_{1}=\mathbb{Z}+\left\{0, \frac{1}{6}\right\}$ and $\mathcal{F}_{2}=\frac{1}{2} \mathbb{Z}$. $\Omega$ also has two distinct tiling sets $\mathcal{T}_{1}=$ $6 \mathbb{Z}+\{0,1,2\}$ and $\mathcal{T}_{2}=2 \mathbb{Z}$. Let $g(x)=\frac{1}{\sqrt{2}} \chi_{\Omega}(x)$. We have an orthonormal Gabor basis $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ by taking $\mathcal{F}=\mathcal{F}_{1}$ and $\mathcal{T}=\mathcal{T}_{2}$, or by taking $\mathcal{F}=\mathcal{F}_{2}$ and $\mathcal{T}=\mathcal{T}_{1}$. In either case, one is a lattice and the other is not.

We conclude this paper with the following conjecture on orthonormal Gabor bases $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$.

Conjecture: Let $g(x) \in L^{2}\left(\mathbb{R}^{d}\right)$ be compactly supported. Let $\mathcal{F}$ and $\mathcal{T}$ be discrete subsets of $\mathbb{R}^{d}$. Suppose that $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ is an orthonormal Gabor basis. Then $\mathbf{G}(\mathcal{F}, \mathcal{T}, g)$ must be standard. In other words, there exists a spectral set $\Omega$ in $\mathbb{R}^{d}$ such that
(a) $\mathcal{F}$ is a spectrum of $\Omega$.
(b) $\mathcal{T}$ is a tiling of $\Omega$.
(c) $|g(x)|=\frac{1}{\sqrt{\mu(\Omega)}} \chi_{\Omega}(x)$.

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