Subdivision Schemes and Refinement Equations with Nonnegative Masks

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Abstract

We consider the two-scale refinement equation $f(x) = \sum_{n=0}^{N} c_n f(2x-n)$ with $\sum_n c_{2n} = \sum_n c_{2n+1} = 1$ where $c_0, c_N \neq 0$ and the corresponding subdivision scheme. We study the convergence of the subdivision scheme and the cascade algorithm when all $c_n \geq 0$. It has long been conjectured that under such an assumption the subdivision algorithm converge, as well as the cascade algorithm converge uniformly to a continuous function, if and only if only if $0 < c_0, c_N < 1$ and the greatest common divisor of $S = \{n : c_n > 0\}$ is 1. We prove the conjecture for a large class of refinement equations.

Keywords: Nonnegative mask, cascade algorithm, subdivision scheme, refinement equation, refinable function.

1 Introduction

The two-scale refinement equation

$$f(x) = \sum_{n\in\mathbb{Z}} c_n f(2x-n), \qquad \sum_n c_{2n} = \sum_n c_{2n+1} = 1 \tag{1.1}$$

plays a central role in the construction of orthonormal wavelet bases and in the subdivision scheme for curve and surface generations. An important question is the continuity of the solutions $f(x)$ and the convergence of the corresponding subdivision scheme. We will assume that only finitely many $c_n \neq 0$, which is the case for virtually all applications. It is well known that under this assumption the refinement equation (1.1) has a unique (up to scalar

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multiplication) compactly solution $f(x)$ in the sense of a tempered distribution. In this paper we study the special class of refinement equations (1.1) in which all $c_n \geq 0$.

We first introduce some notations. For a given refinement equation (1.1) the *mask* is the Laurent polynomial $\mathcal{C}(z) := \frac{1}{2}\sum_n c_n z^n$. The *support* of $\mathcal{C}$ is the set supp $(\mathcal{C}) := \{n \in \mathbb{Z} : \ c_n \neq 0\}$. We say the mask $\mathcal{C}$ is *nonnegative* if all $c_n \geq 0$. A function $f(x) \in L^1(\mathbb{R})$ is the *assocaited refinable function* of the refinement equation (1.1) if it satisfies (1.1) and $\int_{\mathbb{R}} f(x)\,dx = 1$. Not every refinement equation has an associated refinable function, since the requirment $f(x) \in L^1(\mathbb{R})$ can not be met in general. When it does, the associated refinable function is unique, and is compactly supported, see [DL1].

We shall study (1.1) primarily in conjunction with subdivision schemes. A comprehensive discussion of subdivision schemes can be found in [CDM]. The subdivision scheme relates to the refinement equation (1.1) as follows: Start with a set of vectors $\{\mathbf{v}_n^0 : \ n \in \mathbb{Z}\}$ with each $\mathbf{v}_n^0 \in \mathbb{R}^m$, and recursively define the vectors $\{\mathbf{v}_n^k : \ n \in \mathbb{Z}\}$ by

$$\mathbf{v}_n^k = \sum_{j\in\mathbb{Z}} c_{n-2j}\mathbf{v}_j^{k-1}. \tag{1.2}$$

We say that the subdivision scheme with mask $\mathcal{C}$ *converges* if for each bounded set of vectors $\{\mathbf{v}_n^0 : \ n \in \mathbb{Z}\}$ there exists a continuous function $\mathbf{G} : \ \mathbb{R} \longrightarrow \mathbb{R}^m$ such that

$$\lim_{k\to\infty} \sup_{n\in\mathbb{Z}} \left|\mathbf{G}\left(\frac{n}{2^k}\right) - \mathbf{v}_n^k\right| = 0.$$

The function $\mathbf{G}(x)$ can be expressed as

$$\mathbf{G}(x) = \sum_{n\in\mathbb{Z}} f(x-n)\mathbf{v}_n^0, \tag{1.3}$$

where $f(x)$ is the associated refinable function of (1.1). By taking $m = 1$ and $\mathbf{v}_n^0 = \delta_{n,0}$ one can easily check that the subdivision scheme (1.2) is equivalent to the following *cascade algorithm* for finding the associated refinable function $f(x)$:

$$f_0(x) = \chi_{[0,1)}(x)^1, \qquad f_k(x) = \sum_{n\in\mathbb{Z}} c_n f_{k-1}(2x-n). \tag{1.4}$$

More precisely, the two schemes relate to each other by the formula $\mathbf{v}_n^k = f_k(\frac{n}{2^k})$. Therefore a subdivision scheme converges if and only if the corresponding cascade algorithm converges uniformly to a continuous function.

In this paper we study the convergence of subdivision schemes with nonnegative masks. Such schemes arise in many practical applications. Let $\mathcal{C}(z)$ be the nonnegative mask of the

[1]Observe that f_0 is not continuous. In practice it is better to choose f_0 to be the hat function. The uniform convergence of the cascade algorithm are equivalent for both cases [W].

refinement equation (1.1) such that supp $(\mathcal{C})$ is finite. By applying a suitable translation we may without loss of generality assume that supp $(\mathcal{C}) \subseteq \{0, 1, \ldots, N\}$ with $0, N \in$ supp $(\mathcal{C})$ for some $N \geq 1$. Equation (1.1) now becomes

$$f(x) = \sum_{n=0}^{N} c_n f(2x - n), \qquad \sum_n c_{2n} = \sum_n c_{2n+1} = 1, \tag{1.5}$$

where $c_0, c_N \neq 0$. The convergence problem is stated as the following conjecture:

Conjecture: *Suppose that the refinement equation (1.5) satisfies $c_0, c_N \neq 0$ and all $c_n \geq 0$. Let $\mathcal{C}(z)$ be its mask. Then the subdivision scheme with mask $\mathcal{C}$ converges if and only if*

$$0 < c_0,\ c_N < 1 \qquad \textit{and} \qquad \text{gcd } (n:\ n \in \text{supp } (\mathcal{C})) = 1. \tag{1.6}$$

It is known that (1.6) is necessary for the convergence of the cascade algorithm and the subdivision scheme, see e.g. [CDM] or [W]. The sufficiency is still open, and appears to be rather difficult. It has been discussed extensively in Cavaretta, Dahmen and Micchelli [CDM] and is stated as an important unresolved problem. Various partial results have been obtained. Micchelli and Prautzsch [MP1] show that the subdivision scheme converges if supp $(\mathcal{C}) = \{0, 1, \ldots, N\}$ for $N \geq 2$. This condition is weakened by Gonsor [G] to supp $(\mathcal{C}) \supseteq \{0, 1, N-1, N\}$ for $N \geq 2$. Melkman [M] further relaxed the condition to supp $(\mathcal{C}) \supseteq \{0, p, q, p+q\}$ for some gcd $(p, q) = 1$, who also shows that the subdivision scheme converges if supp $(\mathcal{C})$ contains two consecutive integers in addition to (1.6) with $N \geq 2$. It should be pointed out that the related results in [CDM] are all in the higher dimensional setting. In such settings the problem is also studied in Jia and Zhou [JZ], who prove that the convergence of the subdivision scheme depends only on the support of the mask, not on the actual values of the coefficients. An algorithm for checking the convergence is also given in [JZ].

A particularly interesting class of refinement equations (1.5) is those that supp $(\mathcal{C})$ contains a single odd integer $0 < p < N$, the simplest and most intriguing example of which is

$$f(x) = af(2x) + f(2x - p) + (1 - a)f(2x - N) \tag{1.7}$$

where $0 < a < 1$, $N \geq 2$ is even and gcd $(p, N) = 1$. If this happens and if condition (1.6) is met, the associated refinable function $f(x)$ is *interpolatory* in the sense that

$$f(p) = 1 \qquad \text{and} \qquad f(n) = 0 \qquad \text{for } n \in \mathbb{Z} \setminus \{p\}. \tag{1.8}$$

Interpolatory refinement equations are important for applications in computer generated graphics, because the curve $\mathbf{G}(x)$ given by (1.3) from the subdivision scheme actually passes through the points $\mathbf{v}_n^0$, i.e. $\mathbf{G}(p+n) = \mathbf{v}_n^0$. Unfortunately, as pointed out in [M], none of the existing sufficient conditions mentioned above cover, or are even applicable to, the refinement equation (1.7). In fact, other than the condition supp $(\mathcal{C}) \supseteq \{a, a+1\}$ none of them are applicable to any interpolatory subdivision schemes. By numerical computation it is shown in [M] that the subdivision scheme corresponding to (1.7) with $N = 8$ and $p = 3$ converges. However, such method cannot be used for the general setting.

The objective of this paper is to establish a sufficient condition on the convergence of the subdivision scheme that will cover a substantially larger class of schemes, including the interpolatory schemes given by (1.7) and many other interpolatory schemes. We prove:

Theorem 1.1 *Let* $\mathcal{C} = \frac{1}{2}(a + z^p + (1-a)z^N)$ *be the mask for the refinement equation*

$$f(x) = af(2x) + f(2x-p) + (1-a)f(2x-N)$$

where $0 < a < 1$, $N \geq 2$ *is even and* $\gcd(p, N) = 1$. *Then*

(i) *The subdivision scheme with mask* $\mathcal{C}$ *converges.*

(ii) *The cascade algorithm converges uniformly to the associated refinable function, which is continuous.*

Based on Theorem 1.1 we prove the following more general theorem:

Theorem 1.2 *Suppose that the refinement equation*

$$f(x) = \sum_{n=0}^{N} c_n f(2x-n), \qquad \sum_n c_{2n} = \sum_n c_{2n+1} = 1$$

has a nonnegative mask $\mathcal{C}(z)$, *with* $0 < c_0, c_N < 1$. *Suppose that there exist* $r < p < q$ *in* supp $(\mathcal{C})$ *such that* $\gcd(q-r, p-r) = 1$ *and* $2 | q-r$. *Then*

(i) *The subdivision scheme with mask* $\mathcal{C}$ *converges.*

(ii) *The cascade algorithm converges uniformly to the associated refinable function, which is continuous.*

In particular, if there exist an odd p *and an even* q *in* supp $(\mathcal{C})$ *such that* $0 < p < q$ *and* $\gcd(p, q) = 1$. *Then the convergence properties* (i) *and* (ii) *hold.*

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2 Reductions

In this section we make several reductions to transform the problem of convergence of subdivision schemes into one of combinatorics and number theory. Given the refinement equation (1.5) define the $N \times N$ *generating matrices*

$$P_0 = [c_{2j-i}]_{0 \le i,j < N}, \qquad P_1 = [c_{2j-i+1}]_{0 \le i,j < N}, \tag{2.1}$$

where the rows and columns are indexed by $0 \le i, j < N$ (instead of the conventional $1 \le i, j \le N$). Both P_0 and P_1 are row stochastic matrices, i.e. the sum of elements of every row is 1. Suppose that $f(x)$ is the associated refinable function of (1.5). Let $\mathbf{v}_f(x) = [f(x), f(x+1), \ldots, f(x+N-1)]$ for $x \in [0,1)$. Then it is well known (see [DL2]) that

$$\mathbf{v}_f(x) = \mathbf{v}_f(\tau^m x) P_{d_m} \cdots P_{d_2} P_{d_1} \tag{2.2}$$

for any $m \ge 1$, where $x = \sum_{j=1}^{\infty} 2^{-j} d_j$ with $d_j \in \{0,1\}$ and τ is the *shift function* $\tau x = \sum_{j=1}^{\infty} 2^{-j} d_{j+1}$. Note that $[1,1,\ldots,1]^T$ is a common 1-eignevector of P_0 and P_1. So taking a nonsingular matrix C whose first column is $[1,\ldots,1]^T$ yields

$$C^{-1} P_i C = \begin{bmatrix} 1 & * \\ 0 & A_i \end{bmatrix}, \qquad i = 0, 1. \tag{2.3}$$

Lemma 2.1 *The subdivision scheme corresponding to the refinement equation (1.5) converges if and only if the joint spectral radius $\hat{\rho}(A_0, A_1) < 1$.*

Proof. The convergence of the subdivision scheme is equivalent to the convergence of the cascade algorithm, which is shown in [W] to be equivalent to $\hat{\rho}(A_0, A_1) < 1$. See also [DGL]. ■

It is shown in [DL3] that $\hat{\rho}(A_0, A_1) < 1$ if and only if all products $A_{d_m} \cdots A_{d_2} A_{d_1}$ converge to the zero matrix, where $d_j \in \{0,1\}$, which in turn is equivalent to all products $P_{d_m} \cdots P_{d_2} P_{d_1}$ converge to a rank one matrix, c.f. [W]. This leads to our next reduction:

Lemma 2.2 *The subdivision scheme corresponding to the refinement equation (1.5) converges if and only if for all sequences* $(d_1, d_2, \ldots) \in \{0,1\}^{\mathbb{N}}$ *and* $\mathbf{x} \in \mathbb{R}^N$ *we have*

$$\lim_{m\to\infty} \Delta(P_{d_m} \cdots P_{d_2} P_{d_1} \mathbf{x}) = 0, \tag{2.4}$$

where for any vector $\mathbf{y} = [y_1, y_2, \ldots, y_N]^T \in \mathbb{R}^N$,

$$\Delta(\mathbf{y}) := \max_i y_i - \min_i y_i.$$

Proof. Let $V = \mathbb{R}^N/W$ be the quotient space where W is the subspace spanned by the vector $[1,1,\ldots,1]^T$. Since W is invariant under P_0 and P_1, the two matrices induce two linear maps on V, which we denote by $\tilde{P}_0$ and $\tilde{P}_1$, respectively. Observe that the $(N-1)\times(N-1)$ matrices A_0, A_1 are matrix representations of $\tilde{P}_0$, $\tilde{P}_1$ respectively, with respect to the basis represented by $(\mathbf{v}_2, \ldots, \mathbf{v}_N)$ in which $\mathbf{v}_j$ is the j-th column of the matrix C in (2.3). Therefore $\hat{\rho}(A_0, A_1) < 1$ if and only if $\hat{\rho}(\tilde{P}_0, \tilde{P}_1) < 1$, which in turn is equivalent to $\lim_{m\to\infty} \|\tilde{P}_{d_m} \cdots \tilde{P}_{d_2} \tilde{P}_{d_1}(\mathbf{z})\| = 0$ for any sequence $(d_1, d_2, \ldots) \in \{0,1\}^{\mathbb{N}}$ and $\mathbf{z} \in V$, where $\|.\|$ is some norm on V, see [DL3]. The lemma now follows from the fact that $\Delta(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^N$ induces a norm on V. ■

A sufficient condition for (2.4) to hold is that

$$\Delta(P_{d_{m_0}} \cdots P_{d_2} P_{d_1} \mathbf{x}) < \Delta(\mathbf{x}) \tag{2.5}$$

for some fixed m_0 and all sequences $(d_1, \ldots, d_{m_0}) \in \{0,1\}^{m_0}$. This follows from the observation that if (2.5) holds then

$$\Delta(P_{d_{m_0}} \cdots P_{d_2} P_{d_1} \mathbf{x}) \leq \alpha \Delta(\mathbf{x})$$

where $\alpha < 1$ is given by

$$\alpha = \max_{(d_1,\ldots,d_{m_0})} \max_{\Delta(\mathbf{x})=1} \frac{\Delta(P_{d_{m_0}} \cdots P_{d_2} P_{d_1} \mathbf{x})}{\Delta(\mathbf{x})}.$$

We will reduce the convergence problem further. Before doing so we introduce more notations. The vectors in $\mathbb{R}^N$ will be indexed by $0 \leq i < N$ rather than the conventioanl $1 \leq i \leq N$. For $\mathbf{x} \in \mathbb{R}^N$ let $\max \mathbf{x} := \max_i x_i$ and $\min \mathbf{x} := \min_i x_i$. So $\Delta(\mathbf{x}) = \max \mathbf{x} - \min \mathbf{x}$. We shall use Z_N to denote the set $\{0, 1, \ldots, N-1\}$. For any $T \subseteq Z_N$ we let $\mathbf{1}_T$ be the vector $[x_0, \ldots, x_{N-1}]^T$ such that $x_i = 1$ if $i \in T$ and $x_i = 0$ otherwise. Now any nonnegative $N \times N$ row stochastic matrix B induces a map $\Phi_B :\ 2^{Z_N} \longrightarrow 2^{Z_N}$ by

$$\Phi_B(T) = \Big\{ j \in Z_N :\ (B\mathbf{1}_T)_j = 1 \Big\}. \tag{2.6}$$

Lemma 2.3 *Let B be a nonnegative row stochastic matrix. Then*

(i) $\Delta(B\mathbf{x}) \leq \Delta(\mathbf{x})$.

(ii) $\Phi_B(T_1) \cap \Phi_B(T_2) = \emptyset$ *if* $T_1 \cap T_2 = \emptyset$.

(iii) *Let C be another nonnegative row stochastic matrix. Then*

$$\Phi_{BC} = \Phi_B \circ \Phi_C. \tag{2.7}$$

Proof. (i) follows easily from the fact that $\max(B\mathbf{x}) \leq \max(\mathbf{x})$ and $\min(B\mathbf{x}) \geq \min(\mathbf{x})$.

To prove (ii), let $T = T_1 \cup T_2$. Then $\mathbf{1}_T = \mathbf{1}_{T_1} + \mathbf{1}_{T_2}$. So every entry of $B(\mathbf{1}_T) = B(\mathbf{1}_{T_1}) + B(\mathbf{1}_{T_2})$ is no greater than 1. Hence $\Phi_B(T_1) \cap \Phi_B(T_2) = \emptyset$.

Finally, $\Phi_{BC}(T) = \{j : \ (BC\mathbf{1}_T)_j = 1\}$. Since $\max(C\mathbf{1}_T) \leq 1$, unless $\Phi_C(T) = \emptyset$ (in which case the lemma is obviously true) we have

$$\Phi_{BC}(T) = \Phi_B(\{j : \ (C\mathbf{1}_T)_j = 1\}) = \Phi_B \circ \Phi_C(T).$$

■

For simplicity we shall use Φ_i to denote Φ_{P_i} for $i = 0, 1$.

Lemma 2.4 *Suppose that there exists an $m_0 > 0$ such that for all $(d_1, \ldots, d_m) \in \{0,1\}^m$ with $m \geq m_0$ and $T \subseteq Z_N$ we have*

$$\Phi_{d_m} \circ \cdots \circ \Phi_{d_1}(T) = \emptyset \quad \text{or} \quad \Phi_{d_m} \circ \cdots \circ \Phi_{d_1}(T^c) = \emptyset, \tag{2.8}$$

where $T^c := Z_N \setminus T$. Then the subdivision scheme correponding to (1.5) with nonnegative mask converges.

Proof. We prove that $\Delta(P_{d_{m_0}} \cdots P_{d_1}\mathbf{x}) < \Delta(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^N$ with $\Delta(\mathbf{x}) > 0$ and $(d_1, d_2, \ldots, d_{m_0}) \in \{0,1\}^{m_0}$. Without loss of generality we assume that $\max \mathbf{x} = 1$ and $\min \mathbf{x} = 0$, for we may always normalize it to such form. Let

$$T = \{j \in Z_N : \ (\mathbf{x})_j > \min \mathbf{x}\}.$$

Then $\mathbf{x} \leq \mathbf{1}_T$, and hence $P_{d_{m_0}} \cdots P_{d_1}\mathbf{x} \leq P_{d_{m_0}} \cdots P_{d_1}\mathbf{1}_T$. But

$$\begin{aligned} \Big\{j : \ (P_{d_{m_0}} \cdots P_{d_1}\mathbf{1}_T)_j = 1\Big\} &= \Phi_{d_{m_0}} \circ \cdots \circ \Phi_{d_1}(T), \\ \Big\{j : \ (P_{d_{m_0}} \cdots P_{d_1}\mathbf{1}_T)_j = 0\Big\} &= \Phi_{d_{m_0}} \circ \cdots \circ \Phi_{d_1}(T^c). \end{aligned}$$

Condition (2.8) now yields $\Delta(P_{d_{m_0}} \cdots P_{d_1}\mathbf{1}_T) < 1$. Hence $\Delta(P_{d_{m_0}} \cdots P_{d_1}\mathbf{x}) < 1$, which is sufficient for the hypothesis of Lemma 2.2 to hold, proving the convergence. ■

Corollary 2.5 *The subdivision scheme corresponding to (1.5) with nonnegative mask diverges if and only if there exist disjoint proper subsets T and T' of Z_N and a sequence $(d_1, \ldots, d_m) \in \{0,1\}^m$ for some $m \geq 1$ such that*

$$T = \Phi_{d_m} \circ \cdots \circ \Phi_{d_1}(T) \quad \text{and} \quad T' = \Phi_{d_m} \circ \cdots \circ \Phi_{d_1}(T'). \tag{2.9}$$

Proof. Suppose that the subdivision scheme diverges there exist a sequence $(e_1, \ldots, e_n) \in \{0,1\}^n$ with $n > 2^{2N}$ and a proper subset T_0 of Z_N such that

$$\Phi_{e_n} \circ \cdots \circ \Phi_{e_1}(T_0) \neq \emptyset \quad \text{and} \quad \Phi_{e_n} \circ \cdots \circ \Phi_{e_1}(T_0^c) \neq \emptyset.$$

Denote $T_j := \Phi_{e_j} \circ \cdots \circ \Phi_{e_1}(T_0)$ and $R_j := \Phi_{e_j} \circ \cdots \circ \Phi_{e_1}(T_0^c)$ for $0 \leq j \leq n$. Clearly all T_j and R_j are nonempty. Since Z_N has $2^N - 1$ nonempty subsets and $n > 2^{2N}$, there exist $j_1 < j_2 < \cdots < j_k$ with $k > 2^N$ such that $T_{j_i} = T$ for some nonempty $T \subset Z_N$ for all $1 \leq i \leq k$. Now, $k > 2^N$ implies that there exist $j_s < j_t$ such that $R_{j_s} = R_{j_t} = T'$ where T' is nonempty. Hence

$$\Phi_{e_{j_t}} \circ \cdots \circ \Phi_{e_{j_s+1}}(T) = T \quad \text{and} \quad \Phi_{e_{j_t}} \circ \cdots \circ \Phi_{e_{j_s+1}}(T') = T'.$$

Now $T_j \cap R_j = \emptyset$ for all j by Lemma 2.3 (ii). So in particular $T \cap T' = \emptyset$. (2.9) follows by setting $(d_1, \ldots, d_m) = (e_{j_t}, \ldots, e_{j_s+1})$.

Conversely, suppose that (2.9) holds. By taking $\mathbf{x} = \mathbf{1}_T$ we then have

$$\Delta\Big((P_{d_m} \cdots P_{d_1})^n \mathbf{x}\Big) = 1$$

for all $n \geq 0$. This shows that the subdivision scheme diverges. ∎.

We compute Φ_0 and Φ_1 explicitly. For the refinement equation (1.5) with mask supp $(\mathcal{C})$ we denote

$$S = \text{supp } (\mathcal{C}), \qquad S_0 = \text{supp } (\mathcal{C}) \cap 2\mathbb{Z}, \qquad S_1 = \text{supp } (\mathcal{C}) \cap (2\mathbb{Z}+1).$$

Define

$$\Psi(T) = \Big\{ \bigcap_{q \in S_0} (2T - q) \Big\} \cup \Big\{ \bigcap_{p \in S_1} (2T - p) \Big\}.$$

Lemma 2.6 *For any $T \subseteq Z_N$ we have*

$$\Phi_0(T) = \Psi(T) \cap Z_N, \qquad \Phi_1(T) = (\Psi(T) + 1) \cap Z_N. \tag{2.10}$$

Furthermore, for any $(d_1, \ldots, d_m) \in \{0,1\}^m$ we have

$$\Phi_{d_1} \circ \cdots \circ \Phi_{d_m}(T) = (\Psi^m(T) + k) \cap Z_N, \tag{2.11}$$

where $k = \sum_{j=0}^{m-1} d_j \, 2^j$.

Proof. We first prove (2.10). Write $P_0 = [b_{ij}] := [c_{2j-i}]_{i,j=0}^{N-1}$. Let $I_i = \{j : \ b_{ij} \neq 0\}$. It is easy to check that

$$I_i = \begin{cases} \dfrac{S_0 + i}{2}, & i = 2r \\ \dfrac{S_1 + i}{2}, & i = 2r + 1. \end{cases}$$

Now $i \in \Phi_0(T)$ if and only if $(P_0 \mathbf{1}_T)_i = 1$, which in turn holds if and only if $I_i \subseteq T$. For even i, this is equivalent to $\frac{1}{2}(S_0 + i) \subseteq T$, or $i \in \bigcap_{q \in S_0} (2T - q)$. Similarly, for odd i we have $i \in \Phi_0(T)$ if and only if $i \in \bigcap_{p \in S_1} (2T - p)$. This proves (2.10) for $\Phi_0(T)$. For $\Phi_1(T)$ the proof is essentially identical.

To prove (2.11) we first let $\Psi_0 = \Psi$ and $\Psi_1 = \Psi + 1$. Then one easily checks that

$$\Psi_{d_1} \circ \cdots \circ \Psi_{d_m} = \Psi^m + k, \qquad k = \sum_{j=0}^{m-1} d_j \, 2^j.$$

But $\Phi_{d_1} \circ \cdots \circ \Phi_{d_m}(T) = \Psi_{d_1} \circ \cdots \circ \Psi_{d_m}(T) \cap Z_N$, proving (2.11). ∎

3 Proof of Theorems

We first prove Theorem 1.1, which is essential to proving other results in the paper. The refinement equation (1.7) has $S_0 = \{0, N\}$ and $S_1 = \{p\}$, where $N = 2M$ and gcd $(p, N) = 1$. The map Ψ is given by

$$\Psi(T) = \Big(2T \cap (2T - N)\Big) \cup (2T - p). \tag{3.1}$$

A key observation is that if $f(x)$ satisfies (1.7) then the function $\widehat{f}(x) := f(N - x)$ satisfies the "reversed" refinement equation

$$\widehat{f}(x) = (1 - a)\widehat{f}(2x) + \widehat{f}(2x - (N - p)) + a\widehat{f}(2x - N), \tag{3.2}$$

which has mask $\widehat{\mathcal{C}} = e^{2\pi i N} \bar{\mathcal{C}}$ and supp $(\widehat{\mathcal{C}}) = \{0, N - p, N\}$. The subdivision scheme and the cascade algorithm converge for (1.7) if and only if they converge for (3.2). For any subset T of integers define $\widehat{T} := N - T$ and

$$\widehat{\Psi}(T) = \Big(2T \cap (2T - N)\Big) \cup (2T - N + p).$$

Then one verifies that

$$\widehat{\Psi}(\widehat{T}) = \widehat{\Psi(T)}. \tag{3.3}$$

Furthermore, suppose that $T \subseteq Z_N$ then

$$\widehat{\Phi_0(T)} = \widehat{\Psi}(\widehat{T}) \cap (Z_N + 1), \qquad \widehat{\Phi_1(T)} = (\widehat{\Psi}(\widehat{T}) - 1) \cap (Z_N + 1). \tag{3.4}$$

It follows from iterating (3.3) and (3.4) that

$$\widehat{\Phi_{\mathbf{d}}(T)} = (\widehat{\Psi}^m(\widehat{T}) - k) \cap (Z_N + 1), \tag{3.5}$$

for all $\mathbf{d} = (d_1, \ldots, d_m) \in \{0,1\}^m$ and $k = \sum_{j=0}^{m-1} 2^j d_j$, where $\Phi_{\mathbf{d}} := \Phi_{d_1} \circ \cdots \circ \Phi_{d_m}$.

Lemma 3.1 *Let T be a subset of $\mathbb{Z}$ and suppose that $p - r \notin T$. Then $np - 2^m r \notin \Psi^m(T)$ for all $1 \le n \le 2^m$.*

Proof. Since $\Psi(T) \subseteq 2T \cup (2T - p)$ with the union being disjoint, if $a \notin T$ then $2a$, $2a - p \notin \Psi(T)$. Hence $2p - 2r$, $p - 2r \notin \Psi(T)$. This leads to

$$2(p - 2r),\ 2(p - 2r) - p,\ 2(2p - 2r),\ 2(2p - 2r) - p \notin \Psi^2(T).$$

In other words, $np - 4r \notin \Psi^2(T)$ for all $1 \le n \le 4$. This iterative argument proves the lemma, by induction on m. ∎

Proof of Theorem 1.1. Assume that the subdivision scheme diverges. Then by Corollary 2.5 there exist a sequence $(d_1, \ldots, d_{m'})$ and disjoint nonempty sets T, $T' \subset Z_N$ such that

$$\Phi_{d_1} \circ \cdots \circ \Phi_{d_{m'}}(T) = T \quad \text{and} \quad \Phi_{d_1} \circ \cdots \circ \Phi_{d_{m'}}(T') = T'.$$

It follows from Lemma 2.6 that

$$T = (\Psi^{m'}(T) + k') \cap Z_N \ \text{ and } \ T' = (\Psi^{m'}(T') + k') \cap Z_N \tag{3.6}$$

where $k' = \sum_{j=0}^{m'-1} 2^j d_j$. Denote $\mathbf{d} = (d_1, \ldots, d_{m'})$ and $\Phi_{\mathbf{d}} = \Phi_{d_1} \circ \cdots \circ \Phi_{d_{m'}}$. Then iterations yield $\Phi_{\mathbf{d}}^q(T) = T$ and $\Phi_{\mathbf{d}}^q(T') = T'$ for all $q \ge 0$. Set $q = (N-p)p\phi((N-p)p)t$ and $m = qm'$ where $\phi(n)$ is the Euler's function of number of elements in Z_n that are coprime with n, and $t \ge 1$. Then Lemma 2.6 yields

$$T = (\Psi^m(T) + k_m) \cap Z_N \ \text{ and } \ T' = (\Psi^m(T') + k_m) \cap Z_N \tag{3.7}$$

where $k_m = k' \sum_{j=0}^{q-1} 2^{m'j}$. Note that $2^{\phi((N-p)p)} \equiv 1 \pmod{(N-p)p}$. We prove that

$$k_m \equiv 0 \pmod{(N-p)p}, \qquad 2^m \equiv 1 \pmod{(N-p)p}. \tag{3.8}$$

The latter congruence is rather clear. For the first congruence,

$$k_m = k' \sum_{j=0}^{q-1} 2^{m'j}$$

$$
\begin{aligned}
&= k' \sum_{l=0}^{(N-p)pt-1} \sum_{j=0}^{\phi((N-p)p)-1} 2^{m'j+m'l\phi((N-p)p)} \\
&= k' \sum_{l=0}^{(N-p)pt-1} \sum_{j=0}^{\phi((N-p)p)-1} 2^{m'j} 2^{m'l\phi((N-p)p)} \\
&\equiv k' \sum_{l=0}^{(N-p)pt-1} \sum_{j=0}^{\phi((N-p)p)-1} 2^{m'j} \pmod{(N-p)p} \\
&\equiv k'(N-p)pt \sum_{j=0}^{\phi((N-p)p)-1} 2^{m'j} \pmod{(N-p)p} \\
&\equiv 0 \pmod{(N-p)p}.
\end{aligned}
$$

We derive a contradiction using Lemma 3.1. To do so we first without loss of generality assume that $p < N-p$, and divide the proof into three cases.

Case 1. $(d_1, \ldots, d_{m'}) \neq (0, \ldots, 0)$ *and* $(d_1, \ldots, d_{m'}) \neq (1, \ldots, 1)$.

In this case we may choose t sufficiently large so that $k_m > N$ and $2^m - k_m > N$. We prove that either $T = \emptyset$ or $T' = \emptyset$, hence a contradiction.

Suppose that $p - r \notin T$ for some $0 < r \leq p$. Then by Lemma 3.1 $np - 2^m r \notin \Psi^m(T)$ for all $1 \leq n \leq 2^m$. Hence $np - 2^m r + k_m \notin T$ for all $1 \leq n \leq r^m$. The fact that $k_m > N$ together with (3.8) imply that no element of the form $lp - r$ are in T. In other words, if $a \notin T$ for some $0 \leq a < p$ then T contains no element congruent to a modulo p.

The key idea is to apply (3.5) by considering the "reversed" refinement equation (3.2), which yields

$$
\widehat{T} = (\widehat{\Psi}^m(\widehat{T}) - k_m) \cap (Z_N + 1) \quad \text{and} \quad \widehat{T}' = (\widehat{\Psi}^m(\widehat{T}') - k_m) \cap (Z_N + 1). \tag{3.9}
$$

Suppose that $N - p - r \notin \widehat{T}$. Then again Lemma 3.1 implies that $n(N-p) - 2^m r \notin \widehat{\Psi}^m(\widehat{T})$ for all $1 \leq n \leq 2^m$. Therefore by (3.9) $n(N-p) - 2^m r - k_m \notin \widehat{T}$. It follows again from (3.8) and the fact $2^m - k_m > N$ that $\widehat{T} \cap Z_N$ contains no element of the form $n(N-p) - r$. In other words, if $a \notin \widehat{T}$ for some $1 \leq a \leq N-p$ then $\widehat{T}$ contains no element congruent to a modulo $N-p$.

Now by assumption $p < N-p$. Suppose that $a \notin T$ for some $0 \leq a < p$. Then $lp + a \notin T$ for all l. In particular $l_0 p + a \notin T$ where $N - p \leq l_0 p + a < N$. Hence $N - l_0 p - a \notin \widehat{T}$. Observe that $1 \leq N - l_0 p - a \leq p < N - p$. So $(N - l_0 p - a) + (N-p) \notin \widehat{T}$, which yields $a' = (l_0 + 1)p + a - N \notin T$. Since $0 \leq a' < p$, this implies that T contains no element congruent to a' modulo p.

For each $0 \leq a < p$ define the map $g(a) = a'$ as above. Then $0 \leq g(a) < p$ and $a' \equiv a - N \pmod{p}$. But N and p are coprime. This means $\{g^j(a) : \ 0 \leq j < p\} = \{0, 1, \ldots, p-1\}$. It follows that if for some $0 \leq a < p$ we have $a \notin T$ then $\{0, 1, \ldots, p-1\} \cap T = \emptyset$. But we have already shown that if $a \notin T$ for any $0 \leq a < p$ then T contains no element congruent to $a \pmod{p}$. So $T = \emptyset$. The same statement holds for T'. Since T and T' are disjoint, we infer that either $T = \emptyset$ or $T' = \emptyset$, as $a \notin T$ or $a \notin T'$ for any a. This gives a contradiction.

Case 2. $\quad (d_1, \ldots, d_{m'}) = (0, 0, \ldots, 0)$.

In this case $k_m = 0$. We choose m so that $2^m > N$. The proof for Case 1 needs to be modified for Case 2, because now we can only infer that if $a \notin T$ for some $1 \leq a < p$ (as opposed to $0 \leq a < p$ in Case 1) then T contains no element congruent to $a \pmod{p}$. However, in this case, if $p \notin T$ then $lp \notin T$ for $l \geq 1$ (but not necessarily $l = 0$).

Similarly, if $a \notin \widehat{T}$ for some $1 \leq a < N - p$ (as opposed to $1 \leq a \leq N - p$) then $\widehat{T}$ contains no element congruent to a modulo $(N - p)$.

By assumption $p < N - p$. For each $1 \leq a \leq p$ such that $a \notin T$ we define $g(a) \notin T$ with $0 \leq g(a) < p$ as follows: $a + lp \notin T$ for all $l \geq 0$. So $b = a + l_0 p \notin T$ where $N - p \leq b < N$. This yields $N - b \notin \widehat{T}$. But $1 \leq N - b \leq p < N - p$. It follows that $b_1 = N - b + (N - p) \notin \widehat{T}$. Therefore $a' = N - b_1 = a + (l_0 + 1)p - N \notin T$. We set $g(a) = a'$. In addition to $g(a) \notin T$ we have $0 \leq g(a) < p$ and $g(a) \equiv a - N \pmod{p}$. In particular, if $a \not\equiv N \pmod{p}$ then $g(a) > 0$ and hence $g^2(a) = g(g(a)) \notin T$.

Now, $p \notin T$ or $p \notin T'$. So we may assume without loss of generality that $p \notin T$. Observe that $g^j(p) \equiv p - jN \equiv -jN \pmod{p}$. Since N and p are coprime, $g^j(p) \neq 0$ for $0 \leq j < p$. They are all distinct since they are not congruent modulo p, and none of them are in T. So $\{1, 2, \ldots, p\} \cap T = \emptyset$. Finally $g^p(p) = 0$ because $g^p(p) \equiv 0 \pmod{p}$, yielding $0 \notin T$. Therefore $T = \emptyset$, a contradiction.

Case 3. $\quad (d_1, \ldots, d_{m'}) = (1, 1, \ldots, 1)$.

In this case $k_m = 2^m - 1$. We choose m so that $2^m > N$. Again the proof for Case 1 needs to be modified. Note that if $a \notin T$ for some $0 \leq a < p$ then we still infer that T contains no element congruent to $a \pmod{p}$. However we can only infer that if $a \notin \widehat{T}$ for some $1 < a \leq N - p$ (as opposed to $1 \leq a \leq N - p$) then $\widehat{T}$ contains no element congruent to a modulo $(N - p)$.

Nonetheless, this problem can be overcome easily. Note that the assumption $p < N - p$ means there are only two elements in Z_N that are congruent to $1 \pmod{(N - p)}$: 1 and

$N-p+1$. Since one of $\widehat{T}$ and $\widehat{T}'$ does not contain $N-p+1$, say, $N-p+1 \notin \widehat{T}$, it is still a true statement that if $a \notin \widehat{T}$ for some $1 \le a \le N-p$ then $\widehat{T}$ contains no element congruent to $a \pmod{(N-p)}$.

Clearly $N-p+1 \notin \widehat{T}$ is equivalent to $p-1 \notin T$. The proof in Case 1 now carries through to show that $T = \emptyset$, proving the theorem in this case. ■

We now prove the more general Theorem 1.2, which follows from the results of Jia and Zhou [JZ].

Proof of Theorem 1.2. Observe that by Theorem 1.1 the subdivision scheme associated with the mask

$$\mathcal{C}_1(z) = \frac{1}{4} + \frac{1}{2} z^{p-r} + \frac{1}{4} z^{q-r}$$

converges, and hence so does the subdivision scheme with mask

$$\mathcal{C}_2(z) = \frac{1}{4} z^r + \frac{1}{2} z^p + \frac{1}{4} z^q$$

since it is a simple shift from $\mathcal{C}_1$. By Theorem 1.2 of Jia and Zhou [JZ] the subdivision scheme with mask $\mathcal{C}(z) = \sum_{n=0}^{N} c_n z^n$ converges as long as $c_r, c_p, c_q > 0$ and the sum rule $\sum_n c_{2n} = \sum_n c_{2n+1} = 1$ is satisfied, independent of the actual value of $c_n's$. This proves the convergence of the subdivision schemes for the given mask, and the convergence of the cascade algorithm also follows. ■

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