HAUSDORFF DIMENSION OF SELF-SIMILAR SETS WITH OVERLAPS

SZE-MAN NGAI AND YANG WANG

ABSTRACT. We introduce the notion of "finite type" iterated function systems of contractive similitudes, and describe a scheme for computing the exact Hausdorff dimension of their attractors in the absence of the open set condition. This method extends a previous one by Lalley and applies not only to the classes of self-similar sets studied by Edgar, Lalley, Rao and Wen, and others, but also to some new classes that are not covered by the previous ones.

1. INTRODUCTION

Let $\{\phi_j\}_{j=1}^q$ be an *iterated function system* (IFS) of contractive similitudes on $\mathbb{R}^d$ defined by

$$\phi_j(x) = \rho_j R_j x + b_j, \ 1 \le j \le q, \tag{1.1}$$

where for all j, $0 < |\rho_j| < 1$, $b_j \in \mathbb{R}^d$, and R_j is a $d \times d$ orthogonal matrix. Let F be the corresponding *self-similar set* (i.e. the *attractor*) and let $\dim_H(F)$ denote the Hausdorff dimension of F. It is well known that if $\{\phi_j\}_{j=1}^q$ satisfies the *open set condition* (OSC) (see [H], [F2]), then $\dim_H(F) = \alpha$, where α is the unique solution of

$$\sum_{j=1}^q \rho_j^\alpha = 1.$$

In the absence of the OSC the images of F under the ϕ_j have overlaps and the above dimension formula fails in general. We say loosely that the corresponding IFS has *overlaps* and call F a *self-similar set with overlaps.* In this case it is much harder to compute $\dim_H(F)$. Nevertheless a number of results have been obtained.

Edgar [E] has computed the Hausdorff dimension of the so-called Barnsley's wreath, the attractor of the IFS $\phi_j : \mathbb{R}^2 \to \mathbb{R}^2$ defined by six similitudes with a common rotation through π. Three of these similitudes have contraction ratio $\rho = 1/2$ and the other three have $\rho = 1/4$. By decomposing the attractor F, Edgar is able to identify it with a *graph self-similar set* of a nonoverlapping graph-directed construction (see [MW]). $\dim_H(F)$ can hence be computed in terms of the spectral radius of some construction matrix.

An algorithm is described by Lalley [L] to compute the Hausdorff dimension of the attractor of $\phi_j : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi_j(x) = \omega^{-1}x + b_j, \ b_j \in D \subseteq \mathbb{Z}[\omega],$$

Date: April 23, 2000.

1991 *Mathematics Subject Classification.* Primary 28A78; Secondary 28A80.

Key words and phrases. Hausdorff dimension, self-similar set, finite type condition.

The first author is supported in part by the National Science Foundation, grant DMS–96-32032.

The second author is supported in part by the National Science Foundation, grant DMS–9706793. He dedicates this paper to Prof. John Benedetto, on his 60's birthday.

where ω is a Pisot number and D is a finite set. (A real algebraic integer ω is a *Pisot number* if $|\omega| > 1$ and its algebraic conjugates all lie inside the unit disk.)

The Hausdorff dimension of some other overlapping IFS has also been computed by Rao and Wen [RW], Strichartz and Wang [SW], and Zerner [Z].

In view of the above discussions, it is natural to ask under what conditions on an IFS with overlaps can $\dim_H(F)$ be computed. Our main objective is to set up a condition which is as general as possible and will include all the above classes, as well as some interesting new classes that are not covered by the previous methods. Under this condition, we will describe an algorithm to compute the Hausdorff dimension of the attractor. Our algorithm is based on same device as that in Lalley [L] – namely, the construction of a finite directed graph from the IFS whose incidence matrix encodes geometric information about the attractor. Our main objective is to show that this strategy can be adapted to a much larger class of IFSs, including all those studied in [L], [E], [RW], and [SW]. The extension is rather nontrivial mainly due to the fact that we allow the similitudes in the IFS to have different contractions or rotations. As it turns out, our condition is indeed rather general. For example, in §5 we compute the dimension of a self-similar set with overlaps in which the reciprocal of the contraction ratio is neither an integer nor a Pisot number (Example 5.4). To the best of our knowledge, it is the first time that the Hausdorff dimensions of such a family of sets (nontrivial) are explicitly computed.

To describe our condition and algorithm we shall need some standard notation from symbolic dynamics. Let $\Sigma_q = \{1, 2, \ldots, q\}$ and $\Sigma_q^* = \cup_{n\geq 0}\Sigma_q^n$ be the set of all finite words in Σ_q, where Σ_q^n is the set of all words of length n, with Σ_q^0 containing only the empty word $\emptyset$. For $\mathbf{j} \in \Sigma_q^n$ let $|\mathbf{j}| = n$ denote the length of $\mathbf{j}$. For $\mathbf{i} \in \Sigma_q^m$ and $\mathbf{j} \in \Sigma_q^n$ let $\mathbf{ij} \in \Sigma_q^{m+n}$ be the concatenation of $\mathbf{i}$ and $\mathbf{j}$, and call $\mathbf{i}$ an *initial segment* of $\mathbf{ij}$.

We define the iterated maps using these notations. Let $\mathbf{j} = (j_1, \ldots, j_m) \in \Sigma_q^m$. Then

$$\phi_{\mathbf{j}} := \phi_{j_1} \circ \cdots \circ \phi_{j_m}, \quad \rho_{\mathbf{j}} := \rho_{j_1} \cdots \rho_{j_m}, \quad R_{\mathbf{j}} := R_{j_1} \cdots R_{j_m}.$$

Now let $\rho = \min\{\rho_j\}$. For all $k \geq 0$ define

$$(1.2) \quad \Lambda_k = \Big\{\mathbf{j} \in \Sigma_q^* \,\Big|\, \rho_{\mathbf{j}} \leq \rho^k \text{ but } \rho_{\mathbf{i}} > \rho^k \text{ if } \mathbf{i} \text{ is a proper initial segment of } \mathbf{j}\Big\},$$

$$(1.3) \quad \mathcal{P}_k = \Big\{\phi_{\mathbf{j}}(0) \,\Big|\, \mathbf{j} \in \Lambda_k\Big\}.$$

Intuitively, all $\phi_{\mathbf{j}}$ for $\mathbf{j} \in \Lambda_k$ have comparable contraction ratios, which are in the order of ρ^k.

In [L], $\dim_H(F)$ is determined by evaluating the growth rate of $|\mathcal{P}_k|$, the cardinality of $|\mathcal{P}_k|$. This is achieved by constructing a directed graph with elements in $\mathcal{P}_k$ for all $k \geq 0$ as vertices and words of unit length as directed edges going from $\mathcal{P}_k$ to $\mathcal{P}_{k+1}$. This directed graph allows one to obtain a linear recurrence relation for $|\mathcal{P}_k|$. The algebraic properties of Pisot numbers play a central role here.

When the similitudes in an IFS have different linear parts, as in the example in [E], counting $\mathcal{P}_k$ using the method in [L] no longer works. The difficulty arises mainly from the fact that there might be two different $\mathbf{i}, \mathbf{j} \in \Lambda_k$ such that $\phi_{\mathbf{i}}(0) = \phi_{\mathbf{j}}(0)$ but nevertheless $\rho_{\mathbf{i}} R_{\mathbf{i}} \neq \rho_{\mathbf{j}} R_{\mathbf{j}}$. To overcome this difficulty we introduce the set

$$\mathcal{V}_k = \big\{(\rho_{\mathbf{j}} R_{\mathbf{j}}, \phi_{\mathbf{j}}(0), k) \,\big|\, \mathbf{j} \in \Lambda_k\big\}.$$

Later we introduce graphs that use elements of $\mathcal{V}_k$ as vertices, which allows us to evaluate $|\mathcal{V}_k|$ as [L] has done with $|\mathcal{P}_k|$. The dimension of F can then be computed.

Observe that each vertex $\mathbf{v} = (\rho_\mathbf{j} R_\mathbf{j}, \phi_\mathbf{j}(0), k)$ completely determines $\phi_\mathbf{j}(x) = \rho_\mathbf{j} R_\mathbf{j} x + \phi_\mathbf{j}(0)$. So we shall use $\phi_\mathbf{v}(x)$ to denote $\phi_\mathbf{j}(x)$. This notation has the added advantage that if $\mathbf{i}, \mathbf{j} \in \Lambda_k$ satisfy $\phi_\mathbf{i} = \phi_\mathbf{j}$ then they correspond to the same vertex $\mathbf{v}$.

The details of our scheme for computing $\dim_H(F)$ are described in §2. We give an overview here in order to state our main results. Call a nonempty bounded open set $\Omega \subset \mathbb{R}^d$ a *bounded invariant open set* of the IFS $\{\phi_j\}_{j=1}^q$ if $\phi_j(\Omega) \subseteq \Omega$ for all j. Such an Ω always exists. For example, one can check that the open ball $\Omega = B_R(0)$ of radius R for any $R \geq \max_j |b_j|/(1 - |\rho_j|)$ is one. We say two vertices $\mathbf{v}$, $\mathbf{v}' \in \mathcal{V}_k$ are *neighbors (with respect to Ω)* if $\phi_\mathbf{v}(\Omega) \cap \phi_{\mathbf{v}'}(\Omega) \neq \emptyset$. The set

$$\Omega(\mathbf{v}) := \{\mathbf{u} \mid \mathbf{u} \text{ is a neighbor of } \mathbf{v}\}$$

is called the *neighborhood of* $\mathbf{v}$ *(with respect to Ω)*. Two vertices $\mathbf{v} \in \mathcal{V}_k$ and $\mathbf{v}' \in \mathcal{V}_{k'}$ are said to have the same *neighborhood type* if there exists a similitude $\tau(x) = \rho^{k'-k} U x + c$ where U is orthogonal and $c \in \mathbb{R}^d$ such that

$$\{\tau \circ \phi_\mathbf{u} \mid \mathbf{u} \in \Omega(\mathbf{v})\} = \{\phi_{\mathbf{u}'} \mid \mathbf{u}' \in \Omega(\mathbf{v}')\} \quad \text{and} \quad \tau \circ \phi_\mathbf{v} = \phi_{\mathbf{v}'}.$$

The IFS (1.1) is said to be of *finite type* if there are finitely many distinct neighborhood types. In this case we can define the *incidence matrix* $S = [s_{ij}]$ for the IFS as follows: Suppose that there are N neighborhood types. Choose any vertex $\mathbf{v}$ that has neighborhood type i. Its offspring in some reduced graph (defined in §2) will have various neighborhood types. The entry s_{ij} denotes the number of offspring that have neighborhood type j. As we shall prove, s_{ij} is independent of the choice of $\mathbf{v}$. This leads to our main result:

Theorem 1.1. *Let $\{\phi_j\}_{j=1}^q$ be an IFS defined as in (1.1). Suppose that the IFS is of finite type with respect to a bounded invariant open set Ω, and let S be the corresponding incidence matrix. Then the attractor F of the IFS satisfies*

$$\dim_H(F) = \dim_B(F) = \frac{\log \lambda}{-\log \rho}, \tag{1.4}$$

where $\rho = \min_j \rho_j$ and $\lambda = \lambda(S)$ is the spectral radius of S.

We also study some properties of IFS of finite type. By making use of the incidence matrix S in Theorem 1.1, we can show that the attractor F of an IFS of finite type is an s-set. Let $\mathcal{H}^s$ denote the s-dimensional Hausdorff measure.

Theorem 1.2. *Let $\{\phi_j(x)\}_{j=1}^q$ be the IFS defined in (1.1). Suppose that the IFS is of finite type. Then the attractor F of the IFS satisfies*

$$0 < \mathcal{H}^s(F) < \infty, \tag{1.5}$$

where $s = \dim_H(F)$.

It is not known in general whether a self-similar set in $\mathbb{R}^d$ with Hausdorff dimension d and positive measure must have interior points. We show that this is the case if the finite type condition is satisfied.

Theorem 1.3. *Let $\{\phi_j(x)\}_{j=1}^q$ be an IFS in $\mathbb{R}^d$ given by (1.1). Suppose that it is of finite type, and the attractor F has Hausdorff dimension d. Then $F^o \neq \emptyset$ and $\overline{F^o} = F$.*

We remark that Zerner [Z] obtained analogous results of Theorems 1.2 and 1.3 for an IFS satisfying the *weak separation property* (WSP). However, we are not able to show that an IFS of finite type must possess the WSP. We conjecture:

Conjecture. *An IFS of finite type satisfies the weak separation property.*[1]

This paper is organized as follows. In §2 we define the finite type condition and give examples of different classes of IFS of finite type. In §3 we describe the algorithm to compute $\dim_H(F)$, and prove Theorem 1.1. In §4, we study some properties of IFS of finite type and prove Theorems 1.2 and 1.3. In §5 we provide some numerical computations using our algorithm, focusing on IFS of similitudes with different linear parts or with contraction ratios that are not the reciprocal of Pisot numbers.

Acknowledgments. Part of this research was carried out while the authors were visiting the Department of Mathematics, Cornell University. They would like to thank the department for its hospitality and support. They are also very grateful to Professor Strichartz for many valuable discussions and comments. The first author would also like to thank the Southeast Applied Analysis Center and the School of Mathematics, Georgia Institute of Technology, for supporting part of this study.

2. Overlapping IFS Of Finite Type

In this section we describe a scheme for computing the Hausdorff dimension of the attractor F of the IFS. This scheme is a generalization, albeit a nontrivial one, of the scheme described in [L]. As stated in the introduction, the main ingredient of our scheme is the notion of IFS of finite type.

We use the same notation in (1.2) and (1.3). Our objective is to find a way to count $\mathcal{V}_k$. As in [L], we achieve this goal by introducing two directed graphs, $\mathcal{G}$ and $\mathcal{G}_R$. The vertex set of $\mathcal{G}$ is the set of all triples (not counting multiplicity)

$$\mathcal{V} := \Big\{(\rho_{\mathbf{j}} R_{\mathbf{j}}, \phi_{\mathbf{j}}(0), k) \;\Big|\; \mathbf{j} \in \Lambda_k,\ k \geq 0\Big\}. \tag{2.1}$$

Observe that $\phi_{\mathbf{j}}(x) = \rho_{\mathbf{j}} R_{\mathbf{j}} x + \phi_{\mathbf{j}}(0)$, so the first two entries of a vertex triple encode $\phi_{\mathbf{j}}(x)$. Conversely, any given map $\phi_{\mathbf{j}}(x)$ with $\mathbf{j} \in \Lambda_k$ uniquely determines a vertex in the obvious way. Hence we may equate a vertex with some iterated map $\phi_{\mathbf{j}}$. Note that if $\phi_{\mathbf{j}} = \phi_{\mathbf{i}}$ for some $\mathbf{j}, \mathbf{i} \in \Sigma_q^*$ then they determine the same vertex, so the correspondence is well defined. We use the following notation for simplicity:

$$\phi_{\mathbf{v}}(x) := \phi_{\mathbf{j}}(x) \quad \text{if } \mathbf{v} = (\rho_{\mathbf{j}} R_{\mathbf{j}}, \phi_{\mathbf{j}}(0), k).$$

We also let $\pi : \bigcup_{k\geq 0} \Lambda_k \longrightarrow \mathcal{V}$ be the map

$$\pi(\mathbf{j}) = (\rho_{\mathbf{j}} R_{\mathbf{j}}, \phi_{\mathbf{j}}(0), k) \quad \text{for } \mathbf{j} \in \Sigma_q^*.$$

This map is onto but in general not one-to-one.

Given two vertices $\mathbf{v}$ and $\mathbf{v}'$, suppose that there exist $\mathbf{j} \in \Lambda_k$, $\mathbf{j}' \in \Lambda_{k+1}$ and $\mathbf{k} \in \Sigma_q^*$ such that $\mathbf{v} = \pi(\mathbf{j})$, $\mathbf{v}' = \pi(\mathbf{j}')$ and $\mathbf{j}' = \mathbf{j}\mathbf{k}$ for some $\mathbf{k} \in \Sigma_q^*$. Then we connect a directed edge $\mathbf{k} :\ \mathbf{v} \longmapsto \mathbf{v}'$ (so $\mathbf{k}$ is the label of the edge). We call $\mathbf{v}$ a *parent* of $\mathbf{v}'$, and $\mathbf{v}'$ an *offspring* of $\mathbf{v}$. Note that there might be several directed edges going from one vertex to another. For example, for the IFS $\phi_1(x) = \rho x$ and $\phi_2(x) = \rho^2 x$, whenever $\mathbf{j} = (2)$ is a directed edge from one vertex to another, so will be $\mathbf{j}' = (11)$ because $\phi_{(11)} = \phi_2$. The so obtained graph is our graph $\mathcal{G}$. For the reduced graph $\mathcal{G}_R$, fix an order for Σ_q^*; here we use the lexicographical order (although any order would do). The reduced graph $\mathcal{G}_R$ is obtained from $\mathcal{G}$ by removing all but the smallest directed edge going to a vertex. In other words, for each vertex $\mathbf{v}$ let $\mathbf{k}_1, \dots, \mathbf{k}_p$ be all the directed edges going from some vertices to $\mathbf{v}$. Suppose

[1]This conjecture is recently proved by Nguyen [N].

that $\mathbf{k}_1 < \mathbf{k}_2 < \cdots < \mathbf{k}_p$ in the lexicographical order. Then we keep only the directed edge $\mathbf{k}_1$ and remove all other edges. This way we obtain $\mathcal{G}_R$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded invariant open set of the IFS $\{\phi_j\}_{j=1}^q$, i.e. $\phi_j(\Omega) \subseteq \Omega$ for all j. Let $\mathcal{V}_k := \pi(\Lambda_k)$. We say two vertices $\mathbf{v}$, $\mathbf{v}' \in \mathcal{V}_k$ are *neighbors (with respect to Ω)* if $\phi_{\mathbf{v}}(\Omega) \cap \phi_{\mathbf{v}'}(\Omega) \neq \emptyset$. The set

$$\Omega(\mathbf{v}) := \{\mathbf{v}' \mid \mathbf{v}' \text{ is a neighbor of } \mathbf{v}\}$$

is called the *neighborhood of* $\mathbf{v}$ *(with respect to Ω)*.

We define an equivalence relation on the set of neighborhoods. For vertices $\mathbf{v} \in \mathcal{V}_k$ and $\mathbf{v}' \in \mathcal{V}_{k'}$, we say $\Omega(\mathbf{v})$ is equivalent to $\Omega(\mathbf{v}')$, and denoted this relation by $\Omega(\mathbf{v}) \sim \Omega(\mathbf{v}')$ (or more precisely $\Omega(\mathbf{v}) \sim_\tau \Omega(\mathbf{v}')$), if there exists a similarity map $\tau(x) = \rho^{k'-k}Ux + c$, where U is orthogonal and $c \in \mathbb{R}^d$ such that $\phi_{\mathbf{v}'} = \tau \circ \phi_{\mathbf{v}}$ and (counting multiplicity)

$$\{\phi_{\mathbf{u}'} \mid \mathbf{u}' \in \Omega(\mathbf{v}')\} = \{\tau \circ \phi_{\mathbf{u}} \mid \mathbf{u} \in \Omega(\mathbf{v})\}.$$

It is easy to check that $\sim$ is indeed an equivalence relation. We shall use $[\Omega(\mathbf{v})]$ to denote the equivalence class, and call it the *neighborhood type of* $\mathbf{v}$ *(with respect to Ω)*.

Lemma 2.1. *Let $\mathbf{v}_1$, $\mathbf{v}_2 \in \mathcal{V}_k$ and let $\mathbf{u}_1$, $\mathbf{u}_2$ be their offspring, respectively. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are not neighbors, then neither are $\mathbf{u}_1$ and $\mathbf{u}_2$.*

Proof. Observe that $\phi_{\mathbf{u}_j} = \phi_{\mathbf{v}_j} \circ \phi_{\mathbf{k}_j}$ for some $\mathbf{k}_j \in \Sigma_q^*$, $j = 1, 2$. Since $\phi_{\mathbf{k}_j}(\Omega) \subseteq \Omega$, we have

$$\phi_{\mathbf{u}_1}(\Omega) \cap \phi_{\mathbf{u}_2}(\Omega) \subseteq \phi_{\mathbf{v}_1}(\Omega) \cap \phi_{\mathbf{v}_2}(\Omega) = \emptyset,$$

proving the lemma. ∎

Proposition 2.2. *Let Ω be a bounded invariant open set for the IFS $\{\phi_j\}_{j=1}^q$ given by (1.1) and let $\mathcal{G}_R$ be the corresponding reduced graph. Then*

(a) *There exists a unique path in $\mathcal{G}_R$ from the root vertex $\mathbf{v}_{\text{root}} := \pi(\emptyset)$ to any given vertex.*

(b) *Let $\mathbf{v}$ and $\mathbf{v}'$ be two vertices with offspring $\mathbf{u}_1, \ldots, \mathbf{u}_m$ and $\mathbf{u}'_1, \ldots, \mathbf{u}'_l$ in $\mathcal{G}_R$, respectively. Suppose that $[\Omega(\mathbf{v})] = [\Omega(\mathbf{v}')]$. Then*

$$\{[\Omega(\mathbf{u}_i)] \mid 1 \le i \le m\} = \{[\Omega(\mathbf{u}'_i)] \mid 1 \le i \le l\} \tag{2.2}$$

counting multiplicity. In particular, $m = l$.

Proof. (a) The existence of a path is obvious. If there are two different paths in $\mathcal{G}_R$ from $\mathbf{v}_{\text{root}}$ to a vertex $\mathbf{v}$, then the vertex at which the two paths cross will have two parents in $\mathcal{G}_R$, a contradiction.

(b) Let $\mathbf{v} \in \mathcal{V}_k$ and $\mathbf{v}' \in \mathcal{V}_{k'}$. Suppose that $\Omega(\mathbf{v}) \sim_\tau \Omega(\mathbf{v}')$ where $\tau(x) = \rho^{k'-k}Ux + c$ for some orthogonal U and $c \in \mathbb{R}^d$. We label $\Omega(\mathbf{v})$ and $\Omega(\mathbf{v}')$ by

$$\Omega(\mathbf{v}) = \{\mathbf{v}_0 = \mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n\}, \quad \Omega(\mathbf{v}') = \{\mathbf{v}'_0 = \mathbf{v}', \mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$$

such that $\phi_{\mathbf{v}'_j} = \tau \circ \phi_{\mathbf{v}_j}$, $0 \le j \le n$. We prove part (b) of the lemma via two claims below.

Claim 1: *For each $0 \le j \le n$, the vertex $\mathbf{v}_j$ has an offspring in (the non-reduced) graph $\mathcal{G}$ by edge $\mathbf{k} \in \Sigma_q^*$ if and only if the vertex $\mathbf{v}'_j$ does.*

Proof of Claim 1. Fix a j, and assume that $\mathbf{v}_j = \pi(\mathbf{j})$ and $\mathbf{v}'_j = \pi(\mathbf{j}')$ for some $\mathbf{j} \in \Lambda_k$ and $\mathbf{j}' \in \Lambda_{k'}$. Suppose that $\mathbf{v}_j$ has an offspring in $\mathcal{G}$ by edge $\mathbf{k}$. Then $\mathbf{jk} \in \Lambda_{k+1}$. Now, $\rho_{\mathbf{j}'} = \rho^{k'-k}\rho_{\mathbf{j}}$ and $k' - k \in \mathbb{Z}$, so $\mathbf{jk} \in \Lambda_{k+1}$ implies $\mathbf{j}'\mathbf{k} \in \Lambda_{k'+1}$. Hence $\mathbf{v}'_j$ has an offspring in $\mathcal{G}$ by edge $\mathbf{k}$ also. The above argument will prove the converse as well. □

Claim 2: *Let $0 \le i, j \le n$ and suppose that in the graph $\mathcal{G}$ we have edges $\mathbf{k}_1$, $\mathbf{k}_2$ such that*

$$\begin{array}{ll} \mathbf{k}_1 : \ \mathbf{v}_i \longmapsto \mathbf{u}_1, & \mathbf{k}_2 : \ \mathbf{v}_j \longmapsto \mathbf{u}_2, \ \textit{and} \\ \mathbf{k}_1 : \ \mathbf{v}'_i \longmapsto \mathbf{u}'_1, & \mathbf{k}_2 : \ \mathbf{v}'_j \longmapsto \mathbf{u}'_2. \end{array}$$

Then $\mathbf{u}_1 = \mathbf{u}_2$ if and only if $\mathbf{u}'_1 = \mathbf{u}'_2$, and $\mathbf{u}_1$, $\mathbf{u}_2$ are neighbors if and only if $\mathbf{u}'_1$, $\mathbf{u}'_2$ are.

Proof of Claim 2. Observe that

$$\phi_{\mathbf{u}'_1} = \phi_{\mathbf{v}'_i} \circ \phi_{\mathbf{k}_1} = \tau \circ \phi_{\mathbf{v}_i} \circ \phi_{\mathbf{k}_1} = \tau \circ \phi_{\mathbf{u}_1}. \tag{2.3}$$

Similarly we have $\phi_{\mathbf{u}'_2} = \tau \circ \phi_{\mathbf{u}_2}$. Hence $\phi_{\mathbf{u}_1} = \phi_{\mathbf{u}_2}$ if and only if $\phi_{\mathbf{u}'_1} = \phi_{\mathbf{u}'_2}$, and so $\mathbf{u}_1 = \mathbf{u}_2$ if and only if $\mathbf{u}'_1 = \mathbf{u}'_2$. For the second part,

$$\phi_{\mathbf{u}_1}(\Omega) \cap \phi_{\mathbf{u}_2}(\Omega) = \emptyset \quad \text{if and only if} \quad \tau \circ \phi_{\mathbf{u}_1}(\Omega) \cap \tau \circ \phi_{\mathbf{u}_2}(\Omega) = \emptyset.$$

This proves the claim. □

Let $\mathcal{U}$ and $\mathcal{U}'$ be the set of offspring of the vertices in $\Omega(\mathbf{v})$ and $\Omega(\mathbf{v}')$, respectively. We now introduce a map γ from $\mathcal{U}$ to $\mathcal{U}'$ as follows: Suppose that $\mathbf{u}$ is an offspring of $\mathbf{v}_i$ by edge $\mathbf{k}$ then we let $\gamma(\mathbf{u})$ be the offspring of $\mathbf{v}'_i$ by edge $\mathbf{k}$. Claims 1 and 2 show that γ is well defined, and is in fact a one-to-one correspondence. Furthermore, by (2.3) we have

$$\phi_{\gamma(\mathbf{u})} = \tau \circ \phi_{\mathbf{u}}. \tag{2.4}$$

Part (b) of the lemma now follows quite easily. First, by Lemma 2.1 only vertices in $\Omega(\mathbf{v})$ (respectively, $\Omega(\mathbf{v}')$) can be the parents of any offspring of $\mathbf{v}$ (respectively, $\mathbf{v}'$) in $\mathcal{G}$. Again by Claims 1 and 2, $\mathbf{u}$ is an offspring of $\mathbf{v}$ in $\mathcal{G}_R$ if and only if $\gamma(\mathbf{u})$ is an offspring of $\mathbf{v}'$ in $\mathcal{G}_R$. This yields $m = l$. Now, (2.2) follows immediately from (2.4). ∎

Definition 2.1. *We say the IFS (1.1) is of* finite type *if there exists a bounded invariant open set Ω such that $\{[\Omega(\mathbf{v})] \mid \mathbf{v}$ is a vertex of $\mathcal{G}\}$ is a finite set. In this case, we say the IFS is of* finite type with respect to Ω, *and Ω is a* finite type condition set.

In the rest of this section we establish classes of IFS of finite type.

Theorem 2.3. *Suppose that the IFS $\{\phi_j(x)\}_{j=1}^q$ given by (1.1) satisfies the OSC with open set Ω. Assume that* $\log \rho_1, \dots, \log \rho_q$ *are commensurable. Then the IFS is of finite type with respect to Ω.*

Proof. Let $\rho = \min_j \rho_j$. Since $\{\log \rho_j\}_{j=1}^q$ are commensurate the set $\{\rho^{-k}\rho_{\mathbf{j}} \mid \mathbf{j} \in \Lambda_k, \ k \ge 0\}$ is finite. Observe that by the OSC, for a given k all $\phi_{\mathbf{j}}(\Omega)$, $\mathbf{j} \in \Lambda_k$, are disjoint. Hence each $\Omega(\mathbf{v})$, $\mathbf{v} \in \mathcal{V}_k$, consists of the single vertex $\mathbf{v}$. So for $\mathbf{j} \in \Lambda_k$ and $\mathbf{j}' \in \Lambda_{k'}$ we can find a $\tau(x) = \rho^{k'-k}Ux + c$, where U is orthogonal and $c \in \mathbb{R}^d$, such that $\phi_{\mathbf{j}'} = \tau \circ \phi_{\mathbf{j}}$ if and only if $\rho^{-k}\rho_{\mathbf{j}} = \rho^{-k'}\rho_{\mathbf{j}'}$. So there are finitely many equivalence classes among $\Omega(\mathbf{v})$, $\mathbf{v} \in \mathcal{V}$. Therefore the IFS is of finite type. ∎

Let $M_d(\mathbb{R})$ and $M_d(\mathbb{Z})$ be the sets of all $d \times d$ matrices with entries in $\mathbb{R}$ and $\mathbb{Z}$ respectively.

Theorem 2.4. *Let $\{\phi_j(x)\}_{j=1}^q$ be an IFS in $\mathbb{R}^d$ having the form*

$$\phi_j(x) = A^{n_j}x + b_j, \quad 1 \le j \le q,$$

where A is a contractive similitude in $M_d(\mathbb{R})$, $n_j \in \mathbb{N}$ and $b_j \in \mathbb{R}^d$. Assume that $A^{-1} \in M_d(\mathbb{Z})$ and all $b_j \in \mathbb{Z}^d$. Then the IFS is of finite type with respect to any bounded invariant open set Ω of the IFS.

Proof. Let r be the contraction ratio of A, $\rho_j = r^{n_j}$ and $\rho = \min_{1\leq j\leq q}\rho_j = r^N$ where $N := \max_{1\leq j\leq q} n_j$. Hence for all $\mathbf{j}\in\Lambda_k$ we have $\rho_{\mathbf{j}}R_{\mathbf{j}} = A^m$ for some $Nk\leq m< N(k+1)$. This means

$$\left\{A^{-Nk}\rho_{\mathbf{j}}R_{\mathbf{j}}\;\middle|\;\mathbf{j}\in\Lambda_k, k\geq 0\right\}\subseteq\left\{A^i\;\middle|\;0\leq i<N\right\}. \tag{2.5}$$

Suppose that Ω is a bounded invariant open set of the IFS with $\mathrm{diam}(\Omega)=C$. For a vertex $\mathbf{v}\in\mathcal{V}_k$ we look at the "inflated neighborhood"

$$A^{-Nk}\Omega(\mathbf{v}) := \left\{\left(A^{-Nk}\rho_{\mathbf{j}}R_{\mathbf{j}}, A^{-Nk}\phi_{\mathbf{j}}(0)\right)\;\middle|\;\mathbf{j}\in\Lambda_k, \pi(\mathbf{j})\in\Omega(\mathbf{v})\right\}.$$

Observe that by (2.5) $A^{-Nk}\rho_{\mathbf{j}}R_{\mathbf{j}}\in\{A^{-i}\mid 0\leq i<N\}$, so it can take on no more than N possible values. Now, $A^{-Nk}\phi_{\mathbf{j}}(0)\subseteq A^N\mathbb{Z}^d$ for $\mathbf{j}\in\Lambda_k$. Furthermore, $A^{-Nk}\phi_{\mathbf{j}}(\Omega)$ is an open set with diameter no more than C. So it follows from the lattice property of $A^N\mathbb{Z}^d$ that there are only a bounded number of translationally inequivalent sets $\{A^{-Nk}\phi_{\mathbf{j}}(0)\mid\pi(\mathbf{j})\in\Omega(\mathbf{v})\}$ among all $\mathbf{v}\in\mathcal{V}_k$ and all $k\geq 0$. Together with the finiteness of $A^{-Nk}\rho_{\mathbf{j}}R_{\mathbf{j}}$ among all $\mathbf{j}\in\Lambda_k$ and $k\geq 0$ it implies that there are only finitely many equivalence classes among all $\Omega(\mathbf{v})$, $\mathbf{v}\in\mathcal{V}$. ∎

Remark. From the proof it is clear that the conditions $A^{-1}\in M_d(\mathbb{Z})$ and all $b_j\in\mathbb{Z}^d$ in Theorem 2.4 can be replaced by the following more general condition: There exists a full rank lattice $\mathcal{L}$ in $\mathbb{R}^d$ such that $A^{-1}\mathcal{L}\subseteq\mathcal{L}$ and all $b_j\in\mathcal{L}$.

Theorem 2.5. *Let $\{\phi_j(x)\}_{j=1}^q$ be an IFS in $\mathbb{R}^d$ having the form*

$$\phi_j(x) = \omega^{-n_j}R_jx + b_j,\quad 1\leq j\leq q,$$

where $\omega>1$ is a Pisot number and for $1\leq j\leq q$, $n_j\in\mathbb{N}$, $b_j\in\mathbb{R}^d$ and R_j is an orthogonal matrix. Assume that $\{R_j\}_{j=1}^q$ generates a finite matrix group G, and

$$G\{b_j\mid 1\leq j\leq q\}\subseteq r_1\mathbb{Z}[\omega]\times\cdots\times r_d\mathbb{Z}[\omega]$$

for some $r_1,\ldots,r_d\in\mathbb{R}$. Then the IFS is of finite type with respect to any bounded invariant open set Ω of the IFS.

Proof. A well-known property concerning a Pisot number ω is: For any finite $D\subseteq\mathbb{Z}$, there exists an $\varepsilon_0>0$ such that for any polynomials $h(x), g(x)\in D[x]$ we have either $h(\omega)=g(\omega)$ or $|h(\omega)-g(\omega)|\geq\varepsilon_0$. An immediate corollary is that the above property still holds if we assume that D is a finite subset of $r\mathbb{Z}[\omega]$ rather than $\mathbb{Z}$ (see [G], [L], [LN]).

Let $E=G\{b_j\mid 1\leq j\leq q\}$ and $\Delta E := E-E$. ΔE is a finite set in $\mathbb{R}^d$ whose i-th component is in $r_i\mathbb{Z}[\omega]$. It follows that there exists an $\varepsilon_0>0$ such that for any vector coefficient polynomials $h(x), g(x)\in\Delta E[x]$ we have either $h(\omega)=g(\omega)$ or $|h(\omega)-g(\omega)|\geq\varepsilon_0$. This "uniform discreteness" implies that for any given radius $R>0$ there are only a bounded number of translationally inequivalent sets among all $\{f(\omega)\mid f(x)\in E[x]\}\cap B_R(y)$ (not counting multiplicity), $y\in\mathbb{R}^d$.

Now, let $\rho_j=\omega^{-n_j}$ and $\rho=\omega^{-N}$ where $N=\max_{1\leq j\leq q}n_j$. Suppose that Ω is a bounded invariant open set of the IFS with $\mathrm{diam}(\Omega)=C$. For a vertex $\mathbf{v}\in\mathcal{V}_k$ we look at the "inflated neighborhood"

$$\rho^{-k}\Omega(\mathbf{v}) := \left\{\left(\rho^{-k}\rho_{\mathbf{j}}R_{\mathbf{j}}, \rho^{-k}\phi_{\mathbf{j}}(0)\right)\;\middle|\;\mathbf{j}\in\Lambda_k, \pi(\mathbf{j})\in\Omega(\mathbf{v})\right\}.$$

Note that $\rho^{-k}\rho_{\mathbf{j}}R_{\mathbf{j}}\in\{\omega^{-i}S\mid 0\leq i<N, S\in G\}$, so it can take on no more than $N|G|$ possible values. Observe that $\rho^{-k}\phi_{\mathbf{j}}(0)\subseteq E[\omega]$ for $\mathbf{j}\in\Lambda_k$. Furthermore, $\rho^{-k}\phi_{\mathbf{j}}(\Omega)$ is an open set with diameter no more than C. So it follows from having a bounded number of

translationally inequivalent $\{f(\omega) \mid f(x) \in E[x]\} \cap B_R(y)$ that there are only a bounded number of translationally inequivalent sets $\{\rho^{-k}\phi_{\mathbf{j}}(0) \mid \pi(\mathbf{j}) \in \Omega(\mathbf{v})\}$ among all $\mathbf{v} \in \mathcal{V}_k$ and all $k \geq 0$. Together with the finiteness of $\rho^{-k}\rho_{\mathbf{j}}R_{\mathbf{j}}$ among all $\mathbf{j} \in \Lambda_k$ and $k \geq 0$ it implies that there are only finitely many equivalence classes among all $\Omega(\mathbf{v})$, $\mathbf{v} \in \mathcal{V}$. ∎

Note that Theorem 2.5 implies that the classes of IFS in [E], [L], §5 of [RW], and §3 of [SW] are of finite type.

Remark. An interesting special case of the IFS (1.1) is when all $\rho_j R_j = A$ for some similitude A, i.e. the IFS has uniform contraction. Under this assumption, all vertices in $\mathcal{V}_k$ have identical linear part A^k. So each vertex $\mathbf{v}$ is of the form $(A^k, \phi_{\mathbf{j}}(0), k)$ where $\mathbf{j} \in \Lambda_k$ is uniquely determined by $\phi_{\mathbf{v}}(0) = \phi_{\mathbf{j}}(0)$. It is fairly easy to check that two neighborhoods $\Omega(\mathbf{v}) \subseteq \mathcal{V}_k$ and $\Omega(\mathbf{v}') \subseteq \mathcal{V}_{k'}$ are equivalent if and only if the sets

$$\{A^{-k}\phi_{\mathbf{u}}(0) \mid \mathbf{u} \in \Omega(\mathbf{v})\} \quad \text{and} \quad \{A^{-k'}\phi_{\mathbf{u}'}(0) \mid \mathbf{u}' \in \Omega(\mathbf{v}')\}$$

are *translationally* equivalent. This observation simplifies the computation of the Hausdorff dimension of an overlapping IFS with uniform linear part.

3. Proof Of The Main Theorem

We begin by proving two lemmas.

Lemma 3.1. *Suppose that the IFS (1.1) is of finite type with respect to some Ω. Let E, U be any subsets in $\mathbb{R}^d$ with $\operatorname{diam}(U) \leq K_1\rho^k$ and $\operatorname{diam}(E) \leq K_2$. Then there exists an $M = M(K_1, K_2) > 0$ such that for all $k \geq 0$,*

$$\#\Big\{\mathbf{v} \in \mathcal{V}_k \;\Big|\; U \cap \phi_{\mathbf{v}}(E) \neq \emptyset\Big\} \leq M.$$

Proof. Define a norm $\|\cdot\|$ on the set of all affine maps $\varphi(x) = Ax + c$ in $\mathbb{R}^d$ by

$$\|\varphi\| := \|A\|_0 + |c|,$$

where $\|\cdot\|_0$ is any chosen matrix norm and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$. Then it is clear that there exists an $\varepsilon_0 > 0$ such that for any $\mathbf{j}, \mathbf{k} \in \Lambda_k$,

$$\|\phi_{\mathbf{j}} - \phi_{\mathbf{k}}\| \leq \varepsilon_0\rho^k \implies \phi_{\mathbf{j}}(\Omega) \cap \phi_{\mathbf{k}}(\Omega) \neq \emptyset. \tag{3.1}$$

Denote

$$\mathcal{U} = \Big\{\mathbf{v} \in \mathcal{V}_k \;\Big|\; U \cap \phi_{\mathbf{v}}(E) \neq \emptyset\Big\}.$$

Observe that there exists a $K > 0$ (independent of k) such that for any $\mathbf{v}, \mathbf{u} \in \mathcal{U}$ we must have $\|\phi_{\mathbf{v}} - \phi_{\mathbf{u}}\| \leq K\rho^k$. Suppose that the cardinality of $\mathcal{U}$ can be arbitrarily large. Then by the Pigeonhole Principle we may find a subset $\mathcal{W}$ of $\mathcal{U}$ with arbitrarily large cardinality such that for any $\mathbf{v}, \mathbf{u} \in \mathcal{W}$ we have $\|\phi_{\mathbf{v}} - \phi_{\mathbf{u}}\| \leq \varepsilon_0\rho^k$. Now it follows from (3.1) that $\mathcal{W} \subseteq \Omega(\mathbf{v})$ where $\mathbf{v}$ is any element in $\mathcal{W}$. But the finite type property of the IFS implies that the cardinality of all neighborhoods are uniformly bounded. This is a contradiction. ∎

Lemma 3.2. *Suppose that the IFS (1.1) is of finite type. Then the attractor F satisfies*

$$\liminf_{k\to\infty} \frac{\log|\mathcal{V}_k|}{-k\log\rho} \leq \dim_B(F) \leq \limsup_{k\to\infty} \frac{\log|\mathcal{V}_k|}{-k\log\rho}.$$

Proof. Let $N(c\rho^k)$ be the minimal number of balls of radius $c\rho^k$ needed to cover F. Then for any c_1, $c_2 > 0$ there exist positive constants $K^+(c_1, c_2)$ and $K^-(c_1, c_2)$ such that

$$K^-(c_1, c_2)N(c_1\rho^k) \leq N(c_2\rho^k) \leq K^+(c_1, c_2)N(c_1\rho^k).$$

Observe that $F = \bigcup_{\mathbf{j}\in\Lambda_k} \phi_{\mathbf{j}}(F) = \bigcup_{\mathbf{v}\in\mathcal{V}_k} \phi_{\mathbf{v}}(F)$, and there exists a $c_0 > 0$ such that each $\phi_{\mathbf{v}}(F)$ can be covered by a ball of radius $c_0\rho^k$.

Now let $B_1, \ldots, B_{N(\delta)}$ be balls of radius $\delta > 0$ that cover F. We may uniquely write $\delta = c\rho^k$ for some k and $\rho < c \leq 1$. By Lemma 3.1 the cardinality of $\{\mathbf{v} \in \mathcal{V}_k \mid B_j \cap \phi_{\mathbf{v}}(F) \neq \emptyset\}$ is bounded by some fixed $M > 0$ for all $1 \leq j \leq N(\delta)$. Therefore $|\mathcal{V}_k| \leq MN(\delta)$. On the other hand,

$$|\mathcal{V}_k| \geq N(c_0\rho^k) \geq K^-(c, c_0)N(c\rho^k) = K^-(c, c_0)N(\delta). \tag{3.2}$$

Therefore

$$\dim_B(F) \geq \liminf_{\delta\to 0} \frac{\log N(\delta)}{-\log\delta} \geq \liminf_{k\to\infty} \frac{\log(|\mathcal{V}_k|/M)}{-\log(c\rho^k)} = \liminf_{k\to\infty} \frac{\log|\mathcal{V}_k|}{-k\log\rho}.$$

Similarly, applying (3.2) we obtain

$$\dim_B(F) \leq \limsup_{\delta\to 0} \frac{\log N(\delta)}{-\log\delta} \leq \limsup_{k\to\infty} \frac{\log(|\mathcal{V}_k|/K^-(c, c_0))}{-\log(c\rho^k)} = \limsup_{k\to\infty} \frac{\log|\mathcal{V}_k|}{-k\log\rho}.$$

∎

We can evaluate $|\mathcal{V}_k|$ using a matrix S. The matrix S has rows and columns indexed by the neighborhood types. Suppose that the IFS (1.1) is of finite type with respect to Ω. So $\{[\Omega(\mathbf{v})] : \mathbf{v} \in \mathcal{V}\}$ is a finite set, where $\mathcal{V}$ is the set of vertices defined in (2.1). We label the neighborhood types as $\{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_N\}$. The entries of $S = [s_{ij}]_{N\times N}$ are defined as follows: For any $1 \leq i \leq N$, take a vertex $\mathbf{v} \in \mathcal{V}$ of $\mathcal{G}_R$ such that $[\Omega(\mathbf{v})] = \mathcal{T}_i$. Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be the offspring of $\mathbf{v}$ in $\mathcal{G}_R$. Then

$$s_{ij} = \#\big\{l \;\big|\; 1 \leq l \leq k,\ [\Omega(\mathbf{u}_l)] = \mathcal{T}_j\big\}. \tag{3.3}$$

Observe that s_{ij} is independent of the choice of the vertex $\mathbf{v}$ by Proposition 2.2 (b).

Definition 3.1. *We call the matrix S defined above the* incidence matrix *of the a finite type IFS (1.1) with respect to Ω.*

Now without loss of generality we label the neighborhood type $[\Omega(\mathbf{v}_{\text{root}})]$ as $\mathcal{T}_1$. Then Proposition 2.2 implies that

$$|\mathcal{V}_k| = e_1^T S^k \epsilon, \tag{3.4}$$

where $\epsilon = [1, 1, \ldots, 1]^T$ and $e_1 = [1, 0, \ldots, 0]^T$ are vectors in $\mathbb{R}^N$.

Proof of Theorem 1.1. Observe that all vertices in $\mathcal{G}_R$ are descendants of $\mathbf{v}_{\text{root}}$, so all neighborhood types are generated from $\mathcal{T}_1$ (in finitely many steps). Hence there exists a $k_0 \geq 1$ such that $e_1^T S^{k_0} > 0$ (i.e. it is a positive vector). Let $\|x\|_1 = \sum_{j=1}^N |x_j|$ be the L^1-norm on $\mathbb{R}^N$. Then

$$\lim_{k\to\infty} (e_1^T S^k \epsilon)^{1/k} = \lim_{k\to\infty} (e_1^T S^{k_0} S^k \epsilon)^{1/k} = \lim_{k\to\infty} \|S^k\epsilon\|_1^{1/k} = \lambda.$$

Now let $\delta > 0$ be arbitrary. Then for all k sufficiently large,

$$(\lambda - \delta)^k < e_1^T S^k \epsilon < (\lambda + \delta)^k.$$

Using the fact $e_1^T S^k \epsilon = |\mathcal{V}_k|$ from (3.4), and applying Lemma 3.2, we get

$$\frac{\log(\lambda - \delta)}{-\log\rho} \leq \liminf_{k\to\infty} \frac{\log|\mathcal{V}_k|}{-k\log\rho} \leq \dim_B(F) \leq \limsup_{k\to\infty} \frac{\log|\mathcal{V}_k|}{-k\log\rho} \leq \frac{\log(\lambda + \delta)}{-\log\rho}.$$

Letting $\delta \to 0$ yields the second equality in (1.4). Finally, $\dim_B(F) = \dim_H(F)$ always holds ([F1], [F3]). This proves the theorem. ∎

4. Properties Of IFS Of Finite Type

In this section we will establish Theorems 1.2 and 1.3. We shall adopt the notation $\mathcal{G}$, $\mathcal{G}_R$, π, $\mathcal{V}$, $\mathcal{V}_k$ from §2. For simplicity we denote by $\mathbf{v} \longmapsto_R \mathbf{u}$ if $\mathbf{v}$, $\mathbf{u} \in \mathcal{V}$ and $\mathbf{u}$ is an offspring of $\mathbf{v}$ in the reduced graph $\mathcal{G}_R$.

We define a *path* in $\mathcal{G}_R$ to be an infinite sequence $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots)$ such that $\mathbf{v}_j \in \mathcal{V}_j$ and $\mathbf{v}_j \longmapsto_R \mathbf{v}_{j+1}$ for all $j \geq 0$, with $\mathbf{v}_0 = \mathbf{v}_{\rm root} = \pi(\emptyset)$. Let $\mathcal{B}$ be the set of all paths in $\mathcal{G}_R$. For the given vertices $\mathbf{v}_0 = \mathbf{v}_{\rm root}$, $\mathbf{v}_1, \dots, \mathbf{v}_k$ such that $\mathbf{v}_j \longmapsto_R \mathbf{v}_{j+1}$ we call the set

$$\mathcal{I}_{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k} := \big\{ (\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \dots) \in \mathcal{B} \;\big|\; \mathbf{u}_j = \mathbf{v}_j \text{ for } 0 \leq j \leq k \big\}$$

a *branch*. Observe that by Proposition 2.2 (a) the path from $\mathbf{v}_0$ to $\mathbf{v}_k$ is unique, so we may simply denote

$$\mathcal{I}_{\mathbf{v}_k} := \mathcal{I}_{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k}.$$

Proof of Theorem 1.2. Since $\mathcal{H}^s(F) < \infty$ always holds (see [H], [F3]), we only need to show that $\mathcal{H}^s(F) > 0$. We achieve this by constructing a mass distribution on F and applying the standard mass distribution principle. We begin by first defining a mass distribution on $\mathcal{B}$ using the branchs described above.

Suppose that the IFS is of finite type with respect to the bounded invariant open set Ω. Let $\mathcal{T}_1, \dots, \mathcal{T}_N$ be the neighborhood types of the IFS, with $\mathcal{T}_1$ being the neighborhood type $[\Omega(\mathbf{v}_{\rm root})]$. Let S be the corresponding incidence matrix. We have shown that $s = -\log\lambda / \log\rho$ where $\lambda = \lambda(S)$ is the spectral radius of S. Since all vertices in $\mathcal{G}_R$ are descendants of $\mathbf{v}_{\rm root}$, and all neighborhood types are eventually generated by $\mathcal{T}_1$, we may find a λ-eigenvector $x = [b_1, \dots, b_N]^T$ of S such that $b_1 > 0$ and all other $b_j \geq 0$. Let $x_* = [a_1, \dots, a_N]^T$ where $a_j = b_j / b_1$. So we have $Sx_* = \lambda x_*$, $a_j \geq 0$ and $a_1 = 1$.

We now define the mass distribution μ on $\mathcal{B}$. For each branch $\mathcal{I}_{\mathbf{v}_k}$ where $\mathbf{v}_k \in \mathcal{V}_k$ such that $[\Omega(\mathbf{v}_k)] = \mathcal{T}_i$ we let

$$\mu(\mathcal{I}_{\mathbf{v}_k}) = \lambda^{-k} a_i. \tag{4.1}$$

We prove that μ indeed defines a mass distribution on $\mathcal{B}$. Observe that two branchs $\mathcal{I}_{\mathbf{v}}$ and $\mathcal{I}_{\mathbf{v}'}$ with $\mathbf{v} \in \mathcal{V}_k$, $\mathbf{v}' \in \mathcal{V}_l$ and $k \leq l$ intersect if and only if $\mathbf{v}' = \mathbf{v}$ for $k = l$ and $\mathbf{v}'$ is a descendant of $\mathbf{v}$ for $k < l$. In both cases $\mathcal{I}_{\mathbf{v}'} \subseteq \mathcal{I}_{\mathbf{v}}$. So all we need to show is that for any $\mathbf{v} \in \mathcal{V}$,

$$\sum_{\mathbf{u} \in \mathcal{D}} \mu(\mathcal{I}_{\mathbf{u}}) = \mu(\mathcal{I}_{\mathbf{v}}), \tag{4.2}$$

where $\mathcal{D}$ is the set of offspring of $\mathbf{v}$. Suppose that $\mathbf{v} \in \mathcal{V}_k$ and $[\Omega(\mathbf{v})] = \mathcal{T}_i$. Then by (4.1) $\mu(\mathcal{I}_{\mathbf{v}}) = \lambda^{-k} a_i$, and by the definition of S (see (3.3)),

$$\sum_{\mathbf{u} \in \mathcal{D}} \mu(\mathcal{I}_{\mathbf{u}}) = \lambda^{-k-1} (\sum_{j=1}^{N} s_{ij} a_j) = \lambda^{-k-1} \lambda a_i = \lambda^{-k} a_i.$$

So (4.2) holds. It follows now from $\mu(\mathcal{B}) = \mu(\mathcal{I}_{\mathbf{v}_{\rm root}}) = 1$ that μ is indeed a mass distribution on $\mathcal{B}$.

To prove our theorem we must transport μ to a mass distribution on the attractor F. Note that for all $k \geq 1$ we have

$$F = \bigcup_{\mathbf{j} \in \Lambda_k} \phi_{\mathbf{j}}(F) = \bigcup_{\mathbf{v} \in \mathcal{V}_k} \phi_{\mathbf{v}}(F). \tag{4.3}$$

Since $\phi_{\mathbf{v}'}(F) \subseteq \phi_{\mathbf{v}}(F)$ if $\mathbf{v}'$ is a descendant of $\mathbf{v}$ in $\mathcal{G}_R$, each path $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots) \in \mathcal{B}$ corresponds to a point x in F, which is the unique point in $\bigcap_k \phi_{\mathbf{v}_k}(F)$. We shall call $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots) \in \mathcal{B}$ the *address* of x. A point $x \in F$ has at least one address in $\mathcal{B}$ by (4.3), but may have more than one. For any subset U of $\mathbb{R}^d$ let $\mathcal{C}(U)$ be the set of all paths in $\mathcal{B}$ that are addresses of points in $F \cap U$, and define $\mu^*(U) = \mu(\mathcal{C}(U))$. Then clearly μ^* is a mass distribution supported on F.

Finally, let $0 < \delta < \rho$. For any set $U \subset \mathbb{R}^d$ with $\operatorname{diam}(U) \leq \delta$ assume that $\rho^{k+1} \leq \operatorname{diam}(U) < \rho^k$. Lemma 3.1 implies that U can intersect no more than M of all $\phi_{\mathbf{v}}(F)$'s, $\mathbf{v} \in \mathcal{V}_k$, where $M > 0$ is some fixed constant independent of k. For $l \leq M$, let $\mathbf{v}_1, \dots, \mathbf{v}_l$ be the vertices in $\mathcal{V}_k$ such that $U \cap \phi_{\mathbf{v}_j}(F) \neq \emptyset$. Then

$$\mu^*(U \cap F) \leq \sum_{j=1}^{l} \mu(\mathcal{I}_{\mathbf{v}_k}) \leq M\lambda^{-k} \max_{1 \leq i \leq N} \{a_i\}.$$

Now, $\lambda^{-1} = \rho^s$. Hence

$$\lambda^{-k} = \rho^{ks} = \rho^{-s}\rho^{(k+1)s} \leq \rho^{-s}\big(\operatorname{diam}(U)\big)^s.$$

It follows that $\mu^*(U \cap F) \leq c\big(\operatorname{diam}(U)\big)^s$ where $c = M\rho^{-s} \max_i\{a_i\}$. By the mass distribution principle (see Falconer [F2], Chapter 4), we have $\mathcal{H}^s(F) \geq \mu^*(F)/c > 0$. ∎

Proof of Theorem 1.3. By Theorem 1.2 we have $m(F) = \mathcal{H}^d(F) > 0$, where $m(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^d$. Hence there exists a Lebesgue point $x^* \in F$, and in particular

$$c_n = \frac{m\big(F \cap B_{\rho^n}(x^*)\big)}{m\big(B_{\rho^n}(x^*)\big)} \longrightarrow 1 \qquad \text{as } n \longrightarrow \infty, \tag{4.4}$$

where $\rho = \min_j \rho_j$ as usual. Note that $F = \bigcup_{\mathbf{j} \in \Lambda_n} \phi_{\mathbf{j}}(F)$. We define

$$\mathcal{J}_n := \Big\{\mathbf{j} \in \Lambda_n \;\Big|\; \phi_{\mathbf{j}}(F) \cap B_{\rho^n}(x^*) \neq \emptyset\Big\}.$$

Then $|\mathcal{J}_n| \leq M$ for some M independent of n. Let $\varphi_{\mathbf{j}}(x) = \rho^{-n}\phi_{\mathbf{j}}(x) - \rho^{-n}x^*$. So

$$\mathcal{J}_n = \Big\{\mathbf{j} \in \Lambda_n \;\Big|\; \varphi_{\mathbf{j}}(F) \cap B_1(0) \neq \emptyset\Big\}.$$

Let $E_n = \bigcup_{\mathbf{j} \in \mathcal{J}_n} \varphi_{\mathbf{j}}(F)$. Then by (4.4) we have

$$m\big(E_n \cap B_1(0)\big) = c_n m\big(B_1(0)\big).$$

Choose a subsequence $\{n_k\}$ of $\{n\}$ such that E_{n_k} converges to a compact set E in the Hausdorff metric. Then $E_{n_k} \cap B_1(0)$ converges to $E \cap B_1(0)$ in the Hausdorff metric. It follows from

$$m(E \cap B_1(0)) \geq \lim_{k \to \infty} m(E_{n_k} \cap B_1(0)) = m(B_1(0))$$

that $E \cap B_1(0) = B_1(0)$.

Observe that each $\varphi_{\mathbf{j}}$ translates, rotates and dilates. But the dilation scale is between 1 and ρ^{-1}. Now each E_{n_k} is covered by $\{\varphi_{\mathbf{j}}(F)\}_{\mathbf{j} \in \mathcal{J}_{n_k}}$ with $|\mathcal{J}_{n_k}| \leq M$. The diameters of all $\varphi_{\mathbf{j}}(F)$ are uniformly bounded. By the compactness of the Hausdorff metric we have $E = \bigcup_{j=1}^{L} F_j$ where $L \leq M$ and each F_j is the limit of some $\{\varphi_{\mathbf{j}_k}(F)\}$. Hence each $F_j = \tau_j(F)$ for some similarity map τ_j on $\mathbb{R}^d$. Since $E^o \neq \emptyset$, we have $F_j^o \neq \emptyset$ for some j. Therefore $F^o \neq \emptyset$.

Finally, $\tilde{F} := \overline{F^o}$ is also invariant under the IFS in (1.1) and is compact and nonempty. So $F = \tilde{F}$. ∎

Remark. The finite type property is generally destroyed by a small perturbation of the IFS. This is easily seen from the following example: It follows from the projection theorem (see [M] or [F2]) that the Hausdorff dimension of the IFS

$$\phi_1(x) = \frac{1}{3}x, \ \phi_1(x) = \frac{1}{3}(x+1), \ \phi_1(x) = \frac{1}{3}(x+a)$$

is 1 for almost all a. For any irrational a the attractor has $\mathcal{H}^1$-measure (Lebesgue measure) 0 (see [K] or [LW]). Therefore, the IFS is not of finite type for almost all irrational a. But the IFS is of finite type for all rational a. Therefore a small perturbation generally destroys the finite type property.

5. Numerical Computations

In this section we will illustrate our algorithm by some examples. We will also compare $\dim_H(F)$ with the similarity dimension of F. For the IFS in (1.1), the *similarity dimension* of its attractor F, denoted by $\dim_s(F)$, is the unique solution α of the equation $\sum_{j=1}^{q} \rho_j^\alpha = 1$. It is known that $\dim_H(F) \le \dim_s(F)$. Strict inequality holds in each of our examples, which provides another way to see that the OSC fails.

Example 5.1. *The similitudes in the following IFS have non-uniform linear parts, with the second map involving a reflection.*

$$\phi_1(x) = \frac{1}{4}x, \quad \phi_2(x) = -\frac{1}{4}x + \frac{1}{4}, \quad \phi_3(x) = \frac{1}{4}x + \frac{1}{4}, \quad \phi_4(x) = \frac{1}{4}x + \frac{3}{4}.$$

By Theorem 2.5, this IFS is of finite type. We will fix $\Omega = (0,1)$ as a finite type condition set, and let $\mathcal{T}_1$ denote the neighborhood type of the root vertex. Now

$$\Lambda_1 = \{(1),(2),(3),(4)\}$$

and

$$\mathcal{V}_1 = \Big\{(\frac{1}{4},0,1),(-\frac{1}{4},\frac{1}{4},1),(\frac{1}{4},\frac{1}{4},1),(\frac{1}{4},\frac{3}{4},1)\Big\}.$$

Denote by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ the vertices in $\mathcal{V}_1$ according to the above order. Both $\mathbf{v}_3$ and $\mathbf{v}_4$ are of neighborhood type $\mathcal{T}_1$. $\mathbf{v}_1$ and $\mathbf{v}_2$ are neighbors (with respect to $\Omega = (0,1)$). They are also of the same neighborhood type (by considering the similitude $\tau(x) = -x + 1/4$), which will be denoted by $\mathcal{T}_2$. So the root vertex generates two offspring of type $\mathcal{T}_1$ and two offspring of type $\mathcal{T}_2$, all of them belong to $\mathcal{G}_R$. We denote this symbolically as

$$\mathcal{T}_1 \to 2\mathcal{T}_1 + 2\mathcal{T}_2.$$

Note that although $\phi_1(0) = \phi_2(0)$ (see Figure 5.1), it is necessary to distinguish between ϕ_1 and ϕ_2 because they have different linear parts and, as we will see, they generate different offspring.

$\mathbf{v}_1$ and $\mathbf{v}_2$ generate two common offspring. The four offspring in $\mathcal{G}$ generated by $\mathbf{v}_1$ correspond to the words $(11),(12),(13),(14) \in \Lambda_2$; they are of neighborhood types $\mathcal{T}_2,\mathcal{T}_2,\mathcal{T}_1,\mathcal{T}_1$ respectively. Those generated by $\mathbf{v}_2$ are defined by the words $(21),(22),(23),(24) \in \Lambda_2$. Note that (12) and (24) are common offspring, so are (14) and (22). Since $(2) < (4)$ in the lexicographical order, we will keep (12) and remove (14) when constructing $\mathcal{G}_R$. Thus we have

$$\mathcal{T}_2 \to \mathcal{T}_1 + 2\mathcal{T}_2.$$

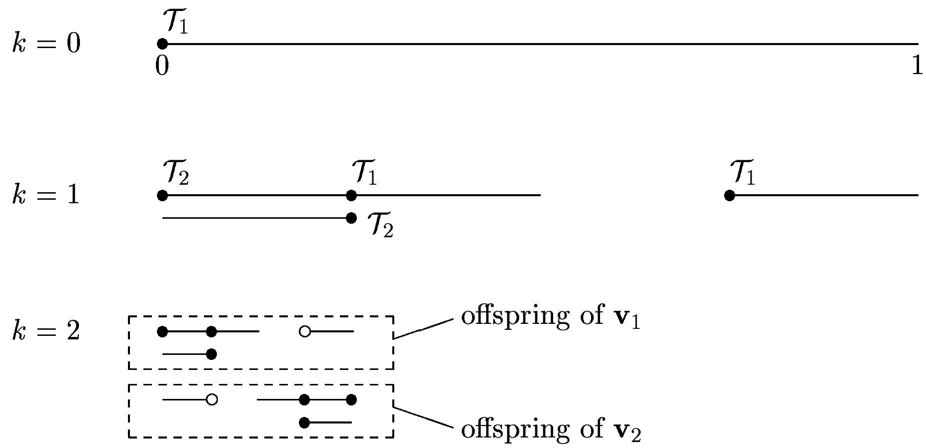

FIGURE 5.1. Figure showing the vertices in $\mathcal{V}_k$ for $k = 0, 1, 2$. Overlapping vertices are separated vertically to show distinction and multiplicity. Points in P_k are represented by dots (or circles). For $k = 2$, only offspring of $\mathbf{v}_1 = (1/4, 0, 1)$ and $\mathbf{v}_2 = (1/4, 3/4, 1)$ are shown, and those indicated by a circle are to be removed when constructing $\mathcal{G}_R$.

$\mathcal{T}_1, \mathcal{T}_2$ are the only neighborhood types and the process above also yields the incidence matrix

$$S = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}.$$

The spectral radius of S is $\lambda = 2 + \sqrt{2}$ and by Theorem 1.1,

$$\dim_H(F) = \frac{\log(2+\sqrt{2})}{\log 4} = 0.8857766515\ldots.$$

The similarity dimension of F is equal to 1.

Example 5.2. *The similitudes in the following IFS have two different contraction ratios,* $1/3$ *and* $1/9$:

$$\phi_1(x) = \frac{1}{3}x, \quad \phi_2(x) = \frac{1}{9}x + \frac{8}{27}, \quad \phi_3(x) = \frac{1}{3}x + \frac{2}{3}.$$

Again by Theorem 2.5, $\{\phi_1, \phi_2, \phi_3\}$ is of finite type. Fix $\Omega = (0, 1)$ as a finite type condition set and let $\mathcal{T}_1$ be the neighborhood type of the root vertex. Now

$$\Lambda_1 = \big\{(11), (12), (13), (2), (31), (32), (33)\big\}$$

and

$$\mathcal{V}_1 = \Big\{(\frac{1}{9}, 0, 1), (\frac{1}{27}, \frac{8}{81}, 1), (\frac{1}{9}, \frac{2}{9}, 1), (\frac{1}{9}, \frac{8}{27}, 1), (\frac{1}{9}, \frac{2}{3}, 1), (\frac{1}{27}, \frac{62}{81}, 1), (\frac{1}{9}, \frac{8}{9}, 1)\Big\}.$$

We classify all vertices in $\mathcal{V}_1$ according to their neighborhood types as follows:

Vertex	Neighborhood Type
$(\frac{1}{9}, \frac{8}{9}, 1)$	$\mathcal{T}_1$
$(\frac{1}{9}, 0, 1)$, $(\frac{1}{9}, \frac{2}{3}, 1)$	$\mathcal{T}_2$
$(\frac{1}{27}, \frac{8}{81}, 1)$, $(\frac{1}{27}, \frac{62}{81}, 1)$	$\mathcal{T}_3$
$(\frac{1}{9}, \frac{2}{9}, 1)$	$\mathcal{T}_4$
$(\frac{1}{9}, \frac{8}{27}, 1)$	$\mathcal{T}_5$

Thus a vertex of neighborhood type $\mathcal{T}_1$ produces one offspring in $\mathcal{G}$ of each of the neighborhood types $\mathcal{T}_1$, $\mathcal{T}_4$, $\mathcal{T}_5$, and two offspring of each of the neighborhood types $\mathcal{T}_2$ and $\mathcal{T}_3$. All of these offspring are also in $\mathcal{G}_R$ and therefore

$$\mathcal{T}_1 \to \mathcal{T}_1 + 2\mathcal{T}_2 + 2\mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5.$$

The vertices $\mathbf{v}_1 = (\frac{1}{9}, 0, 1)$ and $\mathbf{v}_2 = (\frac{1}{27}, \frac{8}{81}, 1)$ have a common offspring. $\mathbf{v}_1$ is defined by the word $(11) \in \Lambda_1$ and is of neighborhood type $\mathcal{T}_2$. It generates seven offspring in $\mathcal{G}$ with corresponding words

$$(1111), (1112), (1113), (112), (1131), (1132), (1133) \in \Lambda_2,$$

which are of neighborhood types $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}_4$, $\mathcal{T}_5$, $\mathcal{T}_2$, $\mathcal{T}_3$ and $\mathcal{T}_2$, respectively. $\mathbf{v}_2$ corresponds to the word $(12) \in \Lambda_1$ and is of neighborhood type $\mathcal{T}_3$. It produces three offspring in $\mathcal{G}$ corresponding to

$$(121), (122), (123) \in \Lambda_2.$$

Note that $\mathbf{u} = (\frac{1}{81}, \frac{8}{81}, 2)$ is a common offspring because $\phi_{(1133)} = \phi_{(121)}$. Since $(1) < (33)$, the edge $\mathbf{k} = (33)$ connecting $\mathbf{v}_1$ to $\mathbf{u}$ is removed when constructing the reduced graph $\mathcal{G}_R$. Consequently we have

$$\mathcal{T}_2 \to 2\mathcal{T}_2 + 2\mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5,$$
$$\mathcal{T}_3 \to \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3.$$

Using the same argument, one can check that

$$\mathcal{T}_4 \to \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5,$$
$$\mathcal{T}_5 \to \mathcal{T}_1 + 2\mathcal{T}_2 + 2\mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5.$$

These are all the neighborhood types and the incidence matrix is

$$S = \begin{bmatrix} 1 & 2 & 2 & 1 & 1 \\ 0 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 \end{bmatrix}.$$

The spectral radius λ of S is the root of the polynomial $x^3 - 6x^2 + 5x - 1$ and

$$\dim_H(F) = \frac{\log \lambda}{\log 9} = 0.7369177683\ldots.$$

The similarity dimension of F is

$$\dim_s(F) = \frac{\log(\sqrt{2} - 1)}{-\log 3} = 0.8022608122\ldots.$$

Remark. The size of the matrix S above can be further reduced. Indeed, note that $\Omega(\mathbf{v}_1) = \Omega(\mathbf{v}_2) = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\Omega(\mathbf{v}_4) = \Omega(\mathbf{v}_5)$, so together with $\Omega(\mathbf{v}_{\text{root}})$, there are only three translationally inequivalent "types" of such neighborhoods. We may generalize the notion of neighborhood types (originally for vertices) to such neighborhoods, and construct an analogous incidence matrix S'. In the above example, this yields

$$S' = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}.$$

With suitable modifications if necessary, this argument can also be applied to other examples. However, in order to avoid additional technical details, we will not pursue such simplifications here.

Example 5.3. *Consider the following variant of the Sierpinski gasket in $\mathbb{R}^2$ defined by the IFS $\{\phi_j(x)\}_{j=1}^3$:*

$$\phi_1(x,y) = (\rho x, \rho y), \quad \phi_2(x,y) = (\rho x + \rho^2, \rho y), \quad \phi_3(x,y) = (\rho^2 x, \rho^2 y + \rho),$$

where $\rho = (\sqrt{5}-1)/2$ (see Figure 5.2).

The IFS $\{\phi_1, \phi_2, \phi_3\}$ is of finite type since ρ^{-1} is a Pisot number and $\{(0,0), (\rho^2,0), (0,\rho)\} \subseteq \mathbb{Z}[\rho] \times \mathbb{Z}[\rho]$ (Theorem 2.5). A finite type condition set Ω can be obtained by taking the filled open triangle with vertices at $(0,0)$, $(1,0)$, and $(0,1)$. Let $\mathcal{T}_1$ denote the neighborhood type of the root vertex. Upon one iteration we get

$$\Lambda_1 = \{(11), (12), (13), (21), (22), (23), (3)\},$$

$$\mathcal{V}_1 = \Big\{ \big(\rho^2, (0,0), 1\big), \big(\rho^2, (\rho^3, 0), 1\big), \big(\rho^3, (0,\rho^2), 1\big), \big(\rho^2, (\rho^2, 0), 1\big), \big(\rho^2, (\rho, 0), 1\big), \big(\rho^3, (\rho^2, \rho^2), 1\big), \big(\rho^2, (0,\rho), 1\big) \Big\}.$$

The vertex $(\rho^2, (0,\rho), 1)$ is of neighborhood type $\mathcal{T}_1$. The rest of the vertices are classified as follows:

Vertex	Neighborhood Type
$\big(\rho^3, (0,\rho^2), 1\big)$, $\big(\rho^3, (\rho^2,\rho^2), 1\big)$	$\mathcal{T}_2$
$\big(\rho^2, (0,0), 1\big)$	$\mathcal{T}_3$
$\big(\rho^2, (\rho^3,0), 1\big)$	$\mathcal{T}_4$
$\big(\rho^2, (\rho^2,0), 1\big)$	$\mathcal{T}_5$
$\big(\rho^2, (\rho,0), 1\big)$	$\mathcal{T}_6$

Hence we have

$$\mathcal{T}_1 \to \mathcal{T}_1 + 2\mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5 + \mathcal{T}_6.$$

It is easy to see that upon one more iteration,

$$\mathcal{T}_2 \to \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_6.$$

Common offspring are generated when iterating the other vertices in $\mathcal{V}_1$. The vertex $\mathbf{v}_1 = (\rho^2, (0,0), 1)$ generates seven offspring in $\mathcal{G}$, defined by the words

$$(113), (1113), (1123), (1111), (1112), (1121), (1122) \in \Lambda_2.$$

The first six of them are of neighborhood types $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5$, respectively. The last one is of a new neighborhood type, which we shall denote by $\mathcal{T}_7$.

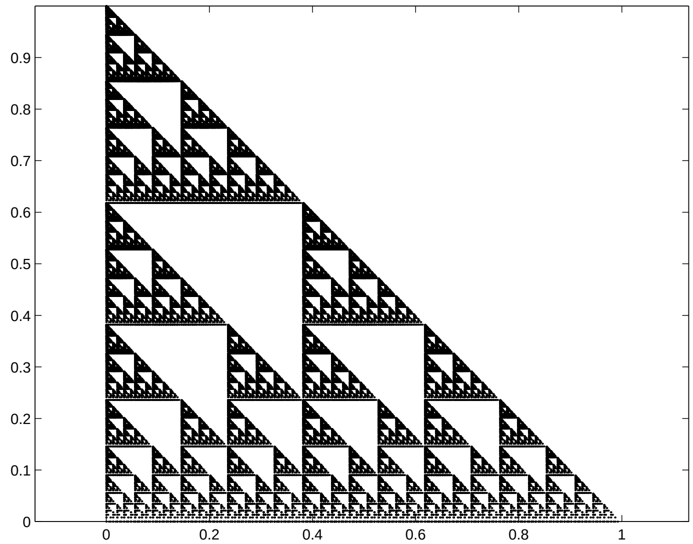

FIGURE 5.2. The attractor F of the IFS in Example 5.3.

The vertex $\mathbf{v}_2 = (\rho^2, (\rho^3, 0), 1)$ also generates seven offspring in $\mathcal{G}$, with the offspring $\mathbf{u} = (\rho^4, (\rho^3, 0), 2)$ (labeled by the edge (11)) coinciding with the offspring generated by $\mathbf{v}_1$ by the edge (22). When constructing $\mathcal{G}_R$, the edge (22) connecting $\mathbf{v}_1$ to $\mathbf{u}$ is removed. As a result,

$$\mathcal{T}_3 \to \mathcal{T}_1 + 2\mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5.$$

The same argument also yields

$$\begin{aligned} \mathcal{T}_4 &\to \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_4 + \mathcal{T}_7, \\ \mathcal{T}_5 &\to \mathcal{T}_1 + 2\mathcal{T}_2 + \mathcal{T}_4 + 2\mathcal{T}_5, \\ \mathcal{T}_6 &\to \mathcal{T}_1 + 2\mathcal{T}_2 + \mathcal{T}_4 + \mathcal{T}_5 + \mathcal{T}_6 + \mathcal{T}_7. \end{aligned}$$

The vertex $(\rho^4, (\rho^3, 0), 2)$ (of neighborhood type $\mathcal{T}_7$) generates seven offspring in $\mathcal{G}$ upon one more iteration. It can be checked as above that in $\mathcal{G}_R$,

$$\mathcal{T}_7 \to \mathcal{T}_1 + 2\mathcal{T}_2 + \mathcal{T}_4 + \mathcal{T}_5 + \mathcal{T}_7.$$

The above process exhausts all possible neighborhood types and yields the incidence matrix

$$S = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

The spectral radius λ is the root of $x^3 - 6x^2 + 5x - 1$ and

$$\dim_H(F) = \frac{\log \lambda}{-2 \log \rho} = 1.6823919818\ldots,$$

which is strictly less than $\dim_s(F) = \frac{\log(\sqrt{2}-1)}{\log \rho} = 1.8315709239\ldots.$

Example 5.4. *We will show that each IFS in the following class is of finite type and compute its Hausdorff dimension:*

$$\phi_1(x) = \rho x, \quad \phi_2(x) = \rho x + \rho, \quad \phi_3(x) = \rho x + 1, \quad 0 < \rho < \frac{3-\sqrt{5}}{2}.$$

It is known that for the given range of values of ρ, $\{\phi_1, \phi_2, \phi_3\}$ does not satisfy the open set condition (see [LNR]). We will show that it is of finite type. Let $\Omega = (0, 1/(1-\rho))$. Let $\mathcal{T}_1$ denote the neighborhood type of the root vertex, and $\mathcal{T}_2$, $\mathcal{T}_3$ denote neighborhood types of the vertices $(\rho, 0, 1)$ and $(\rho, \rho, 1) \in \mathcal{V}_1$, respectively. (The vertex $(\rho, 1, 1)$ is also of neighborhood type $\mathcal{T}_1$.) Obviously,

$$\mathcal{T}_1 \to \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3.$$

$(\rho, 0, 1)$ and $(\rho, \rho, 1)$ generate a common offspring $(\rho^2, \rho, 2)$ in $\mathcal{G}$, defined by the words (13) and (21) respectively. In the reduced graph, the edge connecting $(\rho, 0, 1)$ to $(\rho^2, \rho, 2)$ is removed, and we have

$$\mathcal{T}_2 \to \mathcal{T}_2 + \mathcal{T}_3,$$
$$\mathcal{T}_3 \to \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3.$$

Since no new neighborhood types are generated, we conclude that the IFS is of finite type and obtain the transition matrix

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The the spectral radius of S is $\lambda = (3+\sqrt{5})/2$ and thus

$$\dim_H(F) = \frac{\log(3+\sqrt{5}) - \log 2}{-\log \rho},$$

while $\dim_s(F) = \frac{\log 3}{-\log \rho}$.

References

[E] G. A. Edgar, A fractal puzzle, *Math. Intelligencer* **13** (1991), 44-50.

[F1] K. J. Falconer, Dimensions and measures of quasi self-similar sets, *Proc. Amer. Math. Soc.* **106** (1989), 543-554.

[F2] ———, *Fractal geometry. Mathematical foundations and applications*, John Wiley & Sons, Chichester, 1990.

[F3] ———, *Techniques in fractal geometry*, John Wiley & Sons, Ltd, Chichester, 1997.

[G] A. M. Garsia, Arithmetic properties of Bernoulli convolutions, *Trans. Amer. Math. Soc.* **102** (1962), 409-432.

[H] J. E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30** (1981), 713-747.

[K] R. Kenyon, Projecting the one-dimensional Sierpinski gasket, *Israel J. Math.* **97** (1997), 221-238.

[LW] J. C. Lagarias and Y. Wang, Tiling the line with translates of one tile, *Invent. Math.* **124**, fasc 2 (1996), 341-365.

[L] S. P. Lalley, β-expansions with deleted digits for Pisot numbers, *Trans. Amer. Math. Soc.* **349** (1997), 4355-4365.

[LN] K.-S. Lau and S.-M. Ngai, Multifractal measures and a weak separation condition, *Adv. Math.* **141** (1999), 45-96.

[LNR] K.-S. Lau, S.-M. Ngai, and H. Rao, Iterated function systems with overlaps and self-similar measures, *J. London Math. Soc.* (to appear).

[M] J. M. Marstrand, Some fundamental geometrical properties of plane sets of fractional dimensions, *Proc. London. Math. Soc. (3)* **4** (1954), 257-302.

[MW] R. D. Mauldin and S. C. Williams, Hausdorff dimension in graph directed constructions, *Trans. Amer. Math. Soc.* **309** (1988), 811–829.

[N] N. Nguyen, Iterated function systems of finite type and the weak separation property, preprint (2000).

[RW] H. Rao and Z.-Y. Wen, A class of self-similar fractals with overlap structure, *Adv. in Appl. Math.* **20** (1998), 50-72.

[SW] R. S. Strichartz and Y. Wang, Geometry of self-affine tiles I, *Indiana Univ. Math. J.* **48**, No. 1 (1999), 1-23.

[Z] M. P. W. Zerner, Weak separation properties for self-similar sets, *Proc. Amer. Math. Soc.* **124** (1996), 3529-3539.

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA.

E-mail address: ngai@math.gatech.edu

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA.

E-mail address: wang@math.gatech.edu and http://www.math.gatech.edu/~wang