# Two-Scale Dilation Equations and the Cascade Algorithm 

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## ABSTRACT

We study the two-scaled dilation equation

$$
f(x)=\sum_{k=0}^{n} c_{k} f(2 x-k)
$$

where the coefficients $c_{k}$ are real and $\sum_{k} c_{2 k}=\sum_{k} c_{2 k+1}=1$. By expressing the dilation equation in matrix product form we prove a necessary and sufficient condition for the cascade algorithm (introduced by Daubechies and Lagarias) to converge uniformly to a continuous solution. We also establish several basic relations between the convergence of infinite products of matrices and the existence and regularity of solutions to the two-scale dilation equations.

## INTRODUCTION

Two-scale dilation equations arise in many diverse applications. The general equation is

$$
\begin{equation*}
f(x)=\sum_{k=0}^{N} c_{k} f(2 x-k) \tag{1}
\end{equation*}
$$

where

$$
\sum_{k \text { even }} c_{k}=\sum_{k \text { odd }} c_{k}=1
$$

Often equation (1) has compactly supported solutions $\operatorname{supp}(f) \subseteq[0, N]$, and we usually seek the normalized one

$$
\int f(x) d x=1
$$

If $c_{k}=\binom{N}{k} / 2^{N-1}$ then $f$ is the normalized $B$-spline of degree $N-1$, given by

$$
\begin{equation*}
f(x)=\frac{(-1)^{N}}{(N-1)!} \sum_{k=0}^{N}(-1)^{k}\binom{N}{k}\left[(k-x)^{+}\right]^{N-1} . \tag{2}
\end{equation*}
$$

Given points $\mathbf{v}_{1}, \ldots, \mathbf{v}_{N} \in \mathbb{R}^{m}$ we can use this function $f$ to generate a $C^{N-2}$ curve

$$
\begin{equation*}
\mathbf{V}(x)=\sum_{k=1}^{N} f(x+k-1) \mathbf{v}_{k}, \quad x \in[0,1] . \tag{3}
\end{equation*}
$$

The curve $\mathbf{V}(x)$ is the B-spline with control points $\mathbf{v}_{1}, \ldots, \mathbf{v}_{N}$. More generally if $c_{k}>0$ for $k=0, \ldots, N$ then the normalized solution $f \geq 0$ satisfies

$$
\sum_{k=1}^{N} f(x+k-1)=1
$$

and so the curve $\mathbf{V}(x)$ lies inside the convex hull of the $\mathbf{v}_{k}$ 's.
There is a simple geometric subdivision method ([6], [12], [13]) for constructing this curve $\mathbf{V}(x)$. Given the initial polyhedron with vertices $\mathbf{v}_{1}, \ldots, \mathbf{v}_{N}$, produce the two subpolyhedra with vertices

$$
\begin{equation*}
\mathbf{v}_{k}^{\prime}=\sum_{j=1}^{N} c_{2 j-k-1} \mathbf{v}_{j}, \quad \mathbf{v}_{k}^{\prime \prime}=\sum_{j=1}^{N} c_{2 j-k} \mathbf{v}_{j} . \quad(k=1, \ldots, N) \tag{4}
\end{equation*}
$$

Here we are using the convention that $c_{k}=0$ for $k<0$ or $k>N$. Repeatedly sub-divide each of these two polyhedra into two smaller ones, using the same schemes (4), in a binary tree fashion. The miniscule polyhedra eventually obtained all link together to form the curve V. In particular we can recover $f(x)$ in this way by selecting $\mathbf{v}_{k}$ to be the vector with a one in position $k$ and zeros elsewhere.

We can also use solutions of (1) to construct compactly supported wavelets. Precisely, if the $c_{k}$ 's are chosen so as to satisfy

$$
\sum c_{k} c_{k+2 m}= \begin{cases}2, & m=0 \\ 0, & m \neq 0\end{cases}
$$

then we can generate compactly supported wavelets

$$
\begin{equation*}
\psi(x)=\sum(-1)^{k} c_{1-k} f(2 x-k) . \tag{5}
\end{equation*}
$$

Moreover additional moment-type conditions on the $c_{k}$ 's will guarantee smoothness of $\psi$ ([7], [14]).
This paper aims to study the two-scale dilation equations (1). It addresses the structure of solutions, the existence of discontinuous but bounded solutions, convergence of approximants

$$
f_{n}=\sum c_{k} f_{n-1}(2 x-k)
$$

and presents some examples. It shares a common approach with [9], [10] in that infinite products of matrices are used to construct solutions of (1). As we were revising this paper as suggested by the referees, many exciting new studies on two-scale dilation equations have been done by various authors. We have included some of those studies in the reference ([15], [16], [17], [18]).

## BASIC RESULTS

In [4], [10] the notion of a left convergent product (LCP) set of matrices was developed. The set-up goes as follows. Let $\mathcal{M}=\mathcal{M}_{m}$ denote the algebra of all real $m \times m$ matrices, and let $\|\cdot\|$ be a norm on $\mathbb{R}^{m}$. This norm induces a corresponding operator norm $\|\cdot\|$ on $\mathcal{M}$. Let $\Sigma \subseteq \mathcal{M}$ be a non-empty bounded set of matrices, and denote by $\mathcal{S}(\Sigma)$ the semi-group generated by $\Sigma$. Then $\mathcal{S}(\Sigma)=\bigcup_{n=1}^{\infty} \Sigma^{n}$, where $\Sigma^{n}=\left\{\prod_{i=1}^{n} M_{i}: M_{i} \in \Sigma, 1 \leq i \leq n\right\}$. Let $\mathcal{E}(\Sigma)$ denote the subspace of common right 1-eigenvectors of $\Sigma$,

$$
\mathcal{E}(\Sigma)=\left\{\mathrm{x} \in \mathbb{R}^{m}: M \mathrm{x}=\mathrm{x}, \forall M \in \Sigma\right\} .
$$

We say that $\Sigma$ is LCP if every infinite product from $\Sigma$ left converges; i.e., if $\lim _{n \rightarrow \infty} M_{n} \cdots M_{1}$ exists for any sequence $\left(M_{n}\right)_{n=1}^{\infty}$ in $\Sigma$. In this case denote by $\Sigma^{\infty}$ the set of all such limits. Define

$$
\|\Sigma\|=\sup \{\|M\|: M \in \Sigma\}, \quad \hat{\rho}(\Sigma)=\underset{n \rightarrow \infty}{\lim \sup }\left\|\Sigma^{n}\right\|^{1 / n}
$$

$\hat{\rho}(\Sigma)$ is called the joint spectral radius of $\Sigma$. Observe that $\hat{\rho}(\Sigma)$ does not depend on the particular choice for the norm on $\mathcal{M}$. If $\Sigma=\left\{M_{\omega}: \omega \in J\right\}$ we also use the notation $\mathcal{S}\left(M_{\omega}: \omega \in J\right)$, $\mathcal{E}\left(M_{\omega}: \omega \in J\right), \hat{\rho}\left(M_{\omega}: \omega \in J\right)$.

The following results appear in [4], [10].

Lemma I. (a) For any $\delta>\hat{\rho}(\Sigma)$ there exists an operator norm $\nu=\nu_{\delta}$ on $\mathcal{M}$ such that $\nu(\Sigma) \leq \delta$.
(b) Suppose $M_{\infty}=\lim _{n \rightarrow \infty} M_{n} \cdots M_{1}$ exists, and that $M$ is a limit point of $\left(M_{n}\right)$. Then each column vector v of $M_{\infty}$ satisfies $M \mathbf{v}=\mathbf{v}$.
(c) If $\Sigma$ is LCP then $\mathcal{S}(\Sigma)$ is bounded.
(d) $\Sigma$ is LCP with $\Sigma^{\infty}=0$ if and only if $\hat{\rho}(\Sigma)<1$.

From parts (a) and (d) together we infer that if $\Sigma^{\infty}=0$ then the matrices in $\Sigma$ must all be strict contractions relative to some operator norm. Let $\left\{P_{\omega}: \omega \in J\right\}$ be row stochastic $m \times m$ matrices. (The entries are allowed to be negative.) Since these matrices all have the common (right) eigenvectors $\mathbf{e}=(1, \ldots, 1)^{t}$ we can reduce them under similarity to the form

$$
P_{\omega} \sim\left(\begin{array}{cc}
1 & \mathbf{b}_{\omega}^{t}  \tag{6}\\
0 & A_{\omega}
\end{array}\right), \quad \omega \in J
$$

where $A_{\omega}$ is $(m-1) \times(m-1)$. In this way, for any $\omega_{1}, \ldots, \omega_{n} \in J$,

$$
P_{\omega_{n}} \cdots P_{\omega_{1}} \sim\left(\begin{array}{cc}
1 & \mathbf{b}^{t}\left(\omega_{1}, \ldots, \omega_{n}\right) \\
0 & A_{\omega_{n}} \cdots A_{\omega_{1}}
\end{array}\right)
$$

where $\mathbf{b}\left(\omega_{1}, \ldots, \omega_{n}\right) \in \mathbb{R}^{m-1}$.
Let $\Omega=J^{\infty}$ be code space. If $\hat{\rho}\left(A_{\omega}: \omega \in J\right)<1$, then $\left\{P_{\omega}: \omega \in J\right\}$ is LCP (see [10]) and

$$
P(\underline{\omega})=\lim _{n \rightarrow \infty} P_{\omega_{n}} \cdots P_{\omega_{1}}=\left(\begin{array}{c}
\mathbf{v}^{t}(\underline{\omega})  \tag{7}\\
\cdots \cdots \cdots \\
\mathbf{v}^{t}(\underline{\omega})
\end{array}\right)
$$

where

$$
\begin{equation*}
\mathbf{v}^{t}(\underline{\omega})=\left(1 \mid \mathbf{b}^{t}(\underline{\omega})\right) C^{-1} \tag{8}
\end{equation*}
$$

and $C=\left(\mathbf{e} \mid C^{\prime}\right)$ is the change-of-basis matrix used to effect the similarity transformation in (6). For example if we take

$$
C=\left(\begin{array}{ccccc}
1 & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{array}\right)
$$

then

$$
\mathbf{v}^{t}(\underline{\omega})=\left(\mathbf{b}^{t}(\underline{\omega}) \mid 1-\sum_{i=1}^{m-1} b_{i}(\underline{\omega})\right)
$$

Consider next the special case of just two row-stochastic matrices $P_{0}, P_{1}$. The code space here, $\Omega=\{0,1\}^{\infty}$, can be mapped onto the interval $[0,1]$ through dyadic expansion, whereby $\underline{\omega}=\left(\omega_{1}, \omega_{2}, \ldots\right) \in \Omega$ corresponds to $x=\sum_{n=1}^{\infty} \frac{\omega_{n}}{2^{n}} \in[0,1]$. For non-dyadic $x$ this expansion is unique, and has infinitely many zeros and ones. For dyadic $x$ (aside from $x=0, x=1$ )
there are two expansions: the terminating expansion with only finitely many ones, and the nonterminating expansion with only finitely many zeros. In what follows we adopt the convention that the terminating expansion is used whenever $x$ is dyadic. (This point will be further discussed below.) In this way each $x \in[0,1]$ has a unique dyadic code $\underline{\omega}=\left(\omega_{1}, \omega_{2}, \ldots\right)$ which we denote by $D(x)$.

If $\left\{P_{0}, P_{1}\right\}$ is LCP then we can use $P: \Omega \rightarrow \mathcal{M}_{m}$ from (8) to induce a map $P_{*}=P \circ D:$ $[0,1) \rightarrow \mathcal{M}_{m}$. In order for $P_{*}$ to be well-defined at dyadic points if we were to use both dyadic expansions, there must hold the consistency condition

$$
\begin{equation*}
P_{0}^{\infty} P_{1}=P_{1}^{\infty} P_{0} \tag{C}
\end{equation*}
$$

Let $\tau:[0,1) \rightarrow[0,1)$ be the dyadic shift operator $\tau x=2 x(\bmod 1)$. Note that $\tau$ is ergodic with respect to the Lebesgue measure $\lambda$ on $[0,1$ ). (See e.g., Breiman [5, Sec. $6.4-$ Prob. 7].) For any $x \in[0,1)$ there holds

$$
\begin{equation*}
P_{*}(x)=P_{*}(\tau x) P_{\omega_{1}} \tag{9}
\end{equation*}
$$

where $\omega_{1}$ is the first bit in the dyadic code $D(x)$.

Lemma II. Suppose $\left\{P_{0}, P_{1}\right\}$ is LCP. Then
(a) $P_{*}$ is continuous at all non-dyadic points $x \in(0,1)$.
(b) If the consistency condition (C) holds, then $P_{*}$ is continuous everywhere in ( 0,1 ).

Proof. (a) Let $r=\operatorname{dim} \mathcal{E}\left(P_{0}, P_{1}\right)$. Using a similarity transformation we can assume that

$$
P_{\omega}=\left(\begin{array}{cc}
I_{r} & B_{\omega} \\
0 & A_{\omega}
\end{array}\right), \quad \omega=0,1 .
$$

Then $\mathcal{E}\left(A_{0}, A_{1}\right)=0$. Since $\left\{P_{0}, P_{1}\right\}$ is LCP, so is $\left\{A_{0}, A_{1}\right\}$. Moreover by Lemma $\mathrm{I}(\mathrm{b})$, $\lim _{n \rightarrow \infty} A_{\omega_{n}} \cdots A_{\omega_{1}}=0$ whenever 0 and 1 appear infinitely often in $\underline{\omega}=\left(\omega_{1}, \omega_{2}, \ldots\right)$.

Consider now $P_{*}(x)-P_{*}(y)$, where $x \in(0,1)$ is non-dyadic. Let $k_{0}(n)$ and $k_{1}(n)$ denote respectively the positions of the first zero and first one in the dyadic code $D(x)$, beyond position
$n$. If $|x-y|<2^{-\max \left\{k_{0}(n), k_{1}(n)\right\}}$ then $x$ and $y$ must have identical dyadic codes up to position $n$. In this case

$$
\begin{aligned}
P_{*}(x)-P_{*}(y) & =\left[P_{*}\left(\tau^{n} x\right)-P_{*}\left(\tau^{n} y\right)\right] P_{\omega_{n}} \cdots P_{\omega_{1}} \\
& =\left(\begin{array}{cc}
0 & B^{\prime} \\
0 & A^{\prime}
\end{array}\right)\left(\begin{array}{cc}
I_{r} & B^{\prime \prime} \\
0 & A_{\omega_{n}} \cdots A_{\omega_{1}}
\end{array}\right)=\left(\begin{array}{cc}
0 & B^{\prime} A_{\omega_{n}} \cdots A_{\omega_{1}} \\
0 & A^{\prime} A_{\omega_{n}} \cdots A_{\omega_{1}}
\end{array}\right)
\end{aligned}
$$

where $A^{\prime}, B^{\prime}$ are bounded independently of $y$, by virtue of Lemma $\mathrm{I}(\mathrm{c})$. Since $x$ is non-dyadic $A_{\omega_{n}} \cdots A_{\omega_{1}} \rightarrow 0$ as $n \rightarrow \infty$, and we get that $P_{*}(y) \rightarrow P_{*}(x)$ as $y \rightarrow x$.
(b) Under condition (C) it follows that $A_{0}^{\infty} A_{1}=A_{1}^{\infty} A_{0}=0$, the last equality following again by Lemma $I(b)$. Thus we can let $x \in(0,1)$ be dyadic in the argument above for part (a).

Notice that we do not claim continuity at $x=0$ or $x=1$ in Lemma II(b). Since 0 and 1 have unique dyadic codes, in order for our proof to work for these values of $x$ we would need to know that $A_{0}^{\infty}=A_{1}^{\infty}=0$, which is not necessarily the case.

From now on, we shall always use $P_{0}$ and $P_{1}$ to denote the row stochastic matrices

$$
\begin{equation*}
P_{0}=\left(c_{2 j-i-1}\right)_{i, j=1}^{N}, \quad P_{1}=\left(c_{2 j-i}\right)_{i, j=1}^{N} \tag{10}
\end{equation*}
$$

where $c_{k}=0$ for $k<0$ or $k>N$. These matrices take the form

$$
P_{0}=\left(\begin{array}{cc}
c_{0} & \mathbf{w}_{0}^{t}  \tag{11}\\
0 & M
\end{array}\right), \quad P_{1}=\left(\begin{array}{cc}
M & 0 \\
\mathbf{w}_{1}^{t} & c_{N}
\end{array}\right)
$$

where $M$ is the $(N-1) \times(N-1)$ row stochastic matrix $M=\left(c_{2 j-i}\right)_{i, j=1}^{N-1}$. In particular the spectra look like

$$
\begin{align*}
& \sigma\left(P_{0}\right)=\sigma(M) \cup\left\{c_{0}\right\} \\
& \sigma\left(P_{1}\right)=\sigma(M) \cup\left\{c_{N}\right\} \tag{12}
\end{align*}
$$

Since $1 \in \sigma(M)$ there is a left 1 -eigenvector $\mathbf{u} \in \mathbb{R}^{N-1}, \mathbf{u}^{t} M=\mathbf{u}^{t}$. Then

$$
\begin{equation*}
\mathbf{v}(0)=\left(0 \mid \mathbf{u}^{t}\right), \quad \mathbf{v}(1)=\left(\mathbf{u}^{t} \mid 0\right) \tag{13}
\end{equation*}
$$

are respective left 1-eigenvectors for $P_{0}$ and $P_{1}$.
Let $T: L_{1}(\mathbb{R}) \rightarrow L_{1}(\mathbb{R})$ be the two-scale operator

$$
\begin{equation*}
T f(x)=\Sigma_{k=0}^{N} c_{k} f(2 x-k) \tag{14}
\end{equation*}
$$

Observe that

$$
\begin{equation*}
\widehat{T f}(\xi)=Q(\xi / 2) \hat{f}(\xi / 2) \tag{15}
\end{equation*}
$$

where $\hat{f}$ denotes the Fourier transform of $f$, and $Q$ is the trigonometric polynomial $Q(\xi)=$ $\frac{1}{2} \Sigma c_{k} e^{i k \xi}$. Observe further that if $\sigma_{h}$ denotes the shift operator $\sigma_{h} f(x)=f(x-h)$ then

$$
\begin{equation*}
T \sigma_{h}=\sigma_{h / 2} T \tag{16}
\end{equation*}
$$

For $f \in L_{1}(\mathbb{R})$ define $S f \in L_{1}([0,1))$ by

$$
\begin{equation*}
S f(x)=\sum_{k=-\infty}^{\infty} f(x+k) \tag{17}
\end{equation*}
$$

Observe that this summation operator $S: L_{1}(\mathbb{R}) \rightarrow L_{1}([0,1))$ is a linear operator, $\|S\|=1$, and that

$$
\begin{equation*}
\int_{0}^{1} S f=\int_{-\infty}^{\infty} f \tag{18}
\end{equation*}
$$

Moreover, on account of (14), for any $f \in L_{1}(\mathbb{R})$ there holds

$$
\begin{equation*}
S T f=S f \circ \tau \tag{19}
\end{equation*}
$$

For any function $f \in L_{1}(\mathbb{R})$ with $\operatorname{supp}(f) \subseteq[0, N]$ denote by $\mathbf{V} f \in L_{1}\left([0,1) ; \mathbb{R}^{N}\right)$ the vectorvalued map

$$
\mathbf{V} f(x)=[f(x), f(x+1), \ldots, f(x+N-1)]^{t}
$$

Correspondingly for any vector-valued map $\mathbf{v} \in L_{1}\left([0,1) ; \mathbb{R}^{N}\right)$ denote by $F \mathbf{v} \in L_{1}(\mathbb{R})$ the function

$$
F \mathbf{v}(x)=\left\{\begin{array}{ll}
v_{k}(x-k+1), & k-1 \leq x<k,  \tag{20}\\
0, & x<0 \text { or } x \geq N
\end{array} \quad(k=1, \ldots, N),\right.
$$

with $\operatorname{supp}(F \mathbf{v}) \subseteq[0, N]$. Observe that for any $f(x)$ if $f(x)=0$ for $x<0, x \geq N$, then

$$
\begin{equation*}
\mathbf{V} T f(x)=P_{\omega_{1}}^{t} \mathbf{V} f(\tau x) \tag{21}
\end{equation*}
$$

where $\omega_{1}$ is the first bit in the dyadic code $D(x)$ of $x$. Here is where the convention about choosing the terminating expansion for $D(x)$ enters in. If we were to use instead the non-terminating
expansion, then $P_{*}, \tau$ and $\mathbf{V} f$ would be defined on ( 0,1$]$, and (21) would hold provided $f(x)=0$ for $x \leq 0, x>N$. That is, with the terminating expansion convention (21) requires that $f(N)=0$, and with the non-terminating expansion convention it would require that $f(0)=0$. Of course if $f(0)=f(N)=0$ then (21) holds for either convention.

We are concerned primarily with solutions $f \in L_{1}(\mathbb{R})$ of the two-scale equation (1), which can also be written as

$$
\begin{equation*}
T f=f \tag{22}
\end{equation*}
$$

It follows from (15) that the Fourier transform of such a solution necessarily satisfies

$$
\begin{equation*}
\hat{f}(\xi)=\alpha \prod_{n=1}^{\infty} Q\left(\xi / 2^{n}\right) \tag{23}
\end{equation*}
$$

where $\alpha$ is a constant. Convergence of the infinite product on the right follows from an argument like the one in [8, Thm. 2.1]. Thus, up to normalization there can be at most one solution $f \in L_{1}(\mathbb{R})$ to (1). Moreover if $f$ is non-trivial then $\int f \neq 0$. Otherwise $\alpha$ in (23) would have to be zero. So we can always normalize $f$ by requiring $\int f=1$.

It follows from (17) that if $f \in L_{1}(\mathbb{R})$ satisfies (1), then $S f$ must be constant a.e. This is simply because $S f$ is $\tau$-invariant and $\tau$ is ergodic. Furthermore, according to (18) this constant must be $\int f$; i.e.,

$$
\begin{equation*}
S f \equiv \int f \quad \text { a.e. } \tag{24}
\end{equation*}
$$

We specifically study compactly supported solutions to (1). Observe that if $\operatorname{supp}(f) \subseteq[a, b]$ then $\operatorname{supp}(T f) \subset\left[\frac{a}{2}, \frac{b+N}{2}\right]$. In order for (1) to hold, then, it is clear that we must have $\operatorname{supp}(f) \subseteq[0, N]$. If $f(N)=0$ then it follows from (21) that

$$
\begin{equation*}
\mathbf{V} f(x)=P_{\omega_{1}}^{t} \mathbf{V} f(\tau x) \tag{25}
\end{equation*}
$$

Conversely if $f(x)=0$ for $x<0, x \geq N$ and if (25) holds, then $T f=f$.
The following result relates specifically to the coefficients $c_{0}, c_{N}$.

Lemma III. (a) Suppose the two-scale dilation equation (1) has a non-trivial solution $f$ supported on $[0, N]$ with $f(0+)=0$, where $f(0+)=\lim _{x \downarrow 0} f(x)$. Then $\left|c_{0}\right|<1$. Similarly if $f(N-)=0$ then $\left|c_{N}\right|<1$.
(b) Suppose $\left\{P_{0}, P_{1}\right\}$ is LCP. Then $\left|c_{0}\right|,\left|c_{N}\right| \leq 1$.

Proof. (a) It follows from the two-scale equation that for $x \in(0,1 / 2), f(x)=c_{0} f(2 x)$. Thus

$$
f(2 x)=\frac{1}{c_{0}} f(x)=\frac{1}{c_{0}^{2}} f\left(\frac{x}{2}\right)=\cdots .
$$

Suppose $\left|c_{0}\right| \geq 1$. If $f(0+)=0$ then $\frac{1}{c_{0}^{n}} f\left(\frac{x}{2^{n-1}}\right) \rightarrow 0$, and so $f=0$ on $(0,1)$. Using the two-scale equations again and again this extends to $(1,2),(2,3), \ldots$. This would lead to $f=0$ a.e., and so it must be that $\left|c_{0}\right|<1$. A similar argument works for $c_{N}$.
(b) This follows at once from (12) since $c_{0} \in \sigma\left(P_{0}\right), c_{N} \in \sigma\left(P_{1}\right)$.

Theorem IV. Suppose $\left\{P_{0}, P_{1}\right\}$ is LCP. Then there exists a bounded measurable solution $f$ to (1), supported in $[0, N], \int f=1$. It is determined by

$$
\mathbf{V}^{t} f(x)=(1,0, \ldots, 0) P_{*}(x), \quad x \in[0,1)
$$

Moreover $\mathcal{E}\left(P_{0}, P_{1}\right)$ is one-dimensional.

Proof. When $\left\{P_{0}, P_{1}\right\}$ is LCP it is easy to construct solutions of (22). Indeed it follows straightway from (9) that for any vector $\mathbf{u} \in \mathbb{R}^{N}$

$$
\mathbf{v}(x)=P_{*}^{t}(x) \mathbf{u}
$$

satisfies (25). If we then define $f=F \mathbf{v}$ according to (20), we find that $T f=f$. It follows from Lemma $\mathrm{I}(\mathrm{c})$ that $f$ is bounded. Measurability follows from Lemma $\mathrm{II}(\mathrm{a})$, since $f$ is continuous a.e.

Since $\mathbf{u}$ is arbitrary the uniqueness of solutions $f \in L_{1}(\mathbb{R})$ to (1) then implies that any two rows of $P_{*}(x)$ must be linearly dependent, for a.e. $x \in[0,1)$. Thus $P_{*}(x)$ must be rank one for a.e. $x$. This in turn implies that $\operatorname{dim} \mathcal{E}\left(P_{0}, P_{1}\right)=1$.

It is proved in [9] that if $\hat{\rho}\left(A_{0}, A_{1}\right)<1$ with $A_{0}, A_{1}$ then $f$ is continuous. It is possible that $\left\{P_{0}, P_{1}\right\}$ is LCP but $\hat{\rho}\left(A_{0}, A_{1}\right)=1$. The following is an example where the solutions of (1) are discontinuous at every dyadic point $x$.

$$
c_{0}=1, \quad c_{1}=1 / 2, \quad c_{2}=0, \quad c_{3}=1 / 2
$$

With this choice of coefficients, $\left\{P_{0}, P_{1}\right\}$ is LCP. Indeed, using the change-of-basis matrix

$$
C=\left(\begin{array}{lll}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{array}\right)
$$

we find that

$$
C^{-1} P_{0} C=\left(\begin{array}{cc}
1 & 0 \\
\mathbf{0} & A_{0}
\end{array}\right), \quad C^{-1} P_{1} C=\left(\begin{array}{cc}
1 & 1 / 2 \\
\mathbf{0} & A_{1}
\end{array}\right)
$$

with

$$
A_{0}=\left(\begin{array}{cc}
1 / 2 & 1 / 2 \\
1 & 0
\end{array}\right), \quad A_{1}=\left(\begin{array}{cc}
-1 / 2 & 0 \\
0 & 1 / 2
\end{array}\right) .
$$

Using the $\ell_{\infty}$-norm we see that $\left\|A_{0}\right\|=1,\left\|A_{1}\right\|=1 / 2$. Thus for any sequence $\underline{\omega}=\left(\omega_{1}, \omega_{2}, \ldots\right)$ with infinitely many ones

$$
\lim _{n \rightarrow \infty} P_{\omega_{n}} \cdots P_{\omega_{1}}=\left(\begin{array}{cc}
1 & \mathbf{b}^{t}(\underline{\omega}) \\
\mathbf{0} & 0
\end{array}\right)
$$

where

$$
\mathbf{b}^{t}(\underline{\omega})=\left[\begin{array}{ll}
1 / 2 & 0
\end{array}\right] \sum_{n: \omega_{n}=1} A_{\omega_{n-1}} \cdots A_{\omega_{1}} .
$$

Moreover since $P_{0}^{\infty}$ exists, the $\underline{\omega}$ 's with finitely many ones are covered, too - giving LCP-ness. So in particular according to Theorem IV there exists a non-trivial bounded measurable solution $f$ to the dilation equation.

The consistency condition (C) is violated, since

$$
P_{0}^{\infty} P_{1}=\left(\begin{array}{ccc}
1 / 2 & 1 / 2 & 0 \\
2 / 3 & 1 / 6 & 1 / 6 \\
2 / 3 & 1 / 6 & 1 / 6
\end{array}\right) \neq\left(\begin{array}{ccc}
2 / 3 & 1 / 6 & 1 / 6 \\
2 / 3 & 1 / 6 & 1 / 6 \\
2 / 3 & 1 / 6 & 1 / 6
\end{array}\right)=P_{1}^{\infty} P_{0} .
$$

Observe that $\mathcal{E}\left(P_{0}\right)$ is two-dimensional here. See [12, Fig. 5.4] for a graph of $f$. To see that $\mathbf{V} f$ is in fact discontinuous at every dyadic point, argue in two steps as follows.
$1^{0} f$ is discontinuous at $x=0$ from the right.
To see this use the dilation equation

$$
f(x)=f(2 x)+\frac{1}{2} f(2 x-1)+\frac{1}{2} f(2 x-3)
$$

and the fact that $\operatorname{supp}(f) \subseteq[0,3]$ to notice (take $x \in(0,1 / 2))$ that for $y \in(0,1)$

$$
f(y)=f\left(\frac{y}{2}\right)=f\left(\frac{y}{4}\right)=\cdots
$$

If $\lim _{\epsilon \rightarrow 0^{+}} f(\epsilon)$ existed then $f$ would have to be constant on $(0,1)$, say $f \equiv A$ on $(0,1)$. Then it would follow (take $x \in(1 / 2,1)$ ) that $f \equiv \frac{1}{2} A$ on (1,2), and then (take $x \in(1,3 / 2)$ ) that $f \equiv \frac{1}{4} A$ on $(2,3)$. But then it would follow (take $x \in(3 / 2,2)$ ) that $A=0$.
$2^{0} \mathbf{V} f$ is discontinuous at every dyadic $x \in[0,1)$.
To see this let $x$ be the dyadic number $x=. \omega_{1} \cdots \omega_{k}$, and let $0<\epsilon<2^{-k}$. It follows from (25) that

$$
\mathbf{V} f(x+\epsilon)=P_{\omega_{1}}^{t} \cdots P_{\omega_{k}}^{t} \mathbf{V} f\left(\tau^{k} \epsilon\right)
$$

Since $\lim _{\epsilon \rightarrow 0^{+}} \mathbf{V} f\left(\tau^{k} \epsilon\right)$ does not exist and $P_{0}, P_{1}$ are invertible, we conclude that $\lim _{y \rightarrow x^{+}} \mathbf{V} f(y)$ does not exist either.

Since $A_{1}^{\infty}=0$ in this example, we do get left-continuity of $f$ in ( 0,3$]$. (See the argument in the proof of Lemma II.)

Theorem V. If $\mathcal{S}\left(P_{0}, P_{1}\right)$ is bounded, then the two-scale dilation equation (1) has a non-trivial $L^{1}$ solution.

Proof. Consider the sequence of functions $f_{n}(x)=T f_{n-1}, f_{0}(x)=\chi_{[0,1)}(x)$. It follows from (15) that

$$
\hat{f}(\xi)=\hat{f}_{0}\left(\xi / 2^{n}\right) \prod_{k=1}^{n} Q\left(\xi / 2^{k}\right)
$$

where $Q(\xi)=\frac{1}{2} \Sigma c_{k} e^{i k \xi}$. Notice that $\operatorname{supp}\left(f_{n}\right) \subseteq[0, N]$ and for any $x \in[0,1]$,

$$
\mathbf{V} f_{n}(x)=P_{\omega_{1}}^{t} \cdots P_{\omega_{n}} \mathbf{V} f_{0}\left(\tau^{n} x\right)
$$

where $\left(\omega_{1}, \omega_{2}, \cdots\right)=D(x)$. Hence there exists a constant $C>0$ such that $\left|f_{n}(x)\right|<C$ for all $x$ and $n$.

We have $\hat{f}_{0}(\xi)=\left(1-e^{-i \xi}\right) / i \xi$ and $\hat{f}_{n}(\xi) \rightarrow \phi(\xi)=\prod_{k=1}^{\infty} Q\left(\xi / 2^{k}\right)$ uniformly on any compact subset of $\mathbb{R}$. (Such a convergence is proved in [7].) Let $B \subset \mathbb{R}$ be any compact subset of $\mathbb{R}$. Then

$$
\begin{aligned}
\int_{B}|\phi(\xi)|^{2} d \xi & =\lim _{n \rightarrow \infty} \int_{B}\left|\hat{f}_{n}(\xi)\right|^{2} d \xi \\
& \leq \limsup _{n \rightarrow \infty} \int_{\mathbb{R}}\left|\hat{f}_{n}(\xi)\right|^{2} d \xi \\
& =\limsup _{n \rightarrow \infty} \int_{\mathbb{R}}\left|f_{n}(x)\right|^{2} d x \\
& \leq C^{2} N
\end{aligned}
$$

Hence $\phi(\xi) \in L^{2}(\mathbb{R})$. Let $f(x)$ be the inverse Fourier transform of $\phi(\xi)$. Then $f(x) \in L^{2}(\mathbb{R})$. Since $f(x)$ is compactly supported as a result of Paley-Wiener Theorem (see [7]), $f(x) \in L^{1}(\mathbb{R})$. Finally, the Fourier transform of $f(x)-\sum_{k=0}^{N} c_{k} f(2 x-k)$ is 0 . Hence, $f(x)$ satisfies the dilation equation (1).

If each coefficient $c_{k}>0$ for $k=0, \ldots, N$ then $\left\{P_{0}, P_{1}\right\}$ is LCP and $\hat{\rho}\left(A_{0}, A_{1}\right)<1$ with $A_{0}, A_{1}$ as in (6). This follows from [3, Thm. I], [13, Thm. 2.1] since $P_{0}$ and $P_{1}$ each have a (strictly) positive column. Indeed the first rows of $P_{0}$ and $P_{1}$ contain either $c_{N-1}$ or $c_{N}$, and that column underneath is all positive. In particular then we arrive at the following conclusion.

Proposition VI. Assume $N \geq 2$. Suppose $c_{k}>0$ for $k=0, \ldots, N$. Then $\hat{\rho}\left(A_{0}, A_{1}\right) \leq 1-\epsilon$ where $\epsilon=\min _{k} c_{k}$.

Proof: Let $\mathcal{E} \subseteq \mathbb{R}^{N}$ be the subspace spanned by $\mathbf{e}=(1,1, \ldots, 1)^{t}$. We need to show that $P_{0}, P_{1}$ are both strictly contractive on the quotient space $\mathbb{R}^{N} / \mathcal{E}$, relative to some operator norm. Define a norm $\nu$ on $\mathbb{R}^{N} / \mathcal{E}$ by

$$
\nu(\mathrm{x})=\max _{i} x_{i}-\min _{i} x_{i} .
$$

For any non-negative row stochastic matrix $P=\left(p_{i j}\right)$ and any $i, j$

$$
\begin{aligned}
(P \mathbf{x})_{i} & \leq p_{i j} x_{j}+\left(1-p_{i j}\right) \max _{k} x_{k} \\
& =\max _{k} x_{k}-p_{i j}\left(\max _{k} x_{k}-x_{j}\right) \leq \max _{k} x_{k}-\left(\min _{i} p_{i j}\right)\left(\max _{k} x_{k}-x_{j}\right)
\end{aligned} .
$$

Similarly

$$
(P \mathbf{x})_{i} \geq \min _{k} x_{k}-\left(\min _{i} p_{i j}\right)\left(\min _{k} x_{k}-x_{j}\right) .
$$

In particular

$$
\nu(P) \leq 1-\epsilon
$$

where $\epsilon=\max _{j} \min _{i} p_{i j}=\min _{k} c_{k}$. Applying this to $P_{0}, P_{1}$ we get

$$
\nu\left(P_{0}\right), \nu\left(P_{1}\right) \leq 1-\epsilon .
$$

## CASCADE ALGORITHM

In [7] Daubechies introduced the cascade algorithm for finding solutions to (1). One iterates

$$
f_{n}=T f_{n-1} .
$$

She gives conditions on the trigonometric polynomial $Q(\xi)=\frac{1}{2} \Sigma c_{k} e^{i k \xi}$ which ensure convergence of the cascade algorithm (for $f_{0}=\chi_{[-1 / 2,1 / 2)}$ ) to a solution of (1). The matrix approach raises several questions concerning the convergence of the algorithm. The results below partially analyze the situation.

Theorem VII. Let $f_{0}$ be bounded measurable, $\operatorname{supp}\left(f_{0}\right) \subseteq[0, N]$. Suppose $\left\{P_{0}, P_{1}\right\}$ is $L C P$. Then $T^{n} f_{0} \rightarrow f$ in $L_{1}(\mathbb{R})$ if and only if $S f_{0} \equiv \int f_{0}$ a.e.

Proof. The necessity follows from the fact that $\tau$ is measure-preserving. If $T^{n} f_{0} \rightarrow f$ in $L_{1}(\mathbb{R})$, then it follows form (19) and (24) that $S f_{0} \circ \tau^{n}-\int f \rightarrow 0$ in measure, and so we must have $S f_{0} \equiv \int f$ a.e. because $\tau$ is measure preserving. To prove sufficiency assume that $S f_{0} \equiv 1$ a.e., where we have normalized $\int f_{0}=1$ without any loss of generality. It follows from (21) that

$$
\mathbf{V}^{t} f_{n}(x)=\mathbf{V}^{t} f_{0}\left(\tau^{n} x\right) P_{\omega_{n}} \cdots P_{\omega_{1}}
$$

where $\omega_{1}, \ldots, \omega_{n}$ are the first $n$ bits in the dyadic code $D(x)$ of $x$. Thus

$$
\lim _{n \rightarrow \infty} \mathbf{V}^{t} f_{n}(x)=\lim _{n \rightarrow \infty} \mathbf{V}^{t} f_{0}\left(\tau^{n} x\right) P_{*}(x)=\mathbf{V}^{t} f(x) \quad \text { a.e. }
$$

where we have used Theorem $\operatorname{IV}(\mathrm{a})$ in this last step, together with the facts that (i) the rows of $P_{*}(x)$ are identical a.e.; (ii) the components of $\mathbf{V}^{t} f_{0}\left(\tau^{n} x\right)$ all sum to $S f_{0} \equiv 1$ a.e. This shows that $f_{n} \rightarrow f$ a.e. By Lemma $\mathrm{I}(\mathrm{c})$ we know that the $f_{n}$ 's are uniformly bounded. Thus $f_{n} \rightarrow f$ in $L_{1}(\mathbb{R})$.

Theorem VIII. If the cascade algorithm $f_{n}=T^{n} f_{0}, f_{0}=\chi_{[0,1)}$ converges uniformly to a continuous function $f$, then $\hat{\rho}\left(A_{0}, A_{1}\right)<1$.

Proof. As in the proof of Theorem VII

$$
(1,0, \ldots, 0) P_{\omega_{n}} \cdots P_{\omega_{1}}=\mathbf{V} f_{n}(x) \rightarrow \mathbf{V} f(x)
$$

Furthermore by (16)

$$
T^{n} \sigma_{h} f_{0}=\sigma_{h / 2^{n}} f_{n}
$$

and thus, since $f$ is continuous, $T^{n} \sigma_{h} f_{0} \rightarrow f$ uniformly, too. By choosing $h=1, \ldots, N-1$ we find that

$$
\underbrace{(0, \ldots, 0,1,0, \ldots, 0)}_{\text {position } h+1} P_{\omega_{n}} \cdots P_{\omega_{1}}=\mathbf{V}^{t} \sigma_{h / 2^{n}} f_{n}(x) \rightarrow \mathbf{V}^{t} f(x) .
$$

Thus

$$
P_{\omega_{n}} \cdots P_{\omega_{1}} \rightarrow P_{*}(x)=\left(\begin{array}{c}
\mathbf{V}^{t} f(x) \\
\cdots \cdots \cdots \\
\mathbf{V}^{t} f(x)
\end{array}\right)
$$

On account of our terminating expansion convention about $D(x)$ this proves that all infinite products of $P_{0}, P_{1}$ converge - except for those with finitely many zeros. On the other hand - look carefully at (11). Since we know that $P_{0}^{\infty}=P_{*}(0)$ exists and is rank one, it follows that $M^{\infty}$ also exists and is rank one. Since $f$ is continuous it follows from Lemma $\operatorname{III}(\mathrm{a})$ that $\left|c_{N}\right|<1$. From this we get that

$$
P_{1}^{n}=\left(\begin{array}{cc}
M^{n} & 0 \\
\sum_{k=0}^{n-1} c_{N}^{k} \mathbf{w}_{1}^{t} M^{n-k-1} & c_{N}^{n}
\end{array}\right)
$$

also converges. Moreover since we know that 1 is a simple eigenvalue for $P_{0}$, it follows from (12) that it must also be simple for $P_{1}$. So $P_{1}^{\infty}$ exists and is rank one. From this it follows that $A_{\omega_{n}} \cdots A_{\omega_{1}} \rightarrow 0$ for all codes $\underline{\omega}=\left(\omega_{1}, \omega_{2}, \ldots\right) \in \Omega$. By Lemma $\mathrm{I}(\mathrm{d})$, then, $\hat{\rho}\left(A_{0}, A_{1}\right)<1$.

It can happen that the two-scale equation (1) has a (non-trivial) continuous solution, without $\left\{P_{0}, P_{1}\right\}$ being LCP, and without the cascade algorithm converging at all (say, with $f_{0}=\chi_{[0,1)}$ ). To see this let the coefficients $c_{0}, \ldots, c_{N}$ satisfy $\Sigma_{n} c_{2 n}=\Sigma_{n} c_{2 n+1}=1$ and consider the expanded operator

$$
T^{\prime} g(x)=\sum_{k} c_{k} g(2 x-\ell k)
$$

where $\ell$ is an odd number. This operator is of the form (14), with coefficients $c_{0}^{\prime}, \ldots, c_{\ell N}^{\prime}$ given by

$$
c_{m}^{\prime}= \begin{cases}c_{k}, & \text { if } m=\ell k \\ 0, & \text { otherwise }\end{cases}
$$

The coefficients $c_{m}^{\prime}$ satisfy $\Sigma_{n} c_{2 n}=\Sigma_{n} c_{2 n+1}=1$, since $\ell$ is odd. It is easy to relate fixed points of $T$ and $T^{\prime}$. Observe that if $f(x)=g(\ell x)$ then $T f(x)=T^{\prime} g(\ell x)$. Thus if $f$ is a fixed point for $T$ then $g(x)=f(x / \ell)$ is a fixed point for $T^{\prime}$. In particular, we can get $g$ to be continuous, by choosing the coefficients $c_{k}$ so as to get $f$ continuous. However, the cascade algorithm

$$
g_{n}=T^{\prime} g_{n-1}, \quad g_{0}=\chi_{[0,1)}
$$

cannot converge to $g$ in $L_{1}(\mathbb{R})$. To see this, set

$$
f_{n}=T f_{n-1}, \quad f_{0}=\chi_{[0,1 / \ell)} .
$$

Since $f_{0}(x)=g_{0}(\ell x)$ it follows inductively that $f_{n}(x)=g_{n}(\ell x), \forall n$. If it were true that $g_{n} \rightarrow g$ in $L_{1}(\mathbb{R})$ then $f_{n} \rightarrow f$ in $L_{1}(\mathbb{R})$, too. But $S f_{0}$ is not constant a.e., and so this contradicts Theorem VII.

In [7, Prop. 3.3] Daubechies gave conditions on the trigonometric polynomial $Q(\xi)$ which suffice to ensure uniform convergence of the cascade algorithm when $f_{0}$ is the "tent function"

$$
f_{0}(x)=\left\{\begin{array}{l}
1+x, \quad-1 \leq x<0  \tag{26}\\
1-x, \quad 0 \leq x<1
\end{array}\right.
$$

Precisely, she proved that $\hat{f}_{n} \rightarrow \hat{f}$ in $L_{1}(\mathbb{R})$, and she needed to use this tent function $f_{0}$ so that $\hat{f}_{0}$ would be integrable. We can modify Theorem VI to handle functions $f_{0}$ other than $\chi_{[0,1)}$.

Lemma IX. Let $f_{0}$ be a bounded measurable function, $\operatorname{supp}\left(f_{0}\right) \subseteq[0, N]$, and suppose the cascade algorithm $f_{n}=T f_{n-1}$ converges uniformly to a continuous function $f$. Then for any $y \in[0,1)$

$$
\mathbf{V}^{t} f(x)=\lim _{n \rightarrow \infty} \mathbf{V}^{t} f_{0}(y) P_{\omega_{n}} \cdots P_{\omega_{1}}
$$

where $\omega_{1}, \ldots, \omega_{n}$ are the first $n$ bits in the dyadic code $D(x)$ of $x$.

Proof. Let $x_{n}$ be the number with dyadic expansion $x_{n}=. \omega_{1} \cdots \omega_{n} D(y)$. Then by (21)

$$
\mathbf{V}^{t} f_{n}\left(x_{n}\right)=\mathbf{V}^{t} f_{0}(y) P_{\omega_{n}} \cdots P_{\omega_{1}}
$$

Since $x_{n} \rightarrow x$ and since $f_{n} \rightarrow f$ uniformly with $f$ continuous

$$
\lim _{n \rightarrow \infty} \mathbf{V} f_{n}\left(x_{n}\right)=\mathbf{V} f(x)
$$

Theorem X. Let $\mathcal{F}$ be a family of bounded measurable functions $f_{0}, \operatorname{supp}\left(f_{0}\right) \subseteq[0, N]$. Suppose the cascade algorithm $f_{n}=T f_{n-1}$ converges uniformly to a non-trivial continuous function for every $f_{0} \in \mathcal{F}$. If

$$
\operatorname{span}\left\{\mathbf{V} f_{0}(y): y \in[0,1), f_{0} \in \mathcal{F}\right\}=\mathbb{R}^{N}
$$

then $\hat{\rho}\left(A_{0}, A_{1}\right)<1$ where $A_{0}, A_{1}$ are as in (6).

Proof. It follows from Lemma IX that

$$
\lim _{n \rightarrow \infty} P_{\omega_{n}} \cdots P_{\omega_{1}}=\left[\begin{array}{c}
\mathbf{V}^{t} f(x) \\
\cdots \cdots \cdots \\
\mathbf{V}^{t} f(x)
\end{array}\right]
$$

for all codes $\underline{\omega}=\left(\omega_{1}, \omega_{2}, \ldots\right)$ with infinitely many zeros, where $f$ is a continuous solution of (1), $\operatorname{supp}(f) \subseteq[0, N], \int f=1$. Thus we can apply the argument used in the proof of Theorem VIII.

Suppose the two-scale equation (1) has a continuous solution $f, \operatorname{supp}(f) \subseteq[0, N]$ and let $\mathcal{W}$ be the subspace

$$
\mathcal{W}=\operatorname{span}\{\mathbf{V} f(x): x \in[0,1)\}
$$

It follows from (25) that $\mathcal{W}$ is invariant under $P_{0}^{t}, P_{1}^{t}$. If $\mathcal{W}=\mathbb{R}^{N}$ then by taking $f_{0}=f$ we can conclude from Theorem X that $\hat{\rho}\left(A_{0}, A_{1}\right)<1$. In particular this holds whenever $P_{0}^{t}, P_{1}^{t}$ have no common proper invariant subspace. This was also established in [6, §2].

In general, since $\mathcal{W}$ is invariant, we can reduce $P_{0}, P_{1}$ to the form

$$
P_{\omega} \sim\left(\begin{array}{cc}
\bar{P}_{\omega} & 0 \\
B_{\omega} & C_{\omega}
\end{array}\right), \quad \omega=0,1
$$

where $\bar{P}^{t}=\left.P^{t}\right|_{\mathcal{W}}$. By working in $\mathcal{W}$ we can apply Theorem X to conclude that

$$
\bar{P}_{\omega} \sim\left(\begin{array}{cc}
1 & \overline{\mathbf{b}}_{\omega}^{t} \\
0 & \bar{A}_{\omega}
\end{array}\right), \quad \omega=0,1
$$

where $\hat{\rho}\left(\bar{A}_{0}, \bar{A}_{1}\right)<1$. By considering the Hölder modulus of continuity for the solution $f$ of (1), it can be shown as in [9, Remark 3 following Thm. 2.3] that $\hat{\rho}\left(\bar{A}_{0}, \bar{A}_{1}\right) \geq 1 / 2$.

## APPENDIX: Augmented Matrices

In Theorem VII the convergence (in the $L^{1}(\mathbb{R})$ and uniform topologies) of the iterates $T^{n} f_{0}$ was established for functions $f_{0}$ supported in $[0, N]$ and satisfying $S f_{0} \equiv$ const. If $f_{0}$ is supported in some other interval of length $N$, then the shift $\sigma_{h} f_{0}$ will be supported in $[0, N]$ for some $h$. Since, by (16),

$$
T^{n} f_{0}=\sigma_{-h / 2^{n}} T^{n} \sigma_{h} f_{0}
$$

and since the group of shift operators $\left\{\sigma_{h}: h \in \mathbb{R}\right\}$ is strongly continuous (in the $L^{1}(\mathbb{R})$ and uniform topologies), the convergence still holds.

What if $f_{0}$ is supported in an interval of length more than $N$ ? One thing we can do is enlarge $N$, by adding zero coefficients $c_{N+1}, c_{N+2}, \ldots$ This does not impact the operator $T$, but it does raise the question of whether or not the hypotheses on $P_{0}, P_{1}$ in Theorem VII remain valid for the augmented matrices $\bar{P}_{0}, \bar{P}_{1}$. We will show below that in fact the LCP property and the $\hat{\rho}\left(A_{0}, A_{1}\right)<1$ property carry over to $\bar{P}_{0}, \bar{P}_{1}$. To establish this it suffices to analyze what happens when $N$ is increased by one.

Theorem XI. Let $c_{k}=0$ for $k<0$ or $k>N$, as always. Define analogously two other $(N+1) \times$ ( $N+1$ ) row stochastic matrices

$$
\bar{P}_{0}=\left(c_{2 j-i-1}\right)_{i, j=1}^{N+1}, \quad \bar{P}_{1}=\left(c_{2 j-i}\right)_{i, j=1}^{N+1} .
$$

(a) $\left\{P_{0}, P_{1}\right\}$ is $L C P$ if and only if $\left\{\bar{P}_{0}, \bar{P}_{1}\right\}$ is $L C P$.
(b) $\hat{\rho}\left(A_{0}, A_{1}\right)<1$ if and only if $\hat{\rho}\left(\bar{A}_{0}, \bar{A}_{1}\right)<1$, where $A_{0}, A_{1}$ are as in (6) and $\bar{A}_{0}, \bar{A}_{1}$ are defined analogously for $\bar{P}_{0}, \bar{P}_{1}$.

Proof: (a) Observe first that

$$
\bar{P}_{0}=\left(\begin{array}{cc}
P_{0} & \mathbf{0}  \tag{27}\\
* & c_{N}
\end{array}\right), \quad \bar{P}_{1}=\left(\begin{array}{cc}
P_{1} & \mathbf{0} \\
* & 0
\end{array}\right) .
$$

From this it is clear that if $\left\{\bar{P}_{0}, \bar{P}_{1}\right\}$ is LCP, then so is $\left\{P_{0}, P_{1}\right\}$. Conversely, suppose $\left\{P_{0}, P_{1}\right\}$ is LCP. We proceed in four stages.
$1^{0} \bar{P}_{0}^{\infty}$ and $\bar{P}_{1}^{\infty}$ both exist.
Using (27) we get that $\bar{P}_{1}^{\infty}$ exists, and that $\bar{P}_{0}^{\infty}$ also exists if $\left|c_{N}\right|<1$. If $\left|c_{N}\right| \geq 1$ then it must be that $c_{N}=1$, since $c_{N} \in \sigma\left(P_{1}\right)$ by (12) and $P_{1}^{\infty}$ exists. For any matrix $P$ all of whose eigenvalues $\lambda$ satisfy $|\lambda|<1$ or $\lambda=1, P^{\infty}$ exists if and only if the powers $\left\{P^{n}\right\}$ are bounded. (Just consider the Jordan canonical form.) So if $\bar{P}_{0}^{\infty}$ did not exist then the powers $\left\{\bar{P}_{0}^{n}\right\}$ would be unbounded specifically

$$
\bar{P}_{0}^{n}=\left(\begin{array}{cc}
P_{0}^{n} & 0 \\
\mathbf{b}_{n}^{t} & 1
\end{array}\right)
$$

where the vectors $\left\{\mathbf{b}_{n}\right\}$ in the last row would have to be unbounded. But since $\bar{P}_{0}$ can also be identified as

$$
\bar{P}_{0}=\left(\begin{array}{cc}
c_{0} & * \\
0 & P_{1}
\end{array}\right) .
$$

the last row of $\bar{P}_{0}^{n}$ must in fact be bounded.
$2^{0} S\left(\bar{P}_{0}, \bar{P}_{1}\right)$ is bounded.
Let $\Sigma=\left\{\bar{P}_{0}^{n} \bar{P}_{1}: n=0,1, \ldots\right\}$. Since

$$
\bar{P}_{0}^{n} \bar{P}_{1}=\left(\begin{array}{cc}
P_{0}^{n} P_{1} & \mathbf{0} \\
\mathbf{c}_{n}^{t} & 0
\end{array}\right)
$$

where the vectors $\left\{\mathbf{c}_{n}\right\}$ are bounded, it can be seen that $\mathcal{S}(\Sigma)$ is bounded. Every matrix in $\mathcal{S}\left(\bar{P}_{0}, \bar{P}_{1}\right)$ can be written $\bar{Q} \bar{P}_{0}^{m}$ with $Q \in \mathcal{S}(\Sigma)$. Since the powers of $\bar{P}_{0}$ are bounded, we get that $\mathcal{S}\left(\bar{P}_{0}, \bar{P}_{1}\right)$ is bounded.
$3^{0}$ Every infinite product of $\bar{A}_{\omega}$ 's which contains $\bar{A}_{0}$ and $\bar{A}_{1}$ infinitely often converges to zero.
Using (27) it follows that

$$
\bar{A}_{0}=\left(\begin{array}{cc}
A_{0} & 0  \tag{28}\\
* & c_{N}
\end{array}\right), \quad \bar{A}_{1}=\left(\begin{array}{cc}
A_{1} & 0 \\
* & 0
\end{array}\right) .
$$

By Step 2 above, $\mathcal{S}\left(\bar{A}_{0}, \bar{A}_{1}\right)$ is bounded, and so it suffices to show that every limit point

$$
\bar{A}=\left(\begin{array}{cc}
0 & 0 \\
\mathbf{b}^{t} & 0
\end{array}\right)
$$

of $\left(\bar{A}_{\omega_{n}} \cdots \bar{A}_{\omega_{1}}\right)_{n=1}^{\infty}$ is zero. Let's say $\bar{A}=\lim _{k \rightarrow \infty} \bar{B}_{k} \cdots \bar{B}_{1}$ where $\bar{B}_{k}$ is the block $\bar{B}_{k}=$ $\bar{A}_{\omega_{n_{k}}} \cdots \bar{A}_{\omega_{n_{k-1}+1}}$. Since $\omega_{n}=1$ infinitely often we can always assume WLOG that $\bar{A}_{1}$ occurs
in each block, so that $\bar{B}_{k}$ has the form

$$
\bar{B}_{k}=\left(\begin{array}{ll}
* & 0 \\
* & 0
\end{array}\right)
$$

According to Lemma I (b) $\bar{B} \bar{A}=\bar{A}$ where $\bar{B}$ is a limit point of $\left(\bar{B}_{k}\right)$. Since $\bar{B}$ has the same form as each $\bar{B}_{k}$ we get

$$
\left(\begin{array}{cc}
* & 0 \\
* & 0
\end{array}\right)\left(\begin{array}{cc}
0 & 0 \\
\mathbf{b}^{t} & 0
\end{array}\right)=\left(\begin{array}{cc}
0 & 0 \\
\mathbf{b}^{t} & 0
\end{array}\right)
$$

giving $\mathbf{b}=\mathbf{0}$, as desired.
$4^{0}\left\{\bar{P}_{0}, \bar{P}_{1}\right\}$ is $L C P$.
We already showed in Step 1 above that $\bar{P}_{0}^{\infty}$ and $\bar{P}_{1}^{\infty}$ both exist. Since $\mathcal{S}\left(\bar{P}_{0}, \bar{P}_{1}\right)$ is bounded, it suffices to show that all limit points

$$
\bar{P}=\left(\begin{array}{cc}
1 & \mathbf{b}^{t} \\
0 & 0
\end{array}\right)
$$

of $\left(\bar{P}_{\omega_{n}} \cdots \bar{P}_{\omega_{1}}\right)_{n=1}^{\infty}$ are the same, where $\left(\omega_{n}\right)$ is a sequence containing infinitely many zeros and ones. Suppose, then, that there are two limit points, say with vectors $\mathbf{b}^{\prime}$ and $\mathbf{b}^{\prime \prime}$. As in Step 3 above, write

$$
\begin{aligned}
\bar{P}^{\prime}=\lim \bar{B}_{k}^{\prime} \cdots \bar{B}_{1}^{\prime}, & \bar{B}_{k}^{\prime}=\bar{P}_{\omega_{n_{k}^{\prime}}} \cdots \bar{P}_{\omega_{n_{k-1}^{\prime}}+1} \\
\bar{P}^{\prime \prime}=\lim \bar{B}_{k}^{\prime \prime} \cdots \bar{B}_{1}^{\prime \prime}, & \bar{B}_{k}^{\prime \prime}=\bar{P}_{\omega_{n_{k}^{\prime \prime}}} \cdots \bar{P}_{\omega_{n_{k-1}^{\prime \prime}}+1}
\end{aligned} .
$$

We can assume WLOG that $n_{k}^{\prime \prime} \geq n_{k}^{\prime}, \forall k$, so that

$$
\bar{B}_{k}^{\prime \prime} \cdots \bar{B}_{1}^{\prime \prime}=\bar{Q}_{k} \bar{B}_{k}^{\prime} \cdots \bar{B}_{1}^{\prime}
$$

where $\bar{Q}_{k}=\bar{P}_{\omega_{n_{k}^{\prime \prime}}} \cdots \bar{P}_{\omega_{n_{k}^{\prime}+1}}$ has the form

$$
\bar{Q}_{k}=\left(\begin{array}{ll}
1 & * \\
0 & *
\end{array}\right) .
$$

Then if $\bar{Q}$ is a limit point of $\left(\bar{Q}_{k}\right)$ we get

$$
\left(\begin{array}{cc}
1 & \left(\mathbf{b}^{\prime \prime}\right)^{t} \\
0 & 0
\end{array}\right)=\left(\begin{array}{cc}
1 & * \\
0 & *
\end{array}\right)\left(\begin{array}{cc}
1 & \left(\mathbf{b}^{\prime}\right)^{t} \\
0 & 0
\end{array}\right)
$$

giving $\mathbf{b}^{\prime}=\mathbf{b}^{\prime \prime}$, as desired.
(b) As in [4], if $\Sigma \subseteq \mathcal{M}$ is a non-empty bounded set of matrices, define

$$
\rho(\Sigma)=\sup \{\rho(M): M \in \Sigma\}, \quad \rho_{*}(\Sigma)=\limsup _{n \rightarrow \infty}\left[\rho\left(\Sigma^{n}\right)\right]^{1 / n}
$$

where $\rho(M)$ denotes the spectral radius of $M$. The main result of [4] asserts that $\rho_{*}(\Sigma)=\hat{\rho}(\Sigma)$. Thus it suffices to prove Theorem XI(b) for $\rho_{*}$ instead of $\hat{\rho}$.

Using (28) we derive the relationship

$$
\begin{equation*}
\rho_{*}\left(\bar{A}_{0}, \bar{A}_{1}\right)=\max \left(\rho_{*}\left(A_{0}, A_{1}\right),\left|c_{N}\right|\right) \tag{29}
\end{equation*}
$$

Since $\boldsymbol{c}_{N} \in \sigma\left(A_{1}\right)$ it follows that $\left|c_{N}\right|<1$ whenever $\rho_{*}\left(A_{0}, A_{1}\right)<1$. Thus it becomes clear from (29) that $\rho_{*}\left(A_{0}, A_{1}\right)<1$ if and only if $\rho_{*}\left(\bar{A}_{0}, \bar{A}_{1}\right)<1$.

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