Two–Scale Dilation Equations and the Mean Spectral Radius

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Abstract

We study the two–scale dilation equation

$$f(x) = \sum_{n=0}^{N} f(2x-n).$$

By expanding the matrix product expansion technique introduced by Daubechies and Lagarias, we give a necessary and sufficient condition for the existence of continuous scaling functions and establish an exact formulae for their Hölder exponents. We also introduce the *mean spectral radius* of a set of matrices. Applying the mean spectral radius we prove a sufficient condition for the existence of L^1 solutions to the dilation equation. We conjecture that such a condition is also necessary. Finally, we establish a surprising formulae for the exact fractal dimension of the graph of a continuous scaling function by using the mean spectral radius.

1 Introduction and Notation

In this paper, we study the two-scale dilation equation

$$f(x) = \sum_{n=0}^{N} c_n f(2x-n) \tag{1}$$

where the coefficients c_n are real. We always assume $c_0 \neq 0$ and $c_N \neq 0$.

Two-scale dilation equations arise in many applications. They play crucial roles in *subdivision schemes*, which are algorithms for curve and surface generation ([21], [22], [4], [14], [15] etc.), and in the construction of compactly supported orthonormal wavelet functions ([8]).

Due to those important applications the two scale–dilation equation (1) has been studied extensively in recent years. Daubechies ([8]) applied Fourier analysis to estimate the smoothness of compactly supported orthonormal wavelets. In a later paper Daubechies and Lagarias ([11]) improved the estimates in [8] using a different approach. They noticed that better estimates can be obtained from the matrix product expansion (2) by applying their results ([9]) on infinite products of matrices. The same idea was also exploited by

Caravetta, Dahmen, and Micchelli ([4]) to study the convergence of subdivision schemes. It is mainly this matrix product expansion approach we shall adopt in this paper.

Throughout this paper, we use P_0, P_1 to denote the two N by N matrices

$$P_0 = \begin{bmatrix} c_0 & 0 & 0 & \cdots & 0 & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_N & c_{N-1} \end{bmatrix}, \quad P_1 = \begin{bmatrix} c_1 & c_0 & 0 & \cdots & 0 & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & c_N \end{bmatrix},$$

or $P_0 = (c_{2i-j-1})$ and $P_1 = (c_{2i-j})$. For any $f(x)$ with $\mathrm{supp}(f) \subseteq [0, N]$, we denote

$$\mathbf{v}_f(x) = [f(x), f(x+1), \ldots, f(x+N-1)]^t.$$

By a solution to the dilation equation (1) we always refer to a **nontrivial compactly supported** solution. When a solution $f(x)$ is integrable, it is necessary that $f(x)$ is compactly supported with $\mathrm{supp}(f) \subseteq [0, N]$ ([10]). Sometimes it is convenient to call a solution $f(x)$ to (1) a *scaling function* and $\mathbf{v}_f(x)$ a *scaling vector* when $f(x) \in L^2(\mathbf{R})$ because it is a scaling function of a multiresolution analysis and vice versa. (See [8] for more on multiresolution analysis.)

For any $x \in [0, 1]$, let $x = 0.d_1 d_2 d_3 \cdots$ be a *dyadic expansion* of x (so $d_i \in \{0, 1\}$). Define

$$\tau x = \begin{cases} 2x & x \in [0, \frac{1}{2}], \\ 2x - 1 & x \in (\frac{1}{2}, 1]. \end{cases}$$

Then τ becomes the left shift operator on dyadic expansions of $x \in [0, 1]$, i.e., $\tau x = 0.d_2 d_3 d_4 \cdots$. We have

Proposition 1.1 *Let $f(x)$ be a solution to (1). Then for all $n \geq 1$ and $x \in [0, 1]$,*

$$\mathbf{v}_f(x) = P_{d_1} P_{d_2} \cdots P_{d_n} \mathbf{v}_f(\tau^n(x)) \tag{2}$$

where $x = 0.d_1 d_2 d_3 \cdots$, $d_i \in \{0, 1\}$.

Since dyadic expansions routinely occur in this paper, it is convenient to introduce some notation. For any $x \in [0, 1]$, if x is *dyadic*, i.e. $x = k/2^m$ for some $k, m \in \mathbf{Z}$, then x has two dyadic expansions except for $x = 0$ and $x = 1$,

$$\begin{aligned} x &= 0.d_1 \cdots d_m 1000 \cdots \quad \text{(upper expansion)} \\ &= 0.d_1 \cdots d_m 0111 \cdots \quad \text{(lower expansion).} \end{aligned}$$

for some $d_i \in \{0, 1\}$. When x is non–dyadic, the dyadic expansion of x is unique. In this case the upper and lower dyadic expansion of x is simply the same unique dyadic expansion of x. We use $d_i(x)$ to denote the i–th digit in the *upper dyadic expansion* of x.

Definition 1.2 *For any $x, y \in [0,1]$, $x \neq y$ we denote by $\sigma(x,y)$ the largest integer k with the following property: there exists a dyadic expansion of x and a dyadic expansion of y such that the first k digits of the two expansions coincide. We denote by $d_i(x,y)$, $1 \leq i \leq \sigma(x,y)$, the i–th common digit in the two expansions.*

Lemma 1.3 *If $\sigma(x,y) \geq m$ then $|x-y| \leq 2^{-m}$. Conversely, if $|x-y| \leq 2^{-m}$ and either x or y is dyadic, then $\sigma(x,y) \geq m$.*

Proof: The first statement is obvious. To prove the second statement, assume that $x \in (0,1)$ is dyadic. (In the cases of $x = 0$ or $x = 1$, the lemma can be checked easily.) So x has two dyadic expansions

$$x = 0.d_1 \cdots d_k 1000 \cdots = 0.d_1 \cdots d_k 0111 \cdots .$$

Let $y = e_1 e_2 e_3 \cdots$ be a dyadic expansion of y. If $k > m$ then it is obvious that $e_i = d_i$ for $1 \leq i \leq m$. Suppose $k \leq m$. Let $a = 2^m x$ and $b = 2^m y$. We have $a \in \mathbf{Z}$ and $b = b' + r$ where $b' \in \mathbf{Z}$ and $0 \leq r < 1$. Since $|a - b| \leq 2^m |x - y| \leq 1$, we have $b' = a$ or $b' = a - 1$. If $b' = a$ then the upper expansions of x and y will have the same first m digits. If $b' = a - 1$ then the lower expansion of x and the upper expansion of y have the same first m digits. ■

Definition 1.4 *Let $\Sigma \subset \mathbf{M}_N(\mathbf{R})$ ($\mathbf{M}_N(\mathbf{R})$ is the set of all N by N real matrices). Σ is called* **RCP** *(***right convergent products***) if $\lim_{n\to\infty} A_1 \cdots A_n$ exists for every sequence $\{A_i\}$ in Σ. Σ is called* **product bounded** *if the semigroup generated by Σ is bounded.*

For a single matrix $A \in \mathbf{M}_N(\mathbf{R})$ the spectral radius of A is well–known, namely $\rho(A) = \{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$. It is easy to show that for any matrix norm $\|.\|$ on $\mathbf{M}_N(\mathbf{R})$

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}. \tag{3}$$

We extend the definition of spectral radius to a set of matrices.

Definition 1.5 *Let Σ be a bounded subset of $\mathbf{M}_N(\mathbf{R})$. The* **generalized spectral radius** *of Σ is*

$$\rho(\Sigma) = \limsup_{n\to\infty} \Big(\sup_{A_1,\ldots,A_n \in \Sigma} \rho(A_1 \cdots A_n) \Big)^{\frac{1}{n}}. \tag{4}$$

The following results appeared in [2], [9].

Proposition 1.6 *Let $\Sigma \subset \mathbf{M}_N(\mathbf{R})$ be bounded.*

1. *Let $\|.\|$ be any matrix norm on $\mathbf{M}_N(\mathbf{R})$. Then*

$$\rho(\Sigma) = \limsup_{n\to\infty} \sup_{A_1,\ldots,A_n \in \Sigma} \|A_1 \cdots A_n\|^{\frac{1}{n}}. \tag{5}$$

2. *For any $r > \rho(\Sigma)$, there exists a matrix norm $\|.\|$ on $\mathbf{M}_N(\mathbf{R})$ such that $\|A\| < r$ for every $A \in \Sigma$.*

3. *If Σ is RCP, then Σ is product bounded.*

4. *Σ is product bounded if and only if there exists a norm $\|.\|$ on $\mathbf{M}_N(\mathbf{R})$ such that $\|A\| \leq 1$ for every $A \in \Sigma$.*

Note that (5) is the generalization of (4). The right hand side of (5) is called the *joint spectral radius* of Σ. Frequently it is more convenient to use (5) as the definition of $\rho(\Sigma)$.

Definition 1.7 *Suppose $\Sigma = \{A_1, \ldots, A_p\} \subset \mathbf{M}_N(\mathbf{R})$ where $A_i \neq A_j$ for $i \neq j$. Let $\|.\|$ be a matrix norm defined on $\mathbf{M}_N(\mathbf{R})$. The* **mean spectral radius** *is*

$$\bar{\rho}(\Sigma) = \limsup_{n\to\infty} \frac{1}{p}\Big(\sum_{i_1,\cdots,i_n} \|A_{i_1}\cdots A_{i_n}\|\Big)^{\frac{1}{n}} \tag{6}$$

where the summation is taken over all elements $(i_1, \cdots, i_n) \in \{1, \ldots, p\}^n$.

It is standard to show that the difinition of $\bar{\rho}(\Sigma)$ is independent of the choice of the norm $\|.\|$. We also have the following:

Proposition 1.8 *Suppose $\Sigma = \{A_1, \ldots, A_p\} \subset \mathbf{M}_N(\mathbf{R})$ where $A_i \neq A_j$ for $i \neq j$. Then*

1.
$$\bar{\rho}(\Sigma) = \lim_{n\to\infty} \frac{1}{p}\Big(\sum_{i_1,\cdots,i_n} \|A_{i_1}\cdots A_{i_n}\|\Big)^{\frac{1}{n}} \tag{7}$$

where the summation is taken over all elements $(i_1, \cdots, i_n) \in \{1, \ldots, p\}^n$. In other words, the lim sup *in (6) can be replaced by limit.*

2. *Let $\|.\|$ be any matrix norm defined on $\mathbf{M}_N(\mathbf{R})$. Then*

$$\bar{\rho}(\Sigma) \leq \frac{1}{p}\sum_{i=1}^{p} \|A_i\|.$$

Proof: 1. Let

$$a_n = \log\Big(\sum_{i_1,\cdots,i_n} \|A_{i_1}\cdots A_{i_n}\|\Big).$$

Then

$$\begin{aligned} a_{m+n} &= \log\Big(\sum_{i_1,\cdots,i_{m+n}} \|A_{i_1}\cdots A_{i_{m+n}}\|\Big) \\ &\leq \log\Big(\sum_{i_1,\cdots,i_{m+n}} \|A_{i_1}\cdots A_{i_m}\|\cdot\|A_{i_{n+1}}\cdots A_{i_{m+n}}\|\Big) \\ &= \log\Big(\sum_{i_1,\cdots,i_m} \|A_{i_1}\cdots A_{i_m}\|\Big)\Big(\sum_{j_1,\cdots,j_n} \|A_{j_1}\cdots A_{j_n}\|\Big) \\ &= a_m + a_n. \end{aligned}$$

Hence $\lim_{n\to\infty} a_n/n$ exists and (7) holds as required.

2. It follows easily from

$$\sum_{i_1,\cdots,i_n} \|A_{i_1}\cdots A_{i_n}\| \leq \sum_{i_1,\cdots,i_n} \|A_{i_1}\|\cdots\|A_{i_n}\| = \Big(\sum_{i=1}^{p}\|A_i\|\Big)^n.$$

∎

The rest of the paper is divided into three sections. In Section 2 we give a necessary and sufficient condition for the existence of continuous scaling functions. We also prove a formulae for the exact Hölder exponent of a continuous scaling function based on the generalized spectral radius of two matrices. Some of our results in this section have also been obtained independently by Colella and Heil ([7]). In Section 3, we focus on integrable solutions. We prove that a *bounded* solution exists if $\{P_0, P_1\}$ is product bounded, improving an earlier result in [27]. We also establish a strong link between the mean spectral radius and the existence of an integrable scaling function. Finally, in Section 4 we give an exact formulae for the box dimension (fractal dimension) of the graph of a continuous scaling function using the mean spectral radius.

2 Existence and Regularity of Continuous Solutions

Consider the the general dilation equation (1)

$$f(x) = \sum_{n=0}^{N} c_n f(2x-n).$$

Under the assumption

$$\sum_n c_{2n} = \sum_n c_{2n+1} = 1 \tag{8}$$

the two matrices P_0, P_1 are column stochastic. Since $[1, 1, \ldots, 1]$ is a common left 1-eigenvector of both P_0, P_1, we have

$$P_i \sim \begin{bmatrix} 1 & 0 \\ * & A_i \end{bmatrix}, \quad i = 0, 1$$

simultaneously. Daubechies and Lagarias ([11]) proved that a continuous scaling function exists if $\rho(A_0, A_1) < 1$. This condition is not, however, necessary and cannot be applied to the general dilation equation where condition (8) is not satisfied. Wang ([27]) proved the following result for the general case:

Proposition 2.1 *Suppose the dilation equation (1) has a continuous solution. Then*

1. $|c_0| < 1$, $|c_N| < 1$.

2. $\mathcal{W} = \operatorname{span}\{\mathbf{v}_f(x)\,|\,x \in [0,1]\} \subseteq \mathbf{R}^N$ *is the minimal invariant subspace of* $\{P_0, P_1\}$ *containing* $\mathbf{v}_f(0)$ *and* $\mathbf{v}_f(1)$*. Moreover,*

$$P_i\Big|_{\mathcal{W}} \sim \left[\begin{array}{cc} 1 & 0 \\ * & A_i \end{array}\right], \quad i = 0, 1$$

simultaneously with $\rho(A_0, A_1) < 1$.

Daubechies and Lagarias ([10]) examined the following *cascade algorithm* for generating scaling functions of (1):

$$f_{m+1} = \sum_{n=0}^{N} c_n f_m(2x - n), \quad f_0(x) = \chi_{[0,1)}(x).$$

Wang ([27]) proved that under the condition (8) $\mathcal{W} = \mathbf{R}^N$ is a necessary and sufficient condition for the cascade algorithm to converge uniformly to a continuous solution of (1). In general, equation (1) may have a continuous solution with $\mathcal{W} \neq \mathbf{R}^N$.

Example: The dilation eqation

$$f(x) = \frac{1}{2}f(2x) + f(2x-3) + \frac{1}{2}f(2x-6) \tag{9}$$

has a continuous solution

$$f(x) = \begin{cases} x, & x \in [0,3] \\ 6 - x, & x \in (3,6] \\ 0, & x \notin [0,6]. \end{cases}$$

Clearly $\dim \mathcal{W} = 2$ so $\mathcal{W} \neq \mathbf{R}^6$. Notice that (9) is obtained by "stretching" the equation

$$f(x) = \frac{1}{2}f(2x) + f(2x-1) + \frac{1}{2}f(2x-2)$$

by a factor of 3.

Let $\mathcal{W}$ be an invariant subspace of $\{P_0, P_1\}$, $\dim \mathcal{W} = k$, such that

$$P_i\Big|_{\mathcal{W}} \sim \left[\begin{array}{cc} 1 & 0 \\ * & A_i \end{array}\right], \quad i = 0, 1 \tag{10}$$

simultaneously with $\rho(A_0, A_1) < 1$. Then there exists a $Q \in \mathbf{M}_N(\mathbf{R})$ such that

$$QP_iQ^{-1} = \left[\begin{array}{cc} \tilde{P_i} & * \\ 0 & B_i \end{array}\right], \quad i = 0, 1 \tag{11}$$

where $Q\mathcal{W} = \{[\mathbf{u}^t, 0]^t : \ \mathbf{u} \in \mathbf{R}^k\}$ and

$$\tilde{P_i} = \left[\begin{array}{cc} 1 & 0 \\ * & A_i \end{array}\right], \quad i = 0, 1. \tag{12}$$

Lemma 2.2 *Let $\mathcal{W}$ be an invariant subspace of $\{P_0, P_1\}$ such that (10) is satisfied. Fix $\mathbf{v} \in \mathcal{W}$ and for any $x \in [0,1]$ define*

$$\mathbf{v}(x) = \Big(\prod_{i=1}^{\infty} P_{d_i(x)}\Big)\mathbf{v}$$

where $\prod_i A_{d_i}$ denotes to the right product $A_{d_1}A_{d_2}A_{d_3}\cdots$.

1. *Let Q, $\tilde{P}_1$, $\tilde{P}_1$ be as in (11) and (12). Then $Q\mathbf{v}(x) = [\mathbf{b}^t(x), 0]^t$ where $\mathbf{b}(x) \in \mathbf{R}^k$, $k = \dim \mathcal{W}$. Moreover, $\mathbf{b}(x)$ is bounded and*

$$\mathbf{b}(x) = \tilde{P}_{d_1(x)} \cdots \tilde{P}_{d_m(x)}\mathbf{b}(\tau^m x) \tag{13}$$

   for all $m \geq 0$.

2. *For any $x, y \in [0,1]$,*

$$\mathbf{b}(x) - \mathbf{b}(y) = \begin{bmatrix} 0 \\ \mathbf{c_1}(x,y) \end{bmatrix}, \quad \mathbf{c_1}(x,y) \in \mathbf{R}^{k-1}. \tag{14}$$

   If $\sigma(x,y) \geq m$ then

$$\mathbf{c_1}(x,y) = A_{d_1} \cdots A_{d_m}\mathbf{c_1}(\tau^m x, \tau^m y) \tag{15}$$

   where $d_i = d_i(x,y)$.

3. *Let $\|.\|$ be any matrix norm on $\mathbf{M}_{k-1}(\mathbf{R})$ and $r > \rho(A_0, A_1)$. Suppose $x, y \in [0,1]$ and $\sigma(x,y) \geq m$. Then*

$$|\mathbf{v}(x) - \mathbf{v}(y)| \leq C_2\|A_{d_1} \cdots A_{d_m}\| \leq C_3 r^m \tag{16}$$

   where $d_i = d_i(x,y)$ and C_2, C_3 are independent of x, y, m.

Proof: It is straightforward to check that

$$Q\mathbf{v}(x) = Q\Big(\prod_{i=1}^{\infty} P_{d_i(x)}\Big)Q^{-1}Q\mathbf{v} = \begin{bmatrix} \prod_{i=1}^{\infty} \tilde{P}_{d_i(x)}\tilde{\mathbf{v}} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}(x) \\ 0 \end{bmatrix}$$

where $Q\mathbf{v} = [\tilde{\mathbf{v}}^t, 0]^t$. Since $\{\tilde{P}_1, \tilde{P}_1\}$ is RCP, $\mathbf{b}(x)$ must exist and according to Proposition 1.6 it is bounded. Clearly, $\mathbf{b}(x) = \tilde{P}_{d_1(x)} \cdots \tilde{P}_{d_m(x)}\mathbf{b}(\tau^m x)$ for all $m \geq 0$.

We observe that for any $z \in [0,1]$,

$$\prod_{i=1}^{\infty} \tilde{P}_{d_i(z)} = \begin{bmatrix} 1 & 0 \\ \mathbf{c}(z) & 0 \end{bmatrix}$$

where $\mathbf{c}(z) \in \mathbf{R}^{k-1}$ is bounded. Hence for any $z, w \in [0,1]$,

$$\mathbf{b}(z) - \mathbf{b}(w) = \begin{bmatrix} 1 & 0 \\ \mathbf{c}(z) & 0 \end{bmatrix}\tilde{\mathbf{v}} - \begin{bmatrix} 1 & 0 \\ \mathbf{c}(w) & 0 \end{bmatrix}\tilde{\mathbf{v}} = \begin{bmatrix} 0 \\ \mathbf{c_1}(z,w) \end{bmatrix}$$

where $\mathbf{c_1}(z,w)$ is bounded. If $x, y \in [0,1]$ and $\sigma(x,y) \ge m$, then

$$\mathbf{b}(x) - \mathbf{b}(y) = \begin{bmatrix} 1 & 0 \\ * & \prod_{i=1}^m A_{d_i} \end{bmatrix} (\mathbf{b}(\tau^m x) - \mathbf{b}(\tau^m y)) = \begin{bmatrix} 0 \\ \prod_{i=1}^m A_{d_i} \mathbf{c_1}(\tau^m x, \tau^m y) \end{bmatrix}.$$

where $d_i = d_i(x,y)$. So (15) holds. Finally, since $\mathbf{c_1}$ is bounded,

$$|\mathbf{b}(x) - \mathbf{b}(y)| \le C_1 |\prod_{i=1}^m A_{d_i} \mathbf{c_1}(\tau^m x, \tau^m y)| \le C_2 \| \prod_{i=1}^m A_{d_i} \|.$$

Hence $|\mathbf{v}(x) - \mathbf{v}(y)| \le C_2 \|A_{d_1} \cdots A_{d_m}\|$, and $|\mathbf{v}(x) - \mathbf{v}(y)| \le C_3 r^m$ follows easily from Proposition 1.6. ∎

We shall prove the converse of Proposition 2.1. For any $\mathbf{u} = [u_1, \dots, u_{N-1}]^t \in \mathbf{R}^{N-1}$, we denote by $\mathcal{W}(\mathbf{u})$ the minimal invariant subspace of $\{P_0, P_1\}$ containing the vectors

$$\mathbf{v_0} = \begin{bmatrix} 0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}, \quad \mathbf{v_1} = \begin{bmatrix} u_1 \\ \vdots \\ u_{N-1} \\ 0 \end{bmatrix}. \tag{17}$$

Notice that if $\mathbf{u}$ is an 1-eigenvector of the $(N-1) \times (N-1)$ matrix $(c_{2i-j})_{1 \le i,j \le N-1}$, then $\mathbf{v_0}$ abd $\mathbf{v_1}$ are 1-eigenvectors of P_0 and P_1 respectively.

Theorem 2.3 *Suppose there exists a 1-eigenvector* $\mathbf{u} \in \mathbf{R}^{N-1}$ *of* $M = (c_{2i-j})_{1 \le i,j \le N-1}$ *such that*

$$P_i \Big|_{\mathcal{W}(\mathbf{u})} \sim \begin{bmatrix} 1 & 0 \\ * & A_i \end{bmatrix}, \quad i = 0, 1$$

simultaneously with $\rho(A_0, A_1) < 1$. *Then the dilation equation (1) has a continuous solution* $f(x)$. *Moreover,*

$$\mathcal{W}(\mathbf{u}) = \mathrm{span}\Big\{ \mathbf{v}_f(x) \;\Big|\; x \in [0,1] \Big\}.$$

Proof: Let

$$\mathbf{v}(x) = \Big(\prod_{i=1}^\infty P_{d_i(x)} \Big) \mathbf{w}$$

where $\mathbf{w} \in \mathcal{W}$. We first show that by choosing a suitable $\mathbf{w}$ we shall have $\mathbf{v}(x) \ne 0$.

Let Q, $\tilde{P}_1$, $\tilde{P}_1$ be as in (11) and (12) for $\mathcal{W} = \mathcal{W}(\mathbf{u})$. Then it follows from Lemma 2.2 that there exists a $\tilde{\mathbf{w}} \in \mathbf{R}^k$ such that

$$Q\mathbf{w} = \begin{bmatrix} \tilde{\mathbf{w}} \\ 0 \end{bmatrix}, \quad Q\Big(\prod_{i=1}^\infty P_{d_i} \Big) \mathbf{w} = \begin{bmatrix} (\prod_{i=1}^\infty \tilde{P}_{d_i}) \tilde{\mathbf{w}} \\ 0 \end{bmatrix}.$$

Let $\mathbf{e} = [1, 0, \dots, 0]^t \in \mathbf{R}^k$. Then $\mathbf{e}^t$ is a common left 1-eigenvector of both $\tilde{P}_1$, $\tilde{P}_1$. We choose $\mathbf{w}$ so that $\mathbf{e}^t \tilde{\mathbf{w}} \ne 0$. This is clearly possible because $\tilde{\mathbf{w}}$ can be any vector in $\mathbf{R}^k$.

Thus for any binary sequence $\{d_i\}$,

$$[\mathbf{e}^t, 0]Q\Big(\prod_{i=1}^{\infty} P_{d_i}\Big)\mathbf{w} = \mathbf{e}^t\Big(\prod_{i=1}^{\infty} \tilde{P}_{d_i}\Big)\tilde{\mathbf{w}} = \mathbf{e}^t\tilde{\mathbf{w}} \neq 0; \tag{18}$$

hence $(\prod_{i=1}^{\infty} P_{d_i})\mathbf{w} \neq 0$.

Next we prove $\mathbf{v}(x)$ is continuous on $[0,1]$. Let $1 > r > \rho(A_0, A_1)$. Then for $x, y \in [0,1]$ such that $\sigma(x,y) \geq m$ we have

$$|\mathbf{v}(x) - \mathbf{v}(y)| \leq C_3 r^m.$$

This immediately implies that $\mathbf{v}(x)$ is continuous at nondyadic points, where the dyadic expansions are unique. It remains to be shown that $\mathbf{v}(x)$ is also continuous at dyadic points.

The difficulty at dyadic points comes from the non-uniqueness of dyadic expansions at those points, namely,

$$0.d_1 \cdots d_m 1000 \cdots = 0.d_1 \cdots d_m 0111 \cdots.$$

To show that $\mathbf{v}(x)$ is also continuous at dyadic points we need to show

$$P_0 P_1^{\infty}\mathbf{w} = P_1 P_0^{\infty}\mathbf{w}. \tag{19}$$

Let $\mathbf{v_0}$ and $\mathbf{v_1}$ be as in (17). $\mathbf{v_0}$ and $\mathbf{v_1}$ are 1-eigenvectors of P_0, P_1 respectively. Because 1 is a simple eigenvalue for both $\{P_0|_{\mathcal{W}(\mathbf{u})}, P_1|_{\mathcal{W}(\mathbf{u})}\}$, we must have

$$P_0^{\infty}\mathbf{w} = \lambda\mathbf{v_0}, \qquad P_1^{\infty}\mathbf{w} = \mu\mathbf{v_1}. \tag{20}$$

We want to show that $\lambda = \mu \neq 0$. Notice that $P_1\mathbf{v_0} = P_0\mathbf{v_1} = \mathbf{v}^*$. So

$$P_1 P_0^{\infty}\mathbf{w} = \lambda\mathbf{v}^*, \qquad P_0 P_1^{\infty}\mathbf{w} = \mu\mathbf{v}^*.$$

From (18) we have

$$\begin{aligned} \mathbf{e}^t\tilde{\mathbf{w}} &= [\mathbf{e}^t, \mathbf{0}]QP_1P_0^{\infty}\mathbf{w} = \lambda\mathbf{e}^tQ\mathbf{v}^*, \\ \mathbf{e}^t\tilde{\mathbf{w}} &= [\mathbf{e}^t, \mathbf{0}]QP_0P_1^{\infty}\mathbf{w} = \mu\mathbf{e}^tQ\mathbf{v}^*; \end{aligned}$$

hence $\lambda = \mu \neq 0$ and so (19) holds.

The continuity of $\mathbf{v}(x)$ at dyadic points follows easily from (19). Let $x \in [0,1]$ be any dyadic point. Then (19) implies that by substituting the lower expansion of x for the upper expansion of x the vector $\mathbf{v}(x)$ remains the same. Since for any $m > 0$ and $y \in [0,1]$ such that $|x - y| \leq 2^{-m}$ we have $\sigma(x,y) \geq m$, it follows from (19) that $|\mathbf{v}(y) - \mathbf{v}(x)| < C_3 r^m$.

For any $x \in [0, N)$ define

$$f(x) = ([x]+1)\text{-th component of } \mathbf{v}(\{x\}).$$

(Recall that $[x]$ is the largest integer no greater than x and $\{x\} = x - [x]$.), and let $f(N) = 0$. $f(x)$ is a solution of (1), and it is continuous at all non-integer points $x \in [0, N]$. But from (20) we see that $\mathbf{v}_f(1) = \mu \mathbf{v_1} = [\mu \mathbf{u}, 0]^t$, which is $\mathbf{v}_f(0) = [0, \mu \mathbf{u}]^t$ $(\lambda = \mu)$ shifting up by one position. Hence, $f(x)$ must also be continuous at integer points. ∎

Corollary 2.4 *The dilation equation (1) has a C^m solution if and only if there exists an $1/2^m$-eigenvector $\mathbf{u} \in \mathbf{R}^{N-1}$ of $M = (c_{2i-j})_{1 \le i,j \le N-1}$ such that*

$$P_i \Big|_{\mathcal{W}(\mathbf{u})} \sim \begin{bmatrix} \frac{1}{2^m} & 0 \\ * & A_i \end{bmatrix}, \qquad i = 0, 1$$

simultaneously with $\rho(A_0, A_1) < 1/2^m$.

Proof: The dilation equation (1) has a C^m solution if and only if

$$g(x) = \sum_{n=0}^{N} 2^m c_n g(2x - n)$$

has a continuous solution ([9]). The corresponding matrices for the above dilation equation are $2^m P_0,\ 2^m P_1$. ∎

The following theorem determines the exact Hölder exponent of a continuous scaling function.

Theorem 2.5 *Suppose the dilation equation (1) has a continuous solution $f(x)$. Let $\mathcal{W}$, $\{A_0, A_1\}$ be as in Proposition 2.1. Denote $\rho = \rho(A_0, A_1)$ and $\alpha = -\log_2 \rho$. Then*

1. *$f(x)$ is $C^{\alpha-\varepsilon}$ but not $C^{\alpha+\varepsilon}$ for any $0 < \varepsilon \le \alpha$;*
2. *$f(x)$ is C^{α} if and only if $\{A_0/\rho, A_1/\rho\}$ is product bounded.*

Before proving Theorem 2.5, we shall need some preparations.

Lemma 2.6 *Let $\|.\|$ be any matrix norm defined on $\mathbf{M}_n(\mathbf{R})$ and $\mathcal{S} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a basis for $\mathbf{R}^n$. Then there exists a constant $c = c(\|.\|, \mathcal{S}, n) > 0$ such that for any $A \in \mathbf{M}_n(\mathbf{R})$,*

$$|A\mathbf{u}_k| \ge c\|A\|$$

for some $k = k(A)$.

Proof: Since all norms on $\mathbf{M}_n(\mathbf{R})$ are equivalent, without loss of generality we may assume that $\|A\| = \sup_{|\mathbf{x}|=1} |A\mathbf{x}|$. ($|.|$ is the standard Euclidean norm on $\mathbf{R}^n$.) Choose $b > 0$ sufficiently large so that

$$\Big\{\mathbf{x} \;\Big|\; |x| \le 1\Big\} \subseteq \Big\{\sum_{i=1}^{n} a_i \mathbf{u}_i \;\Big|\; |a_i| \le b\Big\}.$$

For any $A \in \mathbf{M}_n(\mathbf{R})$ there exists an $\mathbf{x}_0 \in \mathbf{R}^n$ such that $|\mathbf{x}_0| = 1$ and $|A\mathbf{x}_0| = \|A\|$. Let $\mathbf{x}_0 = \sum_{i=1}^{n} a_i \mathbf{u}_i$. So

$$\sum_{i=1}^{n} |a_i||A\mathbf{u}_i| \ge |A\mathbf{x}_0| = \|A\|.$$

Hence $|a_k||\mathbf{u}_k| \ge \|A\|/n$ for some $k = k(A)$. So $|A\mathbf{u}_k| \ge c\|A\|$ where $c = 1/bn$. ∎

Lemma 2.7 *Under the assumptions of Theorem 2.5, let $0 < r \le 1$ and let $\beta = -\log_2 r$. Then the scaling function $f(x) \in C^\beta(\mathbf{R})$ if and only if $\{A_0/r, A_1/r\}$ is product bounded.*

Proof: Suppose $\{A_0/r, A_1/r\}$ is product bounded. Then there exists a matrix norm $\|.\|$ on $\mathbf{M}_{N-1}(\mathbf{R})$ such that $\|A_0\| \le r$, $\|A_1\| \le r$. (See Proposition 1.6.)

Since $f(x)$ is a continuous scaling function of (1), $\{P_0|_{\mathcal{W}}, P_1|_{\mathcal{W}}\}$ is RCP and

$$\mathbf{v}_f(x) = \Big(\prod_{i=1}^{\infty} P_{d_i(x)}\Big)\mathbf{v_0}$$

where $\mathbf{v_0} = \mathbf{v}_f(0) \in \mathcal{W}$. For any $x, y \in [0,1]$ such that $2^{-m-1} \le |x-y| < 2^{-m}$, if $\sigma(x,y) \ge m$ then from Lemma 2.2,

$$|\mathbf{v}_f(x) - \mathbf{v}_f(y)| \le C_3\|A_{d_1(x)} \cdots A_{d_m(x)}\| \le C_3 r^m.$$

Hence

$$|\mathbf{v}_f(x) - \mathbf{v}_f(y)| \le \frac{C_3 r^m}{2^{-m-1}} = 2C_3\left(\frac{1}{2}\right)^{m\beta} \le 2C_3|x-y|^\beta.$$

If $\sigma(x,y) < m$ then we choose a dyadic $z \in [0,1]$ such that $|x-z| < 2^{-m}$ and $|y-z| < 2^{-m}$. From Lemma 1.3, $\sigma(x,z) \ge m$ and $\sigma(y,z) \ge m$. So

$$|x-z| \le 2^{-m} \le 2|x-y|, \quad |y-z| \le 2^{-m} \le 2|x-y|.$$

Hence

$$\begin{aligned}
|\mathbf{v}_f(x) - \mathbf{v}_f(y)| &\le |\mathbf{v}_f(x) - \mathbf{v}_f(z)| + |\mathbf{v}_f(z) - \mathbf{v}_f(y)| \\
&\le 2C_3|x-z|^\beta + 2C_3|y-z|^\beta \\
&\le C_4|x-y|^\beta,
\end{aligned}$$

where $C_4 = 2^{\beta+2}C_3$.

Conversely, let Q, $\tilde{P}_0$, $\tilde{P}_1$ be as in (11) and (12). Then from Lemma 2.2, $Q\mathbf{v}_f(x) = [\mathbf{b}^t(x), 0]^t$ where $\mathbf{b}(x) \in \mathbf{R}^k$ and $k = \dim W$. Moreover,

$$\mathbf{b}(x) - \mathbf{b}(y) = \begin{bmatrix} 0 \\ \mathbf{c_1}(x,y) \end{bmatrix}, \quad \mathbf{c_1}(x,y) \in \mathbf{R}^{k-1}.$$

Since

$$\dim \operatorname{span}\{\mathbf{v}_f(x) - \mathbf{v}_f(0) \mid x \in [0,1]\} \geq k-1,$$

we have

$$\operatorname{span}\{\mathbf{c_1}(x,0) \mid x \in [0,1]\} = \mathbf{R}^{k-1}.$$

So for some $z_1, \ldots, z_{k-1} \in [0,1]$, $\{\mathbf{c_1}(z_1,0), \ldots, \mathbf{c_1}(z_{k-1},0)\}$ is a basis of $\mathbf{R}^{k-1}$.

Suppose $\{A_0/r, A_1/r\}$ is not product bounded. Then for any fixed matrix norm $\|.\|$ on $\mathbf{M}_{k-1}(\mathbf{R})$ and any $\lambda > 0$ there exist an m and $d_1, \ldots, d_m \in \{0,1\}$ such that

$$\| \prod_{i=1}^{m} A_{d_i} \| \geq \lambda r^m. \tag{21}$$

It follows from Lemma 2.6 that exists a $c > 0$ (independent of λ, m and d_i) and $1 \leq l \leq k-1$ such that

$$\Big| \Big(\prod_{i=1}^{m} A_{d_i} \Big) \mathbf{c_1}(z_l, 0) \Big| \geq c\lambda r^m$$

Let $x_\lambda = 0.d_1 \cdots d_m 0\, 0 \cdots$ and $y_\lambda = x_\lambda + z_l/2^{m+1}$. Then $\sigma(x_\lambda, y_\lambda) \geq m$. So

$$|\mathbf{c_1}(y_\lambda, x_\lambda)| = \Big| \Big(\prod_{i=1}^{m} A_{d_i} \Big) \mathbf{c_1}(\tau^m y, \tau^m x) \Big| = \Big| \Big(\prod_{i=1}^{m} A_{d_i} \Big) \mathbf{c_1}(z_l, 0) \Big| \geq c\lambda r^m.$$

Hence $|\mathbf{b}(x_\lambda) - \mathbf{b}(y_\lambda)| \geq c\lambda r^m$ and so $|\mathbf{v}_f(x_\lambda) - \mathbf{v}_f(y_\lambda)| \geq c' c\lambda r^m$ for some constant $c' > 0$. But $|y_\lambda - x_\lambda| \leq 2^{-m}$, so

$$|\mathbf{v}_f(y_\lambda) - \mathbf{v}_f(x_\lambda)| \geq c\lambda 2^{-m(\log_2 \frac{1}{r})} \geq c\lambda |y_\lambda - x_\lambda|^\beta.$$

Since $\lambda > 0$ is arbitrarily chosen, it follows that $f(x) \notin C^\beta(\mathbf{R})$. ∎

Proof of Theorem 2.5: For any $r > 0$, we have

$$\rho\Big(\frac{A_0}{r}, \frac{A_1}{r}\Big) = \frac{\rho(A_0, A_1)}{r} = \frac{\rho}{r}.$$

Let $\varepsilon > 0$ and $r = 2^{-(\alpha-\varepsilon)} = \rho 2^\varepsilon$. Since $r > \rho(A_0, A_1)$, $\{A_0/r, A_1/r\}$ is product bounded. Hence $f(x) \in C^{-\log_2 r}(\mathbf{R}) = C^{\alpha-\varepsilon}(\mathbf{R})$. Similarly, let $r = 2^{-(\alpha+\varepsilon)} = \rho 2^\varepsilon$. Then $\{A_0/r, A_1/r\}$ is not product bounded. Hence $f(x) \notin C^{-\log_2 r}(\mathbf{R}) = C^{\alpha-\varepsilon}(\mathbf{R})$. The second part of Theorem 2.5 is a direct consequence of Lemma 2.7. ∎

3 Dilation Equations with Integrable Solutions

It is shown in [27] that the dilation equation (1) has an L^1 solution if $\{P_0, P_1\}$ is RCP. We prove the following stronger result.

Theorem 3.1 *Assume that $\sum_n c_n = 2$. If $\{P_0, P_1\}$ is product bounded, then the dilation equation (1) has a bounded solution.*

Proof: We first prove that (1) has an L^2 solution. Let $f_0(x) = \chi_{[0,1)}(x)$ and

$$f_n(x) = \sum_{k=0}^{N} c_k f_{n-1}(2x-k).$$

Notice that $\text{supp}(f_n) \subseteq [0, N]$ and for any $x \in [0,1]$,

$$\mathbf{v}_{f_n}(x) = P_{d_1(x)} P_{d_2(x)} \cdots P_{d_n(x)} \mathbf{v}_{f_0}(\tau^n x). \tag{22}$$

Hence there exists a $C > 0$ such that $|f_n(x)| < C$ for all x and n.

Consider the Fourier transform of $f_n(x)$

$$\begin{aligned}\hat{f}_n(\xi) &= \int_{\mathbf{R}} e^{-ix\xi} f_n(x)\, dx \\ &= \int_{\mathbf{R}} e^{-ix\xi} \sum_{k=0}^{N} c_k f_{n-1}(2x-k)\, dx \\ &= P(\frac{\xi}{2}) \hat{f}_{n-1}(\frac{\xi}{2})\end{aligned}$$

where $P(\xi) = \frac{1}{2}\sum_{k=0}^{N} c_k e^{-ik\xi}$. Thus

$$\hat{f}_n(\xi) = \Big(\prod_{k=1}^{n} P(\frac{\xi}{2^k})\Big) \hat{f}_0(\frac{\xi}{2^{k+1}}).$$

Since $\hat{f}_0(\xi) = (1-e^{-i\xi})/i\xi$, it follows that as $n \longrightarrow +\infty$, $\hat{f}_0(\xi/2^{n+1}) \longrightarrow 1$ *uniformly* on any compact subset of $\mathbf{R}$. Hence

$$\lim_{n\to+\infty} \hat{f}_n(\xi) = \prod_{k=1}^{\infty} P(\frac{\xi}{2^k}) = \phi(\xi)$$

uniformly on any compact subset of $\mathbf{R}$. (The uniform convergence of $\prod_{k=1}^{\infty} P(\xi/2^k)$ on compact sets is established in [8].) Let $S \subset \mathbf{R}$ be any compact subset of $\mathbf{R}$. Then

$$\begin{aligned}\int_S |\phi(\xi)|^2\, d\xi &= \lim_{n\to+\infty} \int_S |\hat{f}_n(\xi)|^2\, d\xi \\ &\le \limsup_{n\to+\infty} \int_{\mathbf{R}} |\hat{f}_n(\xi)|^2\, d\xi \\ &= \limsup_{n\to+\infty} \int_{\mathbf{R}} |f_n(x)|^2\, dx \\ &\le C^2 N.\end{aligned}$$

Hence $\phi(\xi) \in L^2(\mathbf{R})$. Let $f(x)$ be the inverse Fourier transform of $\phi(\xi)$. Then $f(x) \in L^2(\mathbf{R})$, and from Paley–Wiener Theorem $f(x)$ is compactly supported ([8]). Since the Fourier transform of $f(x) - \sum_{k=0}^{N} c_k f(2x-k)$ is 0. It follows that $f(x)$ must be a solution of (1).

We prove $f(x)$ must be bounded. Since $\{P_0, P_1\}$ is product bounded, there exists a norm $\|.\|$ on $\mathbf{R}^N$ such that $\|P_i \mathbf{x}\| \leq \|\mathbf{x}\|$ for $i = 0, 1$ and all $\mathbf{x} \in \mathbf{R}^N$. For any $b \geq 0$ let

$$S_b = \Big\{ x \in [0,1] \;\Big|\; \|\mathbf{v}_f(x)\| \geq b \Big\}.$$

Since $\mathbf{v}_f(x) = P_{d_1(x)} \mathbf{v}_f(\tau x)$, it follows that $x \in S_b$ implies $\tau x \in S_b$. Hence S_b is invariant under τ. But τ is ergodic, so $\mu(S_b) = 0$ or $\mu(S_b) = 1$ where μ is Lebesgue measure on $\mathbf{R}$. So $\mu(S_b) = 0$ for a sufficiently large $b > 0$. Therefore $f(x)$ must be bounded. ∎

Remark: We have actually proved a stronger results than $\mu(S_b) = 0$ for some $b > 0$. We have shown that $\|\mathbf{v}_f(x)\| = b_0$ a.e. where $b_0 = \inf\{b | \mu(S_b) = 0\}$. This implies that $\mathbf{v}_f(x)$ must lie on the sphere $\|x\| = b_0$.

Corollary 3.2 *Assume that $\sum_n c_{2n} = \sum_n c_{2n+1} = 1$. If $c_n \geq 0$, then the dilation equation (1) has a bounded solution.*

Proof: Because both P_0 and P_1 are column stochastic, all products of them are also column stochastic and non-negative. So $\{P_0, P_1\}$ must be product bounded. ∎

Theorem 3.3 *Assume that $\sum_n c_n = 2$. Suppose there exists an invariant subspace $\mathcal{W} \subseteq \mathbf{R}^N$ of $\{P_0, P_1\}$ such that*

(i) $\{P_0|_{\mathcal{W}}, P_1|_{\mathcal{W}}\}$ is product bounded;

(ii) there exists a $\mathbf{v} = [v_1, v_2, \ldots v_N]^t \in \mathcal{W}$ such that $\sum_{i=1}^{N} v_i \neq 0$.

Then the dilation equation (1) has an L^2 solution.

Proof: Define

$$f_0(x) = \begin{cases} v_k & x \in [k-1, k), \\ 0 & x \notin [0, N). \end{cases}$$

Then $\mathbf{v}_{f_0}(x) = \mathbf{v}$ for $x \in [0,1)$. Let

$$f_n(x) = \sum_{k=0}^{N} c_k f_{n-1}(2x-k), \quad n > 0.$$

As in Theorem 3.1, we have $\mathrm{supp}(f_n) \subseteq [0, N]$ and for any $x \in [0,1)$,

$$\mathbf{v}_{f_n}(x) = P_{d_1(x)} P_{d_2(x)} \cdots P_{d_n(x)} \mathbf{v}.$$

Notice that $f_0(x) = \sum_{i=1}^{N} v_i \chi_{[i-1,i)}(x)$ and $\hat{f}_0(\xi) = \sum_{i=1}^{N} v_i \neq 0$. Therefore the same argument used in the proof of Theorem 3.1 will show that the inverse Fourier transform of $\lim_{n\to\infty} \hat{f}_n(\xi)$ is an L^2 solution to (1). ∎

We are unable to show that under the assumption of Theorem 3.3 there exists a *bounded* solution of (1), although such is likely the case. The main difficulty is to show that $\mathbf{v}_f(x) \in \mathcal{W}$ for almost all x.

Theorem 3.4 *Assume that all proper invariant subspaces of* $\{P_0, P_1\}$ *are contained in the subspace*

$$\mathbf{V}_0 = \Big\{[x_1, x_2, \ldots, x_N]^t \;\Big|\; \sum_{i=1}^{N} x_i = 0\Big\}. \tag{23}$$

Then $\rho(P_0, P_1) \leq 2$ *is a necessary condition for the dilation equation (1) to have an* L^1 *solution.*

Notice that if $\sum_n c_{2n} = \sum_n c_{2n+1} = 1$, then $\mathbf{V}_0$ is an invariant subspace of $\{P_0, P_1\}$.

Lemma 3.5 *Let* $A \in \mathbf{M}_n(\mathbf{R})$ *and* $\|.\|$ *be a norm on* $\mathbf{R}^n$. *Suppose* $|\lambda| > \theta > 0$ *for all eigenvalues* λ *of* A. *Then there exists a constant* $c = c(\|.\|, A, \theta) > 0$ *such that*

$$\|A^m \mathbf{x}\| \geq c\theta^m \|x\|. \tag{24}$$

Proof: We only need to show that for all $m \geq 0$,

$$\inf_{\|x\|=1} \frac{\|A^m \mathbf{x}\|}{\theta^m} \geq c > 0. \tag{25}$$

Let $B = A/\theta$. Then B is *expansive*, namely all eigenvalues of B have modulus strictly greater than 1. Thus there exists an $m_0 = m_0(\|.\|, A, \theta) \geq 0$ such that $B^m(U) \supset U$ where $U = \{\mathbf{x} \in \mathbf{R}^n | \|\mathbf{x}\| \leq 1\}$ for all $m \geq m_0$. So for any $\mathbf{x}$ with $\|\mathbf{x}\| = 1$, $\|B^m \mathbf{x}\| \geq 1$ for all $m \geq m_0$. Let

$$c = \inf_{\|x\|=1, m<m_0} \|B^m \mathbf{x}\|.$$

Then $\inf_{\|x\|=1} \|B^m \mathbf{x}\| \geq c$ for all m. (25) follows immediately. ∎

Lemma 3.6 *Suppose all proper invariant subspaces of* $\{P_0, P_1\}$ *are contained* $\mathbf{V}_0$ *and (1) has an* L^1 *solution. Let* $d_1, \ldots d_k \in \{0, 1\}$ *and* $\rho(P_{d_1} \cdots P_{d_k}) > \theta > 0$. *Then there exists a constant* $c > 0$ *such that*

$$\mu\Big(\Big\{x \in [0,1] \;\Big|\; |(P_{d_1} \cdots P_{d_k})^m \mathbf{v}_f(x)| \geq c\theta^m \text{ for all } m > 0\Big\}\Big) > 0.$$

Proof: Since $\int_{\mathbf{R}} f(x)\,dx \neq 0$ ([10]), we see that there exists a $J \subseteq [0,1]$ and $\mu(J) > 0$ such that $\mathbf{v}_f(x) \notin \mathbf{V}_0$ for all $x \in J$. Let $S \subseteq [0,1]$ such that $\mu(S) = 1$. Then $S^* \subseteq S$ is invariant under τ and $\mu(S^*) = 1$ where

$$S^* = S \setminus \bigcup_{m=0}^{\infty} \tau^m([0,1] \setminus S).$$

This implies span $\{\mathbf{v}_f(x) | x \in S^*\}$ is an invariant subspace of $\{P_0, P_1\}$. But $\mathbf{v}_f(x) \notin \mathbf{V}_0$ for $x \in J \cap S^* \neq \phi$. So span $\{\mathbf{v}_f(x) | x \in S^*\} = \mathbf{R}^N$ and hence

$$\text{span}\,\Big\{\mathbf{v}_f(x) \;\Big|\; x \in S\Big\} = \mathbf{R}^N. \tag{26}$$

Denote $Q = P_{d_1} \cdots P_{d_k}$. Let $\{\lambda_1, \ldots, \lambda_s\}$ be the spectrum of Q. Then

$$\mathbf{C}^N = W_{\lambda_1} \oplus \cdots \oplus W_{\lambda_s}$$

where

$$W_{\lambda_i} = \Big\{\mathbf{x} \in \mathbf{C}^N \;\Big|\; (Q - \lambda_i I)^m \mathbf{x} = \mathbf{0} \text{ for some } m\Big\}.$$

Let $W^1 = \mathbf{R}^N \cap (\bigoplus_{|\lambda_i| \geq \theta} W_{\lambda_i})$ and $W^2 = \mathbf{R}^N \cap (\bigoplus_{|\lambda_i| < \theta} W_{\lambda_i})$. Then

$$\mathbf{R}^N = W^1 \oplus W^2.$$

Define the following norm $\|.\|$ on $\mathbf{R}^N$: for any $\mathbf{x} = \mathbf{y} + \mathbf{z}$ where $\mathbf{y} \in W^1$ and $\mathbf{z} \in W^2$,

$$\|\mathbf{x}\| = |\mathbf{y}| + |\mathbf{z}|.$$

Then from Lemma 3.5 there exists a $c_1 > 0$ such that

$$\|Q^m \mathbf{x}\| = \|Q^m \mathbf{y} + Q^m \mathbf{z}\| = |Q^m \mathbf{y}| + |Q^m \mathbf{z}| \geq |Q^m \mathbf{y}| \geq c_1 \theta^m |\mathbf{y}|.$$

For $n > 0$ let

$$S_n = \Big\{x \in [0,1] \;\Big|\; \mathbf{v}_f(x) = \mathbf{y}_1 + \mathbf{y}_2,\; \mathbf{y}_i \in W^i, \text{ such that } |\mathbf{y}_1| \geq \frac{1}{n}\Big\}.$$

It follows from (26) that $\mu(\bigcup_n S_n) > 0$. Hence there exists an $n_0 > 0$ such that $\mu(S_{n_0}) \geq \delta > 0$. For any $x \in S_{n_0}$,

$$\|Q^m \mathbf{v}_f(x)\| = \|Q^m \mathbf{y}_1 + Q^m \mathbf{y}_2\| \geq c_1 \theta^m |\mathbf{y}_1| \geq \frac{c_1}{n_0} \theta^m.$$

Let $c = c_1/n_0$. We have

$$\mu\Big(\Big\{x \in [0,1] \;\Big|\; |(P_{d_1} \cdots P_{d_k})^m \mathbf{v}_f(x)| \geq c\theta^m \text{ for all } m > 0\Big\}\Big) > 0.$$

∎

Proof of Theorem 3.4: Assume that $\rho(P_0, P_1) > b > 2$. Then there exist $d_1, \dots d_k \in \{0,1\}$ such that $\rho(P_{d_1} \cdots P_{d_k}) \geq b^k$. Choose $c > 0$ so that $\mu(S) > 0$ where

$$S = \Big\{ x \in [0,1] \, \Big| \, |(P_{d_1} \cdots P_{d_k})^m \mathbf{v}_f(x)| \geq c b^{km} \text{ for all } m > 0 \Big\}.$$

Let T_m be the subset of $[0,1]$ that contains all x's such that $\tau^{km} x \in S$ and the first km digits in the dyadic expansion of x are

$$\underbrace{d_1 \cdots d_k d_1 \cdots d_k \cdots d_1 \cdots d_k}_{km}.$$

Clearly $\mu(T_m) = \mu(S)/2^{km}$.

$$\begin{aligned} \int_{T_m} |\mathbf{v}_f(x)| \, dx &= \frac{1}{2^{km}} \int_S |(P_{d_1} \cdots P_{d_k})^m \mathbf{v}_f(y)| \, dy \\ &\geq \frac{1}{2^{km}} \cdot c \cdot b^{mk} \mu(S) \\ &= c\mu(S) \Big(\frac{b}{2}\Big)^{mk}. \end{aligned}$$

But $(b/2)^{km} \longrightarrow \infty$ as $m \longrightarrow \infty$. This contradicts the assumption that $f(x) \in L^1(\mathbf{R})$. ■

In what follows we examine the relationship between $\bar{\rho}(A_0, A_1)$ and L^1 solution of the dilation equation (1). Recall that given $\Sigma = \{A_1, \dots, A_p\} \subset \mathbf{M}_N(\mathbf{R})$, $\bar{\rho}(\Sigma)$ is the *mean spectral radius* defined in (6).

Lemma 3.7 *Suppose $A_0, A_1 \in \mathbf{M}_N(\mathbf{R})$ and $\bar{\rho}(A_0, A_1) < 1$. Let $\|.\|$ be a fixed matrix norm. For any $x \in [0,1]$ define*

$$g(x) = \|A_{d_1(x)}\| + \|A_{d_1(x)} A_{d_2(x)}\| + \cdots = \sum_{n=1}^{\infty} \|A_{d_1(x)} \cdots A_{d_n(x)}\|$$

Then $g(x) < \infty$ for almost all $x \in [0,1]$. Furthermore, $g(x) \in L^1([0,1])$.

Proof: Since $\bar{\rho}(A_0, A_1) < 1$, there exist constants $c > 0$ and λ, $\bar{\rho}(A_0, A_1) < \lambda < 1$, such that for all m,

$$\frac{1}{2^m} \sum_{d_1, \dots, d_m} \|A_{d_1} \cdots A_{d_m}\| \leq c\lambda^m$$

where the summation $\sum_{d_1, \dots, d_m}$ is taken over all $(d_1, \dots, d_m) \in \{0,1\}^m$. Let

$$g_m(x) = \sum_{n=1}^{m} \|A_{d_1} \cdots A_{d_n}\|.$$

For any fixed $x \in [0,1]$, $\{g_m(x)\}$ is an increasing sequence and

$$\int_0^1 |g_m(x)| \, dx = \sum_{n=1}^{m} \int_0^1 \|A_{d_1(x)} \cdots A_{d_n(x)}\| \, dx$$

$$
\begin{aligned}
&= \sum_{n=1}^{m} \sum_{d_1,\ldots,d_n} \Big(\frac{1}{2^n}\|A_{d_1}\cdots A_{d_n}\|\Big) \\
&\le \sum_{n=1}^{m} c\cdot\lambda^n \\
&\le \frac{c}{1-\lambda}.
\end{aligned}
$$

Thus it follows from the Monotone Convergence Theorem that

$$\int_0^1 |g(x)|\,dx \le \lim \int_0^1 g_m(x)\,dx \le \frac{c}{1-\lambda}.$$

Hence $g(x) \in L^1([0,1])$ and $g(x) < \infty$ almost everywhere. ∎

Theorem 3.8 *Suppose* $\sum_n c_{2n+m} = 1$ *and* $\bar{\rho}(A_0, A_1) < 1$ *where*

$$P_i \sim \begin{bmatrix} 1 & 0 \\ \mathbf{c}_i & A_i \end{bmatrix}, \quad i = 0, 1.$$

simultaneously. Then the dilation equation (1) has an L^1 *solution.*

Proof: Let $f_0(x) = \chi_{[0,1)}(x)$ and

$$f_n(x) = \sum_{k=0}^{N} c_k f_{n-1}(2x-k), \quad n > 0.$$

We have $\operatorname{supp}(f_n) \subseteq [0, N]$ and for any $x \in [0,1]$, Let $Q \in \mathbf{M}_N(\mathbf{R})$ such that

$$QP_iQ^{-1} = \begin{bmatrix} 1 & 0 \\ \mathbf{c}_i & A_i \end{bmatrix}, \quad i = 0, 1.$$

Then

$$
\begin{aligned}
\mathbf{v}_{f_n}(x) &= P_{d_1(x)}P_{d_2(x)}\cdots P_{d_n(x)}\mathbf{v}_{f_0}(\tau^n x) \\
&= Q^{-1}\begin{bmatrix} 1 & 0 \\ \mathbf{b}_n(x) & B_n(x) \end{bmatrix} Q\mathbf{v}_0
\end{aligned}
$$

where $\mathbf{v}_0 = \mathbf{v}_{f_0}(x) = [1, 0, \ldots, 0]^t$. It is easy to verify that $B_n(x) = A_{d_1(x)}A_{d_2(x)}\cdots A_{d_n(x)}$ and

$$\mathbf{b}_n(x) = \mathbf{c}_{d_1(x)} + A_{d_1(x)}\mathbf{c}_{d_2(x)} + \cdots + A_{d_1(x)}A_{d_2(x)}\cdots A_{d_{n-1}(x)}c_{d_n(x)}.$$

Let $|\mathbf{b}_i| \le c$, $i = 0, 1$. Then

$$|\mathbf{b}_n(x)| \le c \cdot \sum_{m=1}^{n-1} \|A_{d_1}\cdots A_{d_m}\|.$$

Hence it follows from Lemma 3.7 that $\mathbf{b}_n(x) \to \mathbf{b}(x) \in L^1(\mathbf{R})$ and $B_n(x) \to 0$ as $n \to \infty$ for almost all $x \in [0,1]$. Let

$$\mathbf{v}_f(x) = Q^{-1} \begin{bmatrix} 1 & 0 \\ \mathbf{b}(x) & 0 \end{bmatrix} Q \mathbf{v}_0.$$

Then

$$\lim_{n \to \infty} \mathbf{v}_{f_n}(x) = \mathbf{v}_f(x) \quad a.e.$$

and $\mathbf{v}_f(x) \in L^1([0,1])$. Therefore $f(x)$ must be L^1. ∎

Theorem 3.9 *Assume that $\sum_n c_n = 2$. Let $\mathcal{W} \subseteq \mathbf{R}^N$ be an invariant subspace of $\{P_0, P_1\}$ such that*

(i)

$$P_i|_{\mathcal{W}} \sim \begin{bmatrix} 1 & 0 \\ * & A_i \end{bmatrix}, \quad i = 0, 1$$

simultaneously with $\bar{\rho}(A_0, A_1) < 1$;

(ii) there exists a $\mathbf{v} = [v_1, v_2, \ldots v_N]^t \in \mathcal{W}$ such that $\sum_{i=1}^N v_i \neq 0$.

Then the dilation equation (1) has an L^1 solution.

Proof: Define

$$f_0(x) = \begin{cases} v_k, & x \in [k-1, k) \\ 0, & x \notin [0, N) \end{cases}$$

Then $\mathbf{v}_{f_0}(x) = \mathbf{v}$ for $x \in [0,1)$. Let

$$f_n(x) = \sum_{k=0}^{N} c_k f_{n-1}(2x - k), \quad n > 0.$$

As in Theorem 3.3, we have $\mathrm{supp}(f_n) \subseteq [0, N]$ and for any $x \in [0,1]$,

$$\mathbf{v}_{f_n}(x) = P_{d_1(x)} P_{d_2(x)} \cdots P_{d_n(x)} \mathbf{v}.$$

So $\mathbf{v}_{f_n}(x) \in \mathcal{W}$. Notice that

$$\int_{\mathbf{R}} f_n(x)\, dx = \int_{\mathbf{R}} f_0(x)\, dx = \sum_{i=1}^{N} v_i \neq 0.$$

So the same argument for proving Theorem 3.8 proves $f_n(x) \to f(x)$ a.e. for some $f(x) \in L^1(\mathbf{R})$ and $f(x)$ is a solution of (1). ∎

Conjecture[1]: *Assume that $\sum_n c_{2n+m} = 1$ and let $\mathbf{V}_0$ be as in (23). Suppose all proper invariant subspaces of $\{P_0, P_1\}$ are contained in $\mathbf{V}_0$. Then a necessary and sufficient condition for the dilation equation (1) to have an L^1 solution is $\bar{\rho}(A_0, A_1) < 1$.*

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[1] This conjecture has recently been settled by Lau and Wang ([20]).

Example: Consider the following dilation equation

$$f(x) = af(2x) + f(2x-1) + (1-a)f(2x-2). \tag{27}$$

$\rho(A_0, A_1) = \max\{|a|, |1-a|\}$ and $\bar{\rho}(A_0, A_1) = (|a| + |1-a|)/2$. Hence equation (27) has a continuous solution if and only if $0 < a < 1$. For $-1/2 < a < 3/2$, (27) has an L^1 solution.

4 Fractal Dimension of a Continuous Solution

The fractal properties of scaling functions were discussed in many papers (e.g. [1], [10], [3]). In [1], Berger linked a continuous solution of (1) to an iterated functions system (IFS), and in [10] an estimate of the box dimension of a continuous scaling function was given.

Recall that the *box dimension* (or *fractal dimension*) is defined as follows. For any compact set $E \subset \mathbf{R}^N$, the box dimension of E is

$$\dim_B(E) = \inf\Big\{ s > 0 \colon \limsup_{\varepsilon \to 0+} b_\varepsilon^s(E) < +\infty \Big\},$$

where

$$b_\varepsilon^s(E) = \inf \left\{ \sum_i \varepsilon^s \;\Big|\; E \subseteq \cup B_i,\ \mathrm{diam} B_i = \varepsilon \right\}.$$

Let $\mathcal{N}(\varepsilon, E)$ be the minimal number of balls of radius ε required to cover E. Then we have (see [12])

$$\dim_B(E) = \limsup_{\varepsilon \to 0+} \frac{\log \mathcal{N}(\varepsilon, E)}{-\log \varepsilon}.$$

Note that the box dimension $\dim_B(E)$ of E differs from the *Hausdorff dimension* $\dim_H(E)$, which is typically *much harder* to compute and which is defined as

$$\dim_H(E) = \inf\Big\{ s > 0 \colon \limsup_{\varepsilon \to 0+} h_\varepsilon^s(E) < +\infty \Big\},$$

where

$$h_\varepsilon^s(E) = \inf \left\{ \sum_i (\mathrm{diam} B_i)^s \;\Big|\; E \subseteq \cup B_i,\ \mathrm{diam} B_i \le \varepsilon \right\}.$$

The box dimension of the graph of a function has been studied before. A classical result by Falconer([18]) is that if $f(x)$ is C^α where $0 \le \alpha \le 1$, then

$$\dim_B(\text{graph of } f) \le 2 - \alpha.$$

Before we establish an exact formulae for the box dimesion of a continuous scaling function), we shall need the following result, proved by Deliu and Jawerth ([12]):

Theorem 4.1 *Let* $\mathcal{S}_n = \{[j/2^n, (j+1)/2^n) \colon 0 \le j < 2^n\}$. *For any continuous function* $\mathbf{f}\colon [0,1] \to \mathbf{R}^N$ *and an interval* $I \subseteq [0,1]$ *define*

$$osc_I(\mathbf{f}) = \sup_{x,y \in I} |\mathbf{f}(x) - \mathbf{f}(y)|.$$

Then

$$\dim_B(\text{graph of } \mathbf{f}) = \limsup_{n \to +\infty} \frac{\log_2^+ \left(\sum_{I \in \mathcal{S}_n} osc_I(f) \right)}{n} + 1.$$

(Here $\log_2^+ x = \max(\log_2 x, 0)$ *for* $x > 0$, *and* $\log_2^+ 0 = 0$.*)*

Theorem 4.2 *Suppose* $f(x)$ *is a continuous solution ofthe two–scale dilation equation (1) and* $\mathcal{W} = \text{span}\{\mathbf{v}_f(x) \colon x \in [0,1]\}$. *Let*

$$P_i \Big|_{\mathcal{W}} \sim \begin{bmatrix} 1 & 0 \\ * & A_i \end{bmatrix}, \; i = 0, 1$$

simultaneously with $\rho(A_0, A_1) < 1$. *Then*

$$\dim_B(\text{graph of } f) = \max\Big\{1,\, 2 + \log_2 \bar{\rho}(A_0, A_1)\Big\}.$$

Proof: By using Theorem 4.1 it is rather standard to prove that

$$\dim_B(\text{graph of } f) = \dim_B(\text{graph of } \mathbf{v}_f).$$

So we only need to prove

$$\dim_B(\text{graph of } \mathbf{v}_f) = \max\Big\{1,\, 2 + \log_2 \bar{\rho}(A_0, A_1)\Big\}.$$

Let $I = [j/2^n, (j+1)/2^n) \in \mathcal{S}_n$ and let $j/2^n$ have dyadic expansion

$$j/2^n = 0.d_1 \cdots d_n 0 \cdots.$$

Then $d_i(x) = d_i$, $1 \le i \le n$, for all $x \in I$. Let $k = \dim \mathcal{W}$ and let Q, $\mathbf{c_1}(x,y) \in \mathbf{R}^{k-1}$ be as in Lemma 2.2. It follows from Lemma 2.2 that

$$\text{osc}_I(Q\mathbf{v}_f) = \sup_{x,y \in I} |A_{d_1} \cdots A_{d_n} \mathbf{c_1}(\tau^n x, \tau^n y)| = \sup_{x,y \in [0,1]} |A_{d_1} \cdots A_{d_n} \mathbf{c_1}(x,y)|.$$

But $\mathbf{c_1}(x,y)$ is bounded and $\text{span}\{\mathbf{c_1}(x,y) \,|\, x,y \in [0,1]\} = \mathbf{R}^{k-1}$. So let $\|.\|$ be a matrix norm on $\mathbf{M}_{k-1}(\mathbf{R})$ then there exist constants $c_2 > c_1 > 0$ such that

$$c_1 \|A_{d_1} \cdots A_{d_n}\| \le \text{osc}_I(\mathbf{v}_f) \le c_2 \|A_{d_1} \cdots A_{d_n}\|.$$

$$c_1 \sum_{d_1,\ldots,d_n} \|A_{d_1} \cdots A_{d_n}\| \le \sum_{I \in \mathcal{S}_n} \text{osc}_I(\mathbf{v}_f) \le c_2 \sum_{d_1,\ldots,d_n} \|A_{d_1} \cdots A_{d_n}\| \tag{28}$$

where the summation is taken over all $(d_1, \ldots, d_n) \in \{0,1\}^n$. It follows from Theorem 4.1 that

$$\begin{aligned}
\dim_B(\text{graph of } \mathbf{v}_f) &= \limsup_{n\to+\infty} \frac{\log_2^+\Big(\sum_{I\in\mathcal{S}_n} \operatorname{osc}_I(\mathbf{v}_f)\Big)}{n} + 1 \\
&\geq \limsup_{n\to+\infty} \frac{\log_2^+\Big(c_1 \sum_{d_1,\ldots,d_n} \|A_{d_1}\cdots A_{d_n}\|\Big)}{n} + 1 \\
&= \limsup_{n\to+\infty} \log_2^+\Big(\sum_{d_1,\ldots,d_n} \|A_{d_1}\cdots A_{d_n}\|\Big)^{\frac{1}{n}} + 1,
\end{aligned}$$

and (similarly from the second half of (28))

$$\dim_B(\text{graph of } \mathbf{v}_f) \leq \limsup_{n\to+\infty} \log_2^+\Big(\sum_{d_1,\ldots,d_n} \|A_{d_1}\cdots A_{d_n}\|\Big)^{\frac{1}{n}} + 1.$$

So we have

$$\begin{aligned}
\dim_B(\text{graph of } \mathbf{v}_f) &= \limsup_{n\to+\infty} \log_2^+\Big(\sum_{d_1,\ldots,d_n} \|A_{d_1}\cdots A_{d_n}\|\Big)^{\frac{1}{n}} + 1 \\
&= \limsup_{n\to+\infty} \log_2^+ 2\Big(\frac{1}{2^n}\sum_{d_1,\ldots,d_n} \|A_{d_1}\cdots A_{d_n}\|\Big)^{\frac{1}{n}} + 1 \\
&= \log^+\Big(2\bar{\rho}(A_0, A_1)\Big) + 1 \\
&= \max\Big\{1,\, 2 + \log_2 \bar{\rho}(A_0, A_1)\Big\}.
\end{aligned}$$

■

Example: Consider the dilation equation

$$f(x) = af(2x) + f(2x-1) + (1-a)f(2x-2)$$

where $1/2 < a < 1$. $\rho(A_0, A_1) = \max\{a, 1-a\} = a$ and $\bar{\rho}(A_0, A_1) = 1/2$. The equation has a continuous scaling function that is C^α where $\alpha = -\log_2 a$. Using Falconer's estimate we only get

$$\dim_B(\text{graph of } f) \leq 2 + \log_2 a$$

while in fact we have

$$\dim_H(\text{graph of } f) = \dim_B(\text{graph of } f) = 1.$$

(Note that $\dim_H(E) \leq \dim_B(E)$).

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