ON REFINABLE SETS

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Abstract. A refinable set is a compact set with positive Lebesgue measure whose characteristic function satisfies a refinement equation. Refinable sets are a generalization of self-affine tiles. But unlike the latter, the refinement equations defining refinable sets may have negative coefficients, and a refinable set may not tile. In this paper, we establish some fundamental properties of these sets.

1. Introduction

In recent years we have seen extensive studies on self-affine tiles, fueled in part by their ties to orthonormal wavelets, fractal geometry, number theory, and of course, tiling itself. Let $T \subset \mathbb{R}^d$ be a compact set with $\mu(T) > 0$, where μ denotes the Lebesgue measure. We say T is a *self-affine tile* if there exist an expanding matrix $A \in M_d(\mathbb{R})$, i.e. all eigenvalues of A have $|\lambda| > 1$, and $\{d_j\}_{j=1}^q \subset \mathbb{R}^d$ such that

$$A(T) = \bigcup_{j=1}^{q} (T + d_j), \tag{1.1}$$

where the union on the right is essentially disjoint, namely $\mu((T + d_i) \cap (T + d_j)) = 0$ for $i \neq j$. Note that the essential disjointness implies that $|\det(A)| = q$. It is known that a self-affine tile must have nonempty interior with its boundary having 0 Lebesgue measure, and it is the closure of its interior. Much has been written on the various properties of self-affine tiles as well as on their connections to other areas of mathematics. We shall not discuss them in details here. One may refer to e.g. [1], [12] and [15].

The set theoretic equation (1.1) can be written in the form of a *refinement equation*. Let $f(x) = \chi_T(x)$. Then $f(x)$ satisfies

$$f(x) = \sum_{j=1}^{q} f(Ax - d_j) \quad \text{a.e.} \tag{1.2}$$

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Indeed many fundamental properties of self-affine tiles are derived from this refinement equation, including their wavelet connection.

In general, a refinement equation in $\mathbb{R}^d$ is a functional equation of the form

$$f(x) = \sum_{j=1}^{n} c_j f(Ax - d_j) \quad \text{a.e.} \tag{1.3}$$

where $A \in M_d(\mathbb{R})$ is expanding, $\{d_j\} \subset \mathbb{R}^d$ and all c_j are real. We call A the *dilation* and $\{d_j\}$ the translations of the refinement equations, respectively. The function $f(x)$ is called A-*refinable*. If in addition $A \in M_d(\mathbb{Z})$ and $\{d_j\} \subset \mathbb{Z}^d$ then we call (1.3) an *integral refinement equation* and $f(x)$ *integrally refinable*.

A common misconception is that if χ_T is refinable then T is a self-affine tile. There are in fact T that are not tiles, let alone self-affine tiles, such that χ_T are refinable. A simple example is $T = [0,2] \cup [3,5] \cup [6,8]$, for which $f(x) = \chi_T(x)$ satisfies

$$f(x) = f(2x) + f(2x-2) - f(2x-3) - f(2x-5) + f(2x-6) + f(2x-8) \quad \text{a.e.}$$

Clearly T is not a tile because the gaps in T cannot be filled without overlaps.

Definition 1.1. Let $T \subset \mathbb{R}^d$ be a compact set with $\mu(T) > 0$. We say T is *refinable* if χ_T is refinable. It is A-*refinable* if the refinement equation has dilation matrix A.

In this paper we ask the following questions: What topological properties do refinable sets possess? Under what conditions is a refinable set T a self-affine tile? How do we characterize refinable sets? We shall focus almost entirely on integral refinable sets. In the non-integral setting the study becomes much harder. This is evident from the fact that we know very little about non-integral self-affine tiles.

Theorem 1.1. *Let T be a compact set in $\mathbb{R}^d$ with positive Lebesgue measure. Suppose T is refinable with $f(x) = \chi_T(x)$ satisfying the refinement equation*

$$f(x) = \sum_{j=1}^{n} c_j f(Ax - d_j) \quad \text{a.e.}$$

(a) *All c_j are integers.*
(b) $|\det(A)| = \sum_{j=1}^{n} c_j$. *Hence* $\det(A) \in \mathbb{Z}$.
(c) *If in addition $A \in M_d(\mathbb{Z})$ and all $d_j \in \mathbb{Z}^d$, then $\mu(T) \in \mathbb{Z}$. Furthermore, $T^o \neq \emptyset$ and $\mu(\partial T) = 0$. Thus $T = \overline{T^o}$ up to a measure 0 set.*

An important concept in the study of integral refinable functions is the *linear independence* of their integral translates, see e.g. Jia and Wang [13] and Goodman, Jia and Zhou [11]. An integral refinable function $f(x)$ is *linearly independent* if there exist no constants $\{c_\alpha : \alpha \in \mathbb{Z}^d\}$ not all 0 such that

$$\sum_{\alpha \in \mathbb{Z}^d} c_\alpha f(x-\alpha) = 0 \quad \text{a.e.}$$

We have the following theorem:

Theorem 1.2. *Let T be an integral refinable set in $\mathbb{R}^d$. The following are equivalent:*

(a) $\mu(T) = 1$.
(b) *T is an integral self-affine tile such that T tiles $\mathbb{R}^d$ translationally by the lattice $\mathbb{Z}^d$.*
(c) *The characteristic function $f(x) = \chi_T(x)$ is linearly independent.*

One way to construct *integral* refinable sets T is to take an integral self-affine tile T_0 and let $T = T_0 + \mathcal{A}$ where $\mathcal{A} \subset \mathbb{Z}^d$ is a finite set. By suitably choosing $\mathcal{A}$ we may make T refinable. The example $T = [0,2] \cup [3,5] \cup [6,8]$ is of this form, with $T_0 = [0,2]$ and $\mathcal{A} = \{0,3,6\}$ (or $[0,1]$ and $\mathcal{A} = \{0,1,3,4,6,7\}$).

Definition 1.2. We call an integral refinable set T with dilation $A \in M_d(\mathbb{Z})$ *ordinary* if there exists an integral self-affine tile T_0 with dilation A and a finite set $\mathcal{A} \subset \mathbb{Z}^d$ such that $T = T_0 + \mathcal{A}$.

We shall derive necessary and sufficient conditions for a compact set to be an ordinary integral refinable set. In the special case where the dilation is 2 in $\mathbb{R}$ we obtain a complete classification of integral refinable sets. Let $\mathbb{Z}^+$ denote the set of all positive integers and for each $q \in \mathbb{Z}^+$ let $\Phi_q(z)$ denote the cyclotomic polynomial of order q, i.e. the monic polynomial whose roots consist of all primitive q-th roots of unity.

Theorem 1.3. *Let $T \in \mathbb{R}$ be a compact set with $\mu(T) > 1$. Then the following are equivalent:*

(a) *T is an integral 2-refinable set.*
(b) *$T = [0,1] + \mathcal{A}$ for some finite $\mathcal{A} \subset \mathbb{Z}$ with the property that the Laurent polynomial $q(z) = (z-1)\sum_{a\in\mathcal{A}} z^a$ satisfies $q(z) | q(z^2)$.*

(c) $T = [0,1] + \mathcal{A}$ *for some finite* $\mathcal{A} \subset \mathbb{Z}$ *where there exist odd integers* $1 = a_1 < a_2 \leq a_3 \leq \cdots \leq a_m$, $\alpha_k \in \mathbb{Z}^+$ *and* $b \in \mathbb{Z}$ *such that* $q(z) = (z-1)\sum_{a\in\mathcal{A}} z^a$ *satisfies*

$$q(z) = z^b \prod_{k=1}^{m} \prod_{j=0}^{\alpha_k - 1} \Phi_{2^j a_k}(z) = z^b \prod_{k=1}^{m} \Phi_{a_k}(z^{2^{\alpha_j - 1}}). \tag{1.4}$$

In this case, $f(x) = \chi_T(x)$ *satisfies the refinement equation* $f(x) = \sum_{n\in\mathbb{Z}} c_n f(2x-n)$ *in which*

$$\frac{1}{2}\sum_{n\in\mathbb{Z}} c_n z^n = z^b \prod_{k=1}^{m} \Phi_{2^{\alpha_k} a_k}(z). \tag{1.5}$$

It should be pointed out that when a refinable set is a union of intervals (or polytopes in higher dimensions) the characteristic function can be viewed as a refinable spline. There have been some work on classifying refinable splines, see [17, **?**, **?**] in $\mathbb{R}$ and [21] in higher dimensions. In particular, the polynomial property $q(z)|q(z^2)$ for refinable splines has been established in [17] while the structure result in (c) has been established in [**?**].

A λ-refinable set with $|\lambda| > 2$ do not have to be a union of intervals, as there are many self-affine tiles with dilation λ that are not of this form. We have a classification of all λ-refinable sets of the form $T = [0,1] + \mathcal{A}$ where $\mathcal{A} \subset \mathbb{Z}$, as the characteristic function can be viewed as a refinable spline. Such a classification can be obtained by combining results from [17] and [**?**]. To state the result we first introduce the following notation. Let $m, n \in \mathbb{Z}^+$. Assume that m, n have prime factorizations $m = \prod_p p^{\alpha_p}$ and $n = \prod_p p^{\beta_p}$, where p runs through all primes and $\alpha_p, \beta_p \geq 0$. We define

$$\langle m/n \rangle := \prod_p p^{\gamma_p}, \qquad \text{where } \gamma_p = \max\{\alpha_p - \beta_p, 0\}.$$

Proposition 1.4. *Let* $\lambda > 1$ *be an integer and* $\mathcal{A} \subset \mathbb{Z}$ *be finite. Let* $T = [0,1] + \mathcal{A}$*. The following are equivalent:*

(a) T *is a* λ*-refinable set.*

(b) $\mathcal{A}$ *has the property that the Laurent polynomial* $q(z) = (z-1)\sum_{a\in\mathcal{A}} z^a$ *satisfies* $q(z)|q(z^\lambda)$.

(c) $\mathcal{A}$ *the property that*

$$q(z) := (z-1)\sum_{a\in\mathcal{A}} z^a = z^b \prod_{k=1}^{m} \Phi_{a_k}^{r_k}(z). \tag{1.6}$$

where $1 = a_1 < a_2 < \cdots < a_m$, $r_k \in \mathbb{Z}^+$ *with the property* $r_1 = 1$ *and* $b \in \mathbb{Z}$. *Furthermore, for any* $1 \leq k \leq m$ *there exists some* $1 \leq j \leq m$ *such that* $b_j = \langle b_k/\lambda \rangle$ *and* $r_j \geq r_k$.

For any $\mathcal{B} \subset \mathbb{Z}^d$ define the trigonometric polynomial $f_{\mathcal{B}}(\xi) := \sum_{b \in \mathcal{B}} e^{-2\pi i \langle b, \xi \rangle}$. A general characterization of ordinary integral refinable sets is the following theorem.

Theorem 1.5. *Let* $A \in M_d(\mathbb{Z})$ *be expanding and* T_0 *be an integral self-affine tile given by* $A(T_0) = T_0 + \mathcal{D}$ *where* $\mathcal{D} \subset \mathbb{Z}^d$ *with* $|D| = |\det(A)|$. *Let* $T = T_0 + \mathcal{A}$ *where* $\mathcal{A} \subset \mathbb{Z}^d$ *and* $\{T_0 + a : \; a \in \mathcal{A}\}$ *are essentially disjoint. Then* T *is* A*-refinable if and only if* $f_{\mathcal{A}}(\xi) | f_{\mathcal{A}}(A^T \xi) f_{\mathcal{D}}(\xi)$.

We shall have more discussions on ordinary refinable sets later on. It should be pointed out that so far we are unable to construct a single example of a refinable set that is not ordinary.

2. Basic Properties of Refinable Sets

We begin with the following known result, due to Kolountzakis [14].

Lemma 2.1. *Let* $f(x) \in L^1(\mathbb{R}^d)$ *and* $\int_{\mathbb{R}^d} f = c \neq 0$. *Suppose that* $f(x)$ *satisfies the refinement equation (1.3) in which* $A \in M_d(\mathbb{Z})$ *is expanding and all* $d_j \in \mathbb{Z}^d$. *Then* $\sum_{\alpha \in \mathbb{Z}^d} f(x - \alpha) = c$.

Proof of Theorem 1.1. (a) Clearly we may rewrite the refinement equation as

$$\chi_T(A^{-1}x) = \sum_{j=1}^{n} c_j \chi_T(x - d_j).$$

Hence $\sum_{j=1}^{n} c_j \chi_T(x - d_j) \in \mathbb{Z}$ almost everywhere. We prove the following more general claim: *If* $\sum_{j=1}^{n} c_j \chi_T(x - d_j) \in \mathbb{Z}$ *a.e. then all* $c_j \in \mathbb{Z}$. This claim is proved via induction on n. Obviously the claim is true for $n = 1$. Assume it is true for all $n < k$ where $k > 1$. We prove it is also true for $n = k$. Consider the convex hull of $\{d_j\}$. Without loss of generality we assume that d_k is a vertex of the convex hull. Thus we may find a vector $v \in \mathbb{R}$, $|v| = 1$, such that $\langle d_k, v \rangle > \langle d_j, v \rangle$ for all $j < k$. Let $\varepsilon > 0$ such that $\langle d_k, v \rangle > \langle d_j v \rangle + 2\varepsilon$ for all $j < k$. Set

$$r = \sup\big\{ \langle x, v \rangle : \; x \in T, \, \mu(B_\varepsilon(x) \cap T) > 0 \big\}.$$

Thus there exists an $x^* \in T$ with $\langle x^*, v\rangle > r-\varepsilon$ and $\mu(B_\varepsilon(x^*)\cap T) > 0$. Now let $\Omega = B_\varepsilon(x^*)\cap T$. Observe that $\Omega + d_k \subseteq T + d_k$, but $\Omega + d_k$ is essentially disjoint from $T + d_j$ for all $j < k$ because for every $x \in T$ we have

$$\langle x + d_j, v\rangle = \langle x, v\rangle + \langle d_j, v\rangle < r + \langle d_k, v\rangle - 2\varepsilon,$$

while for every $y \in \Omega$ we have

$$\langle y + d_k, v\rangle = \langle y, v\rangle + \langle d_k, v\rangle > r - 2\varepsilon + \langle d_k, v\rangle.$$

Since $\sum_{j=1}^{k} c_j\chi_T(x-d_j) \in \mathbb{Z}$ almost everywhere, in particular on $\Omega+d_k$ we have $\sum_{j=1}^{k} c_j\chi_T(x-d_j) = c_k\chi_T(x-d_k) \in \mathbb{Z}$ almost everywhere. It follows from $\mu(\Omega) > 0$ that $c_k \in \mathbb{Z}$. Thus $\sum_{j=1}^{k-1} c_j\chi_T(x-d_j) \in \mathbb{Z}$ almost everywhere. The induction hypothesis now yields all $c_j \in \mathbb{Z}$.

(b) Integrating both sides of the refinement equation yields

$$\int_{\mathbb{R}^d} \chi_T = \sum_{j=1}^{n} c_j |\det(A)|^{-1} \int_{\mathbb{R}^d} \chi_T.$$

Since $\mu(T) > 0$ by assumption, we have $|\det(A)| = \sum_{j=1}^{n} c_j$. In particular it is an integer.

(c) By Lemma 2.1 we have $\sum_{\alpha\in\mathbb{Z}^d} \chi_T(x-\alpha) = \mu(T)$ almost everywhere. Hence $\mu(T) \in \mathbb{Z}$. This also shows that the integer translations of T gives a homogeneous covering of $\mathbb{R}^d$.

We next prove $T^o \neq \emptyset$ and $\mu(\partial T) = 0$. Set $g(x) = \sum_{\alpha\in\mathbb{Z}^d} \chi_T(x-\alpha)$. So $g(x) = L := \mu(T)$ almost everywhere. Let $x_0 \in T$ such that $g(x_0) = L$. We prove that x_0 is an interior point of T. Let $\alpha_1 = 0, \alpha_2, \dots, \alpha_L \in \mathbb{Z}^d$ such that $\chi_T(x_0 - \alpha_j) = 1$, in other words, $x_0 \in T + \alpha_j$, for these j's. It follows from $g(x_0) = L$ that $x_0 \notin T + \alpha$ for all α not in $\{\alpha_j\}$. Since T is compact, there exists an $\varepsilon > 0$ such that $B_\varepsilon(x_0) \cap (T + \alpha) = \emptyset$ for all α not in $\{\alpha_j\}$. Now

$$\int_{B_\varepsilon(x_0)} g(x)\,dx = \sum_{\alpha\in\mathbb{Z}^d} \int_{B_\varepsilon(x_0)} \chi_T(x-\alpha)\,dx.$$

This yields

$$L\mu(B_\varepsilon(x_0)) = \sum_{\alpha\in\mathbb{Z}^d} \mu(B_\varepsilon(x_0)\cap(T+\alpha)) \sum_{j=1}^{L} \mu(B_\varepsilon(x_0)\cap(T+\alpha_j)).$$

Thus $\mu(B_\varepsilon(x_0)\cap(T+\alpha_j)) = \mu(B_\varepsilon(x_0))$ for all j, and the compactness of T now yields $B_\varepsilon(x_0)\cap(T+\alpha_j) = B_\varepsilon(x_0)$. Thus x_0 is an interior point of $T+\alpha_j$ for all j, and in particular it is an interior point of T. Hence $T^o \neq \emptyset$. Furtheremore, since $g(x) = L$ for almost all $x \in T$, almost all points of T are in the interior of T. Hence $\mu(\partial T) = 0$. ∎

3. Linear Independence

In this section we prove Theorem 1.2. To do so we show that if $\mu(T) > 1$ then χ_T is linearly dependent. Furthermore, there exist integers $\{c_\alpha : \alpha \in \mathbb{Z}^d\}$ where not all $c_\alpha = 0$ such that $\sum_{\alpha\in\mathbb{Z}^d} c_\alpha \chi_T(x-\alpha) = 0$ for almost all $x \in \mathbb{R}^d$.

For the purpose of simplicity we shall identify a sequence $\{c_\alpha : \alpha \in \mathbb{Z}^d\}$ with a function on $\mathbb{Z}^d$. Let $\mathcal{F}$ denote the $\mathbb{Q}$-vector space of all functions on $\mathbb{Z}^d$ whose values are rationals.

Lemma 3.1. *Let T be an integral refinable set with $\mu(T) = k > 1$. Then for each $R > 0$ there exists a $g \in \mathcal{F}$ with $g(\alpha) \in \mathbb{Z} - \frac{1}{k}$ for all $\alpha \in \mathbb{Z}^d$ such that*

$$\sum_{\alpha\in\mathbb{Z}^d} g(\alpha)\chi_T(x-\alpha) = 0$$

for a.e. $x \in B_R(0)$. Note that $g(\alpha) \neq 0$ for all α.

Proof. Since T is integrably refinable we have

$$\chi_T(x) = \sum_{\alpha\in\mathbb{Z}^d} c_\alpha \chi_T(Ax-\alpha) \tag{3.1}$$

where $A \in M_d(\mathbb{Z})$ is expanding and only finitely many $c_\alpha \neq 0$. (The equality here, as well as other similar ones in the paper, should be interpreted in the a.e. sense.) Iterating (3.1) m times yields

$$\chi_T(x) = \sum_{\alpha\in\mathbb{Z}^d} c_\alpha^{(m)} \chi_T(A^m x-\alpha) \tag{3.2}$$

for some $\{c^{(m)}\}$, where again only finitely many $c^{(m)} \neq 0$. Hence

$$\chi_{A^mT}(x) = \chi_T(A^{-m}x) = \sum_{\alpha\in\mathbb{Z}^d} c_\alpha^{(m)} \chi_T(x-\alpha).$$

It follows from $\sum_{\alpha\in\mathbb{Z}^d} \chi_T(x-\alpha) = k$ that

$$\chi_{A^mT}(x) - 1 = \sum_{\alpha\in\mathbb{Z}^d} \Big(c_\alpha^{(m)} - \frac{1}{k}\Big)\chi_T(x-\alpha).$$

But T^o is nonempty, so for sufficiently large m there exists a $\beta_0 \in \mathbb{Z}^d$ such that $B_R(\beta_0) \subset A^m T^o$. Hence $\chi_{A^mT}(x) - 1 = 0$ on $B_R(\beta_0)$. Thus

$$\sum_{\alpha\in\mathbb{Z}^d} \Big(c_\alpha^{(m)} - \frac{1}{k}\Big)\chi_T(x-\alpha-\beta_0) = 0 \qquad \text{on } B_R(\beta_0).$$

Set $g \in \mathcal{F}$ by $g(\alpha) = c^{(m)}_{\alpha-\beta_0} - \frac{1}{k}$. Note that $g(\alpha) \neq 0$ because all $c^{(m)}_{\alpha-\beta_0} \in \mathbb{Z}$. This lead to the desired result. ■

We introduce the notion of a *arbitrarily extendable zero patch of* T. For this we consider functions f in $\mathcal{F}$ with $\operatorname{supp}(f) \subseteq B_m(0)$, and let Δ_m denote the indicator function of $\mathbb{Z}^d \cap B_m(0)$, i.e. $\Delta_m(\alpha) = 1$ if $\alpha \in B_m(0)$ and $\Delta_m(\alpha) = 0$ otherwise.

Definition 3.1. A function $g \in \mathcal{F}$ with $\operatorname{supp}(g) \subseteq B_m(0)$ is an *arbitrarily extendable zero patch of* T if for any $R > 0$ there exists an $f \in \mathcal{F}$ such that $g = f\Delta_m$ and $\sum_{\alpha\in\mathbb{Z}^d} f(\alpha)\chi_T(x-\alpha) = 0$ on $B_R(0)$ almost everywhere. We use $\mathcal{F}_m$ to denote the $\mathbb{Q}$-space of all arbitrarily extendable zero patch of T.

Lemma 3.2. *Let T be an integral refinable set with $\mu(T) > 1$. Then $\dim(\mathcal{F}_m) \geq 1$ for any $m > 0$.*

Proof. A function $f \in \mathcal{F}$ is called a $B_k(0)$-zero patch of T if $\sum_{\alpha\in\mathbb{Z}^d} f(\alpha)\chi_T(x-\alpha) = 0$ on $B_R(0)$ almost everywhere. Let

$$\mathcal{N}^k_m := \{g\,\Delta_m : \ g \text{ is a } B_k(0)\text{-zero patch of } T.\}$$

We have $\mathcal{N}^1_m \supseteq \mathcal{N}^2_m \supseteq \mathcal{N}^3_m \supseteq \cdots$. Clearly

$$\mathcal{F}_m = \bigcap_{k\geq 1} \mathcal{N}^k_m.$$

Suppose T is a refinable set with $\mu(T) > 1$. By Lemma 3.1 each $\mathcal{N}^k_m$ is nonzero, hence $\dim(\mathcal{N}^k_m) \geq 1$. Since they are all finite dimensional, there exists a k_0 such that $\mathcal{N}^k_m = \mathcal{N}^{k_0}_m$ for all $k \geq k_0$. Thus $\mathcal{F}_m = \mathcal{N}^{k_0}_m$ and $\dim \mathcal{F}_m \geq 1$. ■

Lemma 3.3. *Let T be an integral refinable set with $\mu(T) > 1$. For any $n \geq m \geq 1$ and $g \in \mathcal{F}_m$, there exists an $h \in \mathcal{F}_n$ such that $g = h\,\Delta_m$.*

Proof. Consider the subspace $\mathcal{G}^k_n$ of $\mathcal{N}^k_n$ given by

$$\mathcal{G}^k_n := \Big\{h \in \mathcal{N}^k_n : \ h\,\Delta_m = cg \text{ for some } c \in \mathbb{Q}\Big\}.$$

Since $g \in \mathcal{F}_n \subseteq \mathcal{N}^k_n$, there exists a $B_k(0)$-zero patch $f \in \mathcal{F}$ of T such that $f\,\Delta_m = g$. Thus

$$g = f\,\Delta_m = f\,\Delta_n\,\Delta_m = (f\,\Delta_n)\,\Delta_m.$$

Set $h = f\,\Delta_n$ we have $h \in \mathcal{N}^k_n$. Furthermore if $g \neq 0$ then $h \neq 0$. Thus $\dim(\mathcal{G}^k_n) > 0$. Again, it follows from $\mathcal{G}^1_n \supseteq \mathcal{G}^2_n \supseteq \cdots$ and the fact that all are finite dimensional that $\mathcal{G}_n := \bigcap_{k\geq 1} \mathcal{G}^k_n$

is a subspace with $\dim(\mathcal{G}_n) > 0$. Obviously, $\mathcal{G}_n \subseteq \mathcal{F}_n$. Thus there exists some $h \in \mathcal{G}_n \subseteq \mathcal{F}_n$ such that $h\,\Delta_m = g$. ∎

Proposition 3.4. *Let T be an integral refinable set with $\mu(T) > 1$. Then $\{\chi_T(x-\alpha) :\ \alpha \in \mathbb{Z}^d\}$ are linearly dependent. Furthermore, if $d = 1$ then there exist integers $\{c_\alpha :\ \alpha \in \mathbb{Z}\}$ where not all $c_\alpha = 0$ such that $\sum_{\alpha\in\mathbb{Z}} c_\alpha \chi_T(x-\alpha) = 0$ almost everywhere.*

Proof. Assume that $\mu(T) = k > 1$. By Lemma 3.2 we may find a nonzero $g_1 \in \mathcal{F}_1$. This g_1 can be extended to a $g_2 \in \mathcal{F}_2$ such that $g_1 = g_2\,\Delta_1$ by Lemma 3.3. Repeatedly applying Lemma 3.3 yields a sequence $g_n \in \mathcal{F}_n$ such that $g_m = g_n\,\Delta_m$ for all $m \leq n$. Thus $g_n \to g$ pointwise for some $g \in \mathcal{F}$. Clearly $g \neq 0$.

Now T is bounded, and suppose that $T \subseteq B_L(0)$. Then $\sum_{\alpha\in\mathbb{Z}^d} g_n(\alpha)\chi_T(x-\alpha) = 0$ a.e. on $B_{n-L}(0)$. Hence $\sum_{\alpha\in\mathbb{Z}^d} g(\alpha)\chi_T(x-\alpha) = 0$ a.e. on $\mathbb{R}^d$. Hence $\chi_T(x)$ is linearly dependent.

Finally we prove that in the case $d = 1$, g can be chosen so that $g(\alpha) \in \mathbb{Z}$ for all α. Since $g(\alpha) \in \mathbb{Q}$, there exists a $K \in \mathbb{Z}$ such that $Kg(\alpha) \in \mathbb{Z}$ for all $\alpha \in B_L(0) = (-L, L)$. Let $h = Kg$. We prove that $h(\alpha) \in \mathbb{Z}$ for all $\alpha \in \mathbb{Z}$. Assume it is false. Then $h(\alpha) \notin \mathbb{Z}$ for some $\alpha \in \mathbb{Z}$. Without loss of generality we assume that $h(\alpha) \notin \mathbb{Z}$ for some $\alpha > 0$. Let β be the smallest positive integer such that $h(\beta) \notin \mathbb{Z}$. Set $s = \operatorname{ess\,inf}(T)$. So $s + \beta = \operatorname{ess\,inf}(T + \beta)$. In the interval $(s+\beta, s+\beta+1)$ we have

$$\sum_{\alpha\in\mathbb{Z}^d} h(\alpha)\chi_T(x-\alpha) = 0.$$

However, besides $\chi_T(x-\beta)$, only $\chi_T(x-\alpha)$ with $-L < \alpha < \beta$ have support possibly intersecting the interval $(s+\beta, s+\beta+1)$. For those α we have $h(\alpha) \in \mathbb{Z}$. Thus $h(\beta) \in \mathbb{Z}$. This is a contradiction. ∎

Proof of Theorem 1.2. Clearly (b) implies (a). We first prove (a) implies (b). Since $f(x) = \chi_T(x)$ satisfies $\sum_{\alpha\in\mathbb{Z}^d} f(x-\alpha) = 1$ a.e. we know that $\{T + \alpha :\ \alpha \in \mathbb{Z}^d\}$ are essentially disjoint and T tiles $\mathbb{R}^d$ translationally by $\mathbb{Z}^d$. Next, from

$$f(A^{-1}x) = \sum_{\alpha\in\mathbb{Z}^d} c_\alpha f(x-\alpha)$$

we conclude that either $c_\alpha = 0$ or $c_\alpha = 1$. Let $\mathcal{D} = \{\alpha : \; c_\alpha = 1\}$. Then $|\mathcal{D}| = |\det(A)|$ by integrating both sides. Now, $A(T) = T + \mathcal{D}$. So T is an integral self-affine tile that tiles $\mathbb{R}^d$ by $\mathbb{Z}^d$.

We now prove that (c) is equivalent to (a) and (b). The direction (c) implies (a) is proved in Proposition 3.4. In the other direction, that (a), (b) implies (c), we notice that $\{T + \alpha : \; \alpha \in \mathbb{Z}^d\}$ are essentially disjoint. Thus, if $\sum_{\alpha \in \mathbb{Z}^d} c_\alpha f(x - \alpha) = 0$ then we must have all $c_\alpha = 0$. It follows that $f(x) = \chi_T(x)$ must be linearly independent. ∎

4. Ordinary Refinable Sets

Recall that an integral refinable set T is ordinary if it is an essenitally disjoint union of integral translates of a self-affine tile. We establish several results characterizing ordinary refinable sets.

Theorem 4.1. *Let T be an integral refinable set in $\mathbb{R}$, $a = \operatorname{ess\,inf}(T)$. Suppose that $[a, a+\varepsilon] \subset T$ for some $\varepsilon > 0$. Then T is ordinary. In fact,*

$$T = \bigcup_{j=1}^{m} ([a, a+1] + k_j)$$

for some $k_1, k_2, \ldots, k_m \in \mathbb{N}$.

Proof. Let T be λ-refinable. Since it is also λ^2-refinable we may without loss of generality assume that $\lambda > 0$. Furthermore, by a simple translation we may without loss of generality assume that $a = \operatorname{ess\,inf}(T) = 0$. In this case χ_T satisfies

$$\chi_T(x) = \sum_{j=0}^{n} c_j \chi_T(\lambda x - d_j)$$

where $0 = d_0 < d_1 < \cdots < d_n$ are in $\mathbb{Z}$.

We claim that $[0, 1] \subset T$. If not, set $\varepsilon^* = \sup\{\varepsilon : \; [0, \varepsilon] \subset T\}$. We have $\varepsilon^* < 1$ and $[0, \varepsilon^*] \subset T$ since T is closed. The refienment equation can be rewritten as

$$\chi_{\lambda T}(x) = \sum_{j=0}^{n} c_j \chi_T(x - d_j).$$

Since $d_j \geq 1$ for $j > 0$, we clearly have $\chi_{\lambda T}(x) = c_0 \chi_T(x)$ on $[0, 1]$. This yields $c_0 = 1$ and $[0, \delta] \subset T$ where $\delta = \min(1, \lambda\varepsilon^*)$, a contradiction. Hence $[0, 1] \subset T$.

Next we prove $\mu(T \cap [k, k+1]) = 1$ or 0 for all $k \in \mathbb{Z}$. This clearly holds for all $k < 0$, where T does not intersect $(-\infty, 0)$. We prove that for $k \geq 0$, $\chi_T(x)$ is constant on the interval $[k, k+1]$. Observe that $c_0 = 1$ and the refinement equation yields

$$\chi_T(x) = \chi_{\lambda T}(x) - \sum_{j=1}^{l} c_j \chi_T(x - d_j)$$

where $d_l \leq k+1 < d_{l+1}$. The fact that χ_T is a constant on $[k, k+1]$ now follows easily from induction on $k \geq$. ∎

Lemma 4.2. *Let T be an integral 2-refinable set in $\mathbb{R}$, $a = \operatorname{ess\,inf}(T)$. Then $a \in \mathbb{Z}$ and $[a, a+1] \subset T$.*

Proof. It is well known that $a = \min\{d_j\}$, where χ_T satisfies

$$\chi_T(x) = \sum_{j=0}^{n} c_j \chi_T(2x - d_j)$$

with $\{d_j\} \subset \mathbb{Z}$. Hence $a \in \mathbb{Z}$. Again, without loss of generality we may assume that $0 = d_0 < d_1 < \cdots < d_n$ are in $\mathbb{Z}$. We now only need to show that $[0, 1] \subset T$.

By a result of Ron [18] there exists an integral 2-refinable compactly supported distribution $\phi(x)$ with linearly independent integer translates $\{\phi(x - k) : \ k \in \mathbb{Z}\}$ such that

$$\chi_T(x) = \sum_{k \in \mathbb{N}} \alpha_k \phi(x - k)$$

for some real $\{\alpha_k\}$ with finitely many $\alpha_k \neq 0$. Assume that $\phi(x)$ is given by

$$\phi(x) = \sum_{j=0}^{m} a_j \chi_T(2x - b_j)$$

with $0 = b_0 < b_1 < \cdots < b_m$ in $\mathbb{Z}$. Now a result of Sun [19] (see also Bi, Huang and Sun [22]) states that $\operatorname{supp} \phi = [0, b_m]$, i.e. ϕ does not vanish on any open interval in $[0, b_m]$. Thus the support of χ_T contains $[0, 1]$, proving the lemma. ∎

Proof of Theorem 1.3. First, by Theorem 4.1 and Lemma 4.2, if T is 2-refinable then $T = [0, 1] + \mathcal{A}$ for some finite $\mathcal{A} \subset \mathbb{Z}$. This means $\chi_T(x)$ is actually a refinable spline of degree 0. Conversely, if $\chi_T(x)$ is a 2-refinable spline then it is 2-refinable, and hence T is a 2-refinable set. Thus characterizing all 2-refinable set T is equivalent to characterizing all 2-refinable splines of the form $f(x) = \sum_{\alpha \in \mathcal{A}} B_0(x - \alpha)$ with $\mathcal{A} \subset \mathbb{Z}$, where $B_0(x) := \chi_{[0,1]}$. The characterization of refinable splines in $\mathbb{R}$ are already given in Dai, Feng and Wang [**?**].

Theorem 1.3 follows directly from [?], Theorem 2.1. (Observe that using the terminology in [?] χ_T is a rational refinable spline because the refinement equation defining χ_T has rational coefficients – in fact integer coefficients.) ∎

Proof of Theorem 1.4. The proof of this theorem is virtually identical to that of Theorem 1.3. The set $T = [0,1] + \mathcal{A}$ is λ-refinable for some finite $\mathcal{A} \subset \mathbb{Z}$ if and only if $\chi_T(x) = \sum_{\alpha\in\mathcal{A}} B_0(x-\alpha)$ is a λ-refinable spline of degree 0, where $B_0(x) := \chi_{[0,1]}$. Thus characterizing all λ-refinable set T in $\mathbb{R}$ is equivalent to characterizing all λ-refinable splines of the form $f(x) = \sum_{\alpha\in\mathcal{A}} B_0(x-\alpha)$ with $\mathcal{A} \subset \mathbb{Z}$. The characterization of refinable splines in $\mathbb{R}$ are give by Theorem 2.2 in Dai, Feng and Wang [?], which immediately yields this theorem. ∎

Proof of Theorem 1.5. It is well known that $\widehat{\chi}_{T_0}(B\xi) = f_{\mathcal{D}}(\xi)\widehat{\chi}_{T_0}(\xi)$, where $B = A^T$. Now $\widehat{\chi}_T(\xi) = f_{\mathcal{A}}(\xi)\widehat{\chi}_{T_0}(\xi)$. Again, it is known that $\chi_T(x)$ is integrally refinable with dilation A if and only if there exists a trigonometric polynomial $H(\xi)$ such that $\widehat{\chi}_T(B\xi) = H(\xi)\widehat{\chi}_T(\xi)$. In other words,

$$f_{\mathcal{A}}(B\xi)\widehat{\chi}_{T_0}(B\xi) = H(\xi)f_{\mathcal{A}}(\xi)\widehat{\chi}_{T_0}(\xi).$$

Now, it follows from $\widehat{\chi}_{T_0}(B\xi) = f_{\mathcal{D}}(\xi)\widehat{\chi}_{T_0}(\xi)$ that

$$f_{\mathcal{A}}(B\xi)f_{\mathcal{D}}(\xi)\widehat{\chi}_{T_0}(\xi) = H(\xi)f_{\mathcal{A}}(\xi)\widehat{\chi}_{T_0}(\xi).$$

Hence the trigonometric polynomial $H(\xi)$ exists if and only if $f_{\mathcal{A}}(\xi)|f_{\mathcal{A}}(B\xi)f_{\mathcal{D}}(\xi)$. ∎

So far the results in this section have focused on when $T = T_0 + \mathcal{A}$ is a refinable set, where T_0 is an integral self-affine tile and $\mathcal{A} \subset \mathbb{Z}^d$ is a finite set. But an equally important questions is: Given a refinement equation, when will its solution be a characteristic function. It appears that this question is difficult in general. However for dilation 2 in $\mathbb{R}$ we have a complete classification. To state the result we first revisit a well-known method in the study of 2-refinable functions. Starting with the integral 2-refinement equation

$$f(x) = \sum_{j=0}^{N} c_j f(2x-j) \quad \text{a.e.} \tag{4.1}$$

where $c_0, c_N \neq 0$, we know that up to scalar multiples it has a unique distribution solution that is supported in $[0,N]$. Corresponding to the refinement equation (4.1) are two $N \times N$ *companion matrices*

$$A_0 = [c_{2i-j}]_{0\le i,j<N}, \quad A_1 = [c_{2i-j+1}]_{0\le i,j<N}$$

as well as a vector valued function

$$\mathbf{v}_f(t) = [f(t), f(t+1), \ldots, f(t+N-1)]^T, \quad t \in [0,1).$$

(Note that in A_0, A_1 the indices go from 0 to $N-1$ instead of the more traditional 1 to N. Also, we set $c_k = 0$ if $k < 0$ or $k > N$.) It is well-known that the refinement equation (4.1) is equivalent to the vector equation

$$\mathbf{v}_f(x) = A_{\tau(t)} \mathbf{v}_f(\sigma(t)) \quad \text{a.e.} \tag{4.2}$$

where $\tau(t) := \lfloor 2t \rfloor$ and $\sigma(t) = \{2t\}$ are the integer and fractional parts of $2t$, respectively. This vector equation was originally introduced in Daubechies and Lagarias [9, 10], and it is a powerful tool for studying many different properties of 2-refinable functions, see e.g. Wang [23].

Theorem 4.3. *Let $f(x)$ be a nonzero distribution in $\mathbb{R}$ satisfying the refinement equation*

$$f(x) = \sum_{j=0}^{N} c_j f(2x - j) \quad \text{a.e.}$$

where $c_0, c_N \neq 0$. The following are equivalent:

(a) *$f(x) = a\chi_T(x)$ a.e. for some constant $a \neq 0$ and compact set $T \subset \mathbb{R}$.*

(b) *The companion matrices $A_0 = [c_{2i-j}]_{0 \leq i,j < N}$ and $A_1 = [c_{2i-j+1}]_{0 \leq i,j < N}$ have a common eigenvector $\mathbf{v}$ with eigenvalue 1 whose entries are 0 or 1.*

(c) *The polynomial $P(z) = \sum_{j=0}^{N} c_j z^j$ has the form*

$$P(z) = \prod_{k=1}^{m} \Phi_{2^{\alpha_k} a_k}(z), \tag{4.3}$$

where all $\alpha_k > 0$ and $1 = a_1 < a_2 \leq a_3 \leq \cdots \leq a_m$ are all odd. Furthermore $\frac{q(z)}{z-1}$ is a 0-1 polynomial, where

$$q(z) := \prod_{k=1}^{m} \prod_{j=0}^{\alpha_k - 1} \Phi_{2^j a_k}(z) = \prod_{k=1}^{m} \Phi_{a_k}(z^{2^{\alpha_j - 1}}).$$

Proof. (a) $\Rightarrow$ (b). In this case T is an integral refinable set. By Theorem 1.3 we know that $T = [0,1] + \mathcal{A}$ for some finite set $\mathcal{A} \subset \mathbb{Z}$. Since $T \subseteq [0, N]$ it is clear that $\mathcal{A} \subseteq \{0, 1, \ldots, N-1\}$. Now it follows that $\mathbf{v}_f(t) = \mathbf{v}$ a.e. for some constant vector $\mathbf{v} \in \mathbb{R}^N$ whose entries are 0 or 1. Thus by (4.2) $\mathbf{v}$ is a 1-eigenvector for both A_0 and A_1.

(b) $\Rightarrow$ (a). Let $\mathbf{v} = [v_0, v_1, \ldots, v_{N-1}]^T$ be the common 1-eigenvector for A_0 and A_1 whose entries are 0 or 1. Let $T = [0,1] + \mathcal{A}$ where $\mathcal{A}$ is defined by $\mathcal{A} = \{j : \ 0 \leq j \leq N-1, v_j = 1\}$. For $f(x) = \chi_T(x)$ we clearly have $\mathbf{v}_f(t) = \mathbf{v}$ a.e. and furthermore, $\mathbf{v}_f(t)$ satisfies the equation (4.2). It follows that $f(x)$ satisfies the refinement equation.

(c) $\Rightarrow$ (a). By Theorem 2.1 in Dai, Feng and Wang [?], (4.3) implies that $f(x)$ is an integral refinable spline of degree 0. Hence $f(x) = \sum_{k\geq 0} b_k \chi_{[0,1]}(x-k)$ a.e. where all but finitely many $b_k \neq 0$. In addition, up to a constant multiple

$$\sum_{k\geq 0} b_k z^k = \frac{q(z)}{z-1}.$$

Since $\frac{q(z)}{z-1}$ is a 0-1 polynomial, up to a constant multiple $f(x) = \chi_T(x)$ a.e. where $T = [0,1] + \{k : \ b_k = 1\}$.

(a) $\Rightarrow$ (c). Again by Theorem 2.1 in Dai, Feng and Wang [?], since $f(x)$ is an integral refinable spline of degree 0 the polynomial $P(z)$ must be of the form (4.3). Furthermore, Write $T = [0,1] + \mathcal{A}$ and $f(x) = \chi_T(x) = \sum_{k\geq 0} b_k \chi_{[0,1]}(x-k)$ where $b_k = 1$ if and only if $k \in \mathcal{A}$. Theorem 2.1 in [?] also states that

$$\sum_{k\geq 0} b_k z^k = \frac{q(z)}{z-1}.$$

Hence $\frac{q(z)}{z-1}$ is a 0-1 polynomial. ∎

Example 1. We consider the refinement equation

$$f(x) = f(2x) + f(2x-2) - f(2x-3) - f(2x-5) + f(2x-6) + f(2x-8).$$

In this case $P(z) = \sum_{k\geq 0} c_k z^k = 1 + z^2 - z^3 - z^5 + z^6 + z^8$. One may verify that $P(z) = \Phi_4(z)\Phi_{18}(z)$. It satisfies the hypothesis in (c) of Theorem 4.3 with $a_1 = 1$, $a_2 = 9$, $\alpha_1 = 2$, $\alpha_2 = 1$. The corresponding $q(z) = \Phi_1(z^2)\Phi_9(z)$, yielding

$$\frac{q(z)}{z-1} = (1+z)(1+z^3+z^6) = 1 + z + z^3 + z^4 + z^6 + z^7,$$

which is a 0-1 polynomial. Hence we know the solution to the refinement equation is the charcteristic function of $T = [0,1] + \{0, 1, 3, 4, 6, 7\}$.

Alternatively we may also use the companion matrices in part (b) of Theorem 4.3, which are

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

It is easy to check that the vector $\mathbf{v} = [1,1,0,1,1,0,1,1]^T$ is a common 1-eigenvector of both A_0 and A_1. This yields the same conclusion as by part (c) of Theorem 4.3

Example 2. The coefficients of the refinement equation satisfied by a refinable set do not have to be in the set $\{0,1,-1\}$. Set $T = [0,3] + \{0,5,10,14,19,24,28,33,38\}$. Then T is a 4-refinable set. The function $f(x) = \chi_T(x)$ satisfies the refinement equation

$$f(x) = \sum_j c_j f(4x - j)$$

where the mask polynomial $p(z) = \sum_j c_j z^j$ is given by $p(z) = Q(z^4)/Q(z)$, with

$$Q(z) = (z-1)(z^2+z+1)(z^{10}+z^5+1)(1+z^{14}+z^{28}).$$

One may check with Mathematica that $c_{14} = -2$ and $c_{39} = 2$.

5. Some Open Questions

(1) Although a refinable set may not tile in the ordinary sense, it is natural to speculate that it may have a *signed tiling* of $\mathbb{R}^d$. In a signed tiling one is allowed to add as well as subtract. This concept was first introduced in Conway and Lagarias [**?**]. Here we use a broader definition of a signed tiling [1].

Definition 5.1. Let T be a compact set in $\mathbb{R}^d$ with positive Lebesgue measure. We say T is a *signed tile* of $\mathbb{R}^d$ if there exist translations $\{d_j\}$ in $\mathbb{R}^d$ and integers $\{r_j\}$ such that

$$\sum_j r_j \chi_T(x - d_j) = 1 \quad \text{a.e.} \tag{5.1}$$

We call (5.1) a *signed tiling* of $\mathbb{R}^d$ by T.

[1] In [**?**] the coefficients $\{r_j\}$ in (5.1) must be 1 or -1. We allow them to be any integers.

This leads to the following open question: *Is it true that all integral refinable sets are signed tiles?*

(2) We have proved that an integral refinable set T has nonempty interior. Is this true for all refinable sets?

(3) As mentioned in the introduction, we have not been able to find a refinable set that is not ordinary. This leads to the following question: *Is it true that all integral refinable sets are ordinary?*

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