CLASSIFICATION OF REFINABLE SPLINES

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Abstract. A refinable spline is a compactly supported refinable function that is piecewise polynomial. Refinable splines, such as the well known B-splines, play a key role in computer aided geometric designs. So far all studies on refinable splines have focused on positive integer dilations and integer translations, and under this setting a rather complete classification was obtained in [12]. However, refinable splines do not have to have integer dilations and integer translations. The classification of refinable splines with non-integer dilations and arbitrary translations are studied in this paper. We classify completely all refinable splines with integer translations with arbitrary dilations. Our study involves techniques from number theory and complex analysis.

1. Introduction

This paper studies the classification of refinable splines. Refinable functions and splines are among the most important functions, used extensively in applications such as numerical solutions to differential and integral equations, digital signal processing, image compression, and many others. Refinable splines such as the B-splines are the cornerstones in computer aided geometric designs. We aim to characterize compactly supported refinable splines under the general settings.

A compactly supported real function $f(x)$ on $\mathbb{R}$ with $\operatorname{supp}(f) = [a, b]$ is called a *spline* if there exist $a = x_0 < x_1 < \cdots < x_L = b$ and polynomials $P_j(x)$ such that $f(x) = P_j(x)$ on $[x_{j-1}, x_j)$ for $1 \leq j \leq L$. In other words, a spline is a compactly supported piecewise polynomial function. Notice that we do not assume that a spline is continuous. The points $\{x_j\}$ are called the *knots* of the spline, and $\max\{\deg(P_j)\}$ is called the *degree* of the spline.

There are a large variety of splines. Among the most useful ones are those that are also refinable. A compactly supported function $f(x)$ is *refinable* if it satisfies a *refinement*

The second author is supported in part by the Special Funds for Major State Basic Research Projects in China.

The third author is supported in part by the National Science Foundation, grants DMS-0070586 and DMS-0139261.

equation

$$f(x) = \sum_{j=0}^{n} c_j f(\lambda x - d_j), \qquad \sum_{j=0}^{n} c_j = |\lambda|, \tag{1.1}$$

where all $c_j, d_j \in \mathbb{R}$, λ is real and $|\lambda| > 1$. We call λ the *dilation factor* or simply the *dilation* for the refinable function $f(x)$, and $\{d_j\}$ the *translations* of $f(x)$. Throughout this paper we shall simply say f is λ*-refinable with translations* $\{d_j\}$. It should be noted that a refinable function has neither a unqiue dilation factor, nor a unique set of translations. A simple but important fact one observes is that if $f(x)$ satisfies (1.1) then $g(x) = f(x - \frac{b}{\lambda-1})$ satisfies the refinement equation

$$g(x) = \sum_{j=0}^{n} c_j g(\lambda x - d_j + b),$$

which has the same dilation but a new translation set $\{d_j - b\}$. As a result it is often convenient to make a translation so that 0 is the smallest element in the translations. We shall call a refinement equation in which the translation set $\{d_j\}_{j=1}^{n}$ satisfies $d_0 = 0$ and $d_j > 0$ for $j > 0$ a *normalized* refinement equation.

Refinable functions form the foundation for the theory of compactly supported wavelets and the theory of subdivision schemes. There is a vast literature on both subjects. We refer the readers to Daubechies [4] and Cavaretta, Dahmen and Micchelli [2] as well as other sources for more details. Other areas refinable functions play important roles are fractal geometry and self-affine tilings, cf. Falconer [9] and Lagarias and Wang [11].

The simplest refinable spline is the Haar function $B_0(x) = \chi_{[0,1)}(x)$, which satisfies the refinement equation $f(x) = f(2x) + f(2x-1)$. In fact B_0 is m-refinable for any integer $m > 1$, as

$$B_0(x) = \sum_{j=0}^{m-1} B_0(mx - j).$$

It is easily checked that the convolution of λ-refinable functions (resp. spline) remains a λ-refinable function (resp. spline). Thus $B_k := B_0 * B_0 * \cdots * B_0$ where B_0 convolves with itself k times is also an m-refinable spline. The spline B_k is known as the B-spline of degree k, which has knots at $0, 1, \cdots, k+1$ and is $k-1$ times differentiable. In the case $m = 2$, B_k satisfies

$$B_k(x) = \frac{1}{2^k} \sum_{j=0}^{k+1} \binom{k+1}{j} B_k(2x - j).$$

Compactly supported refinable splines, with the additional assumption that the dilation factor is a positive integer and the translations are all integers, have been classified by Lawton, Lee and Shen [12]. They proved the following theorem:

Theorem 1.1 ([12])**.** *Let $f(x)$ be a compactly supported spline of degree d. Then $f(x)$ satisfies a normalized refinement equation with integer dilation $m > 1$ and integer translations if and only if $f(x) = \sum_{n=0}^{K} p_n B_d(x-n)$ for some $K \geq 0$ and $\{p_n\}$ with $p_0 \neq 0$ such that the polynomial $Q(x) = (\sum_{n=0}^{K} p_n z^n)(z-1)^{d+1}$ satisfies $Q(z)|Q(z^m)$.*

Since any refinement equation can be normalized by a suitable translation, Theorem 1.1 in fact has also classified all refinable splines with positive integer dilations and integer translations (in fact, lattice translations, as any lattice can be transformed into $\mathbb{Z}$ by translation and scaling).

Although the above result is quite complete in its particular setting, it is interesting to note a fact — a simple fact yet it was to our knowledge hardly if ever mentioned in the wavelet and spline literature: Refinable splines do not have to have integer dilations or integer translations. One simple example is the refinement equation

$$f(x) = \frac{1}{\sqrt{2}} f(\sqrt{2}x) + \frac{1}{\sqrt{2}} f(\sqrt{2}x - 1), \tag{1.2}$$

which has the solution $f(x) = \chi_{[0,1)} * \chi_{[0,\sqrt{2})}$. By overlooking refinable functions with non-integer dilations one will miss out on certain beautiful interplay between analysis and number theory. One will also miss out on the connection of refinable functions to some of the most elegant studies on algebraic numbers and self-similar measures that began with Erdös [5, 6], followed by others such as Garcia [7], Kahane [8], Solomyak [15], Peres and Schlag [13], and many others.

The classification of refinable splines with integer dilations and integer translations reduces eventually to a problem on polynomials, which can then be solved. However, the classification of refinable splines with non-integer dilations possesses no such luxury. This is where the main difficulty lies in the classification. In this paper we develop techniques to circumvent the problem. A main theorem of ours is:

Theorem 1.2. *Suppose that $f(x)$ is a compactly supported spline satisfying the refinable equation*

$$f(x) = \sum_{j=0}^{n} c_j f(\lambda x - d_j), \qquad \sum_j c_j = |\lambda| \tag{1.3}$$

such that $\lambda \in \mathbb{R}$ and $\{d_j\} \subset \mathbb{Z}$. Then

(A) *There exists an integer $k > 0$ such that $\lambda^k \in \mathbb{Z}$.*

(B) *Let K be the smallest positive integer such that $\lambda^K \in \mathbb{Z}$. Then the compactly supported distribution solution $\phi(x)$ to the refinement equation*

$$\phi(x) = \sum_{j=0}^{n} c_j |\lambda|^{K-1} \phi(\lambda^K x - d_j) \tag{1.4}$$

is a spline.

(C) *There exists a constant α such that the spline $f(x)$ has*

$$f(x) = \alpha\phi(x) * \phi(\lambda^{-1}x) * \cdots * \phi(\lambda^{-(K-1)}x). \tag{1.5}$$

where ϕ is the spline given in (1.4).

Conversely, if the refinement equation (1.3) satisfies (A) and (B) then the compactly supported distribution solution f is a spline given by (1.5).

The above theorem classifies refinable splines for arbitrary dilations, but still requires that the translations be integers. For refinement equations with integer translations strong relations between analytic properties of the refinable functions and algebraic properties of dilation factors have been established in various contexts by the aforementioned authors. Erdös [5] proved that the refinement equation (1.4) cannot have a solution in L^1 if the dilation factor λ is a *Pisot number*, i.e. an algebraic integer whose algebraic conjugates are all inside the unit circle. In the special case of *Bernoulli Convolution*

$$f(x) = \frac{1}{2|\lambda|} f(\lambda x) + frac12|\lambda| f(\lambda x - 1),$$

Garsia [7] proved that the refinable solution $f(x)$ is in L^∞ if all algebraic conjugates of λ are outside the unit circle. Feng and Wang [10] constructed a class of non-Pisot algebraic integers λ for which the Bernoulli convolution does not have an L^2 solution. Theorem 1.2, as well as Theorem 1.3 stated next, are results in this direction.

The case with non-integer translations appears to be far more difficult. So far we do not even have an example of a λ-refinable spline that is λ-indecomposable and that has non-integer translations, whether λ is an integer or not. Here we call a λ-refinable spline $f(x)$ *λ-indecomposable* if it cannot be written as the convolution of two λ-refinable splines. We do prove the following theorem, in which for any function $f(\xi)$ defined on $\mathbb{C}$ we denote $\mathcal{Z}_f := \{\xi \in \mathbb{C} : \ f(\xi) = 0\}$.

Theorem 1.3. *Suppose that $f(x)$ is a compactly supported spline of degree d satisfying the refinable equation*

$$f(x) = \sum_{j=0}^{n} c_j f(\lambda x - d_j), \qquad \sum_j c_j = |\lambda|.$$

Then we have:

(A) *$f(x)$ is symmetric, and it is $d-1$ times continuously differentiable.*
(B) *There exists a $G(\xi) = \sum_{j=0}^{N} a_j e^{-2\pi i b_j \xi}$ where $b_j \in \mathbb{R}$ such that $\widehat{f}(\xi) = \xi^{-d-1} G(\xi)$. Furthermore, $\mathcal{Z}_G \subset \mathbb{R}$ and it contains at least one nonzero element.*
(C) *λ is an algebraic integer.*
(D) *Suppose that $\mathcal{Z}_H$ contains an arithmetic progression of length $N+1$, where $H(\xi) := |\lambda|^{-1} \sum_{j=0}^{n} c_j e^{-2\pi d_j \xi}$. Then $\lambda^k \in \mathbb{Z}$ for some positive integer k.*

We conclude this section by stating the following conjecture which, if true, classifies all refinable splines as a result of Theorem 1.2.

Conjecture 1.4. *Suppose $f(x)$ is a λ-refinable spline that is λ-indecomposable. Then the translation set for f must be contained in a lattice, i.e. a set of the form $a\mathbb{Z} + b$ for some $a \neq 0$.*

The proof of the two main theorems will be divided up and proved separately in the next three sections of this paper.

We would like to thank Qiyu Sun for helpful discussions.

2. Refinable Splines and Quasi-Trigonometric Polynomials

Let $f(x)$ be a compactly supported refinable function satisfying

$$f(x) = \sum_{j=0}^{n} c_j f(\lambda x - d_j), \qquad \sum_j c_j = |\lambda|. \tag{2.1}$$

It is well known that $\widehat{f}(\xi) = H(\lambda^{-1}\xi)\widehat{f}(\lambda^{-1}\xi)$ where $H(\xi) := |\lambda^{-1}|\sum_{j=0}^{n} c_j e^{-2\pi i d_j \xi}$, and is called the *mask* of the refinement equation. A mask is a *quasi-trigonometric polynomial*, which are functions of the form $G(\xi) = \sum_{j=0}^{N} a_j e^{-2\pi i b_j \xi}$ where $a_j \in \mathbb{C}$, $a_j \neq 0$ for all $0 \leq j \leq n$ and $b_j \in \mathbb{R}$ are distinct. We call $\max\{b_j\} - \min\{b_j\}$ the *degree* of G and denote it by $\deg(G)$. Clearly for any quasi-trigonometric polynomials we have $\deg(G_1G_2) = \deg(G_1) + \deg(G_2)$. When all $b_j \in \mathbb{Z}$, we call G a *trigonometric polynomial*.

We shall establish a stronger link between refinable splines and quasi-trigonometric polynomials. First we prove the following result, which is well known in the case of integer dilation and translations:

Proposition 2.1. *Let $f(x)$ be a compactly supported refinable spline of degree $d \geq 1$. Then $f(x)$ is $d-1$ times continuously differentiable.*

Proof. By a suitable translation we may without loss of generality assume that $f(x)$ satisfies (2.1) and it is normalized with $0 = d_0 < d_1 < \cdots < d_n$. Assume that $\lambda > 1$ (if $\lambda < 0$ we can simply iterate the refinement equation (2.1) once to make the dilation factor positive). It is well known that the $\operatorname{supp}(f) \subseteq [0, \frac{d_n}{\lambda-1}]$, and $f(x)$ is not identically 0 on $[0, \varepsilon]$ for any $\varepsilon > 0$.

Now let $0 = x_0 < x_1 < \cdots < x_L$ be the knots for $f(x)$, and $f(x) = p_j(x)$ for some polynomial $p_j(x)$ on $[x_{j-1}, x_j)$, $1 \leq j \leq n$. For sufficiently small $\varepsilon > 0$ we have $f(x) = p_1(x)$ on $[0, \varepsilon]$. However, by (2.1) we also have $f(x) = c_0 f(\lambda x - d_0) = c_0 f(\lambda x)$ on $[0, \varepsilon]$. It follows that $p_1(x) = c_0 p_1(\lambda x)$, and hence $p_1(x) = \alpha x^q$ where $\alpha \in \mathbb{R}$ and $q = -\log c_0 / \log \lambda$. So $0 \leq q \leq d$ is an integer and $c_0 = \lambda^{-q}$.

We prove that $q = d$ by proving that $\deg(p_j) \leq q$ for all j. To see this we first note that

$$f(\lambda^{-1}x) = \sum_{j=0}^{n} c_j f(x - d_j). \tag{2.2}$$

Assume that some $\deg(p_m) > q$, and without loss of generality assume that m is the smallest j with $\deg(p_j) > q$. We derive a contradiction. On the interval $I = [x_{m-1}, x_{m-1} + \varepsilon]$ where $\varepsilon > 0$ is sufficiently small we note that $f(x - d_j)$ for $j > 0$ is piecewise polynomial with degree $\leq q$. The same holds for $f(\lambda^{-1}x)$ as a result of the dilation factor. However, $f(x) = f(\lambda^{-1}x) - \sum_{j=1}^{n} c_j f(x - d_j)$ is a polynomial of degree $> q$, a contradiction. Hence $q = d$.

We next prove that $f(x)$ is $d-1$ times continuously differentiable using essentially the same argument. Note that $f(x) = p_1(x) = \alpha x^d$ on $[0, x_1)$, which implies $f(x)$ is $d-1$ times continuously differentiable on $(-\infty, x_1)$. Assume that $f(x)$ is not $d-1$ times continuously differentiable. Then the singularities must occur at some knots. Again, let x_m be the first knot at which $f^{(d-1)}(x)$ either does not exist, or is discontinuous. Then $f(x-d_j)$ with $j > 0$ and $f(\lambda^{-1}x)$ are $d-1$ times continuously differentiable on $(-\infty, x_m+\varepsilon)$ for some $\varepsilon > 0$. Again, a contradiction follows from $f(x) = f(\lambda^{-1}x) - \sum_{j=1}^n c_j f(x-d_j)$. ∎

Corollary 2.2. *Let $f(x)$ be a compactly supported refinable spline of degree $d \geq 1$. Then there exists a quasi-trigonometric polynomial $G(\xi)$ such that $G^{(j)}(0) = 0$ for all $0 \leq j \leq d$ and $\widehat{f}(\xi) = \xi^{-d-1}G(\xi)$.*

Proof. It follows from Proposition 2.1 that $f^{(d-1)}(x)$ is a compactly supported continuous spline of degree 1, which is Lipschitz. Hence $f^{(d)}(x)$ is piecewise constant. Let $f^{(d-1)}(x) = \sum_{j=0}^m \alpha_j \chi_{I_j}(x)$ where $I_j = [s_j, t_j)$. Therefore $\widehat{f^{(d)}}(\xi) = \xi^{-1}G_1(\xi)$ for some quasi-trigonometric polynomial G_1. Now $\widehat{f^{(d)}}(\xi) = (2\pi i\xi)^d \widehat{f}(\xi)$. Therefore $\widehat{f}(\xi) = \xi^{-d-1}G(\xi)$. The fact that $G^{(j)}(0) = 0$ for all $0 \leq j \leq d$ follows immediately from the fact that $\widehat{f}(\xi)$ is smooth at $\xi = 0$. ∎

Returning to quasi-trigonometric polynomials, for $G(\xi) = \sum_{j=0}^n a_j e^{-2\pi i b_j \xi}$ it is easy to see that we have a decomposition

$$G(\xi) = e^{-2\pi i s_1 \xi}G_1(\xi) + e^{-2\pi i s_2 \xi}G_2(\xi) + \cdots + e^{-2\pi i s_r \xi}G_r(\xi), \tag{2.3}$$

where each G_j is a trigonometric polynomial and $0 \leq s_j < 1$ are distinct. Furthermore up to a permutation of terms this decomposition is unique. By assuming that $0 \leq s_1 < s_2 < \cdots < s_r$ we shall call (2.2) the *standard decomposition* of $G(\xi)$. If G is a trigonometric polynomial then $r = 1$, $s_1 = 0$ and $G_1 = G$. We use $\mathbf{A}_G(\xi) = \sum_{j=0}^L c_j e^{-2\pi k_j \xi}$ to denote the greatest common divisor of the trigonometric polynomials $\{G_j\}$ normalzied so that $k_0 = 0$, $c_0 = 1$ and $k_j \geq 0$ are distinct. $\mathbf{A}_G(\xi)$ is called the *algebraic part* of $G(\xi)$. It will play an important role in this paper via the following proposition.

Proposition 2.3. *Let $G(\xi)$ be a quasi-trigonometric polynomial and $P(\xi)$ be a trigonometric polynomial. Assume that for each zero $\xi_0 \in \mathbb{C}$ of $P(\xi)$ of multiplicity k, $\xi_0 + m$ are zeros of G of multiplicity not less than k for all sufficient large integers m. Then $P(\xi)$ divides $\mathbf{A}_G(\xi)$. In particular, let $G(\xi) = \sum_{j=0}^r e^{-2\pi i s_j \xi}G_j(\xi)$ be the standard decomposition of $G(\xi)$. Then each $G_j(\xi) = P(\xi)\tilde{G}_j(\xi)$ for some trigonometric polynomial $\tilde{G}_j(\xi)$.*

Proof. Let $p(z), q(z)$ and $q_j(z)$ be Laurent polynomials such that $P(\xi) = p(e^{-2\pi i\xi})$, $\mathbf{A}_G(\xi) = q(e^{-2\pi i\xi})$ and $G_j(\xi) = q_j(e^{-2\pi i\xi})$. It suffices to prove that each root $z_0 \neq 0$ of $p(z)$ of multiplicity $k \geq 1$ must be a root of $q(z)$ of multiplicity not less than k.

Pick $\xi_0 \in \mathbb{C}$ so that $e^{-2\pi i\xi_0} = z_0$. So ξ_0 is a zero of $P(\xi) = p(e^{-2\pi i\xi})$ with multiplicity k. Thus there exists $m_0 \in \mathbb{N}$ such that $\xi_0 + m$ are zeros of $G(\xi)$ with multiplicity k for all integers $m \geq m_0$.

Now observe that for $m \geq m_0$ we have

$$G(\xi_0 + m) = \sum_{j=0}^{r} e^{-2\pi i s_j(\xi_0+m)} G_j(\xi_0 + m) = \sum_{j=0}^{r} e^{-2\pi i(m-m_0)s_j} A_j = 0, \tag{2.4}$$

where $A_j = e^{-2\pi i s_j(\xi_0+m_0)} G_j(\xi_0 + m) = e^{-2\pi i s_j(\xi_0+m_0)} G_j(\xi_0)$ are all independent of m. Let M denote the $(r+1) \times (r+1)$ Vandermonde matrix

$$M = \left(e^{-2\pi i\ell s_j}\right)_{0\leq \ell,j\leq r}.$$

Then M is non-singular because the columns are distinct. However, by taking $m = m_0, m_0 + 1, \ldots, m_0 + r$ in (2.4) it becomes $M\mathbf{v} = 0$ where $\mathbf{v} := [A_0, A_1, \ldots, A_r]^T$. Hence all $A_j = 0$, and thus all $G_j(\xi_0) = q_j(z_0) = 0$. It follows that z_0 is a root of $q(z)$. By successively dividing out the factor $(z - z_0)$ the same Vandermonde matrix argument yields that z_0 is a root of multiplicity of at least k for all $q_j(z)$ and $q(z)$. This proves the proposition. ∎

3. Algebraic Properties of Dilation Factors

In this section we establish the algebraic properties stated in the main theorems for the dilation factors of refinable splines. We shall assume throughout the section the following: $f(x)$ is a refinable spline of degree d satisfies the refinement equation (2.1) and $H(\xi) = |\lambda|^{-1} \sum_{j=0}^{n} c_j e^{-2\pi i d_j\xi}$ is the mask. By Corollary 2.2 $\widehat{f}(\xi) = \xi^{-d-1}G(\xi)$. It follows from $\widehat{f}(\xi) = H(\lambda^{-1}\xi)\widehat{f}(\lambda^{-1}\xi)$ that

$$G(\lambda\xi) = \lambda^{d+1}H(\xi)G(\xi) = P(\xi)G(\xi) \tag{3.1}$$

where $P(\xi) := \lambda^{d+1}H(\xi)$. We write $G(\xi) = \sum_{j=0}^{N} a_j e^{-2\pi i b_j\xi}$ with $a_j \neq 0$ and $b_0 < b_1 < \ldots < b_N$ For simplicity we let $\mathcal{B} := \{b_j\}$ be the set of coefficients of the exponents in G, and $\mathcal{D} := \{d_j\}$.

Lemma 3.1. *Let $\mathbb{Z}(\mathcal{B}) = \{\sum_{b\in\mathcal{B}} m_b b : \ m_b \in \mathbb{Z}\}$ be the additive subgroup of $\mathbb{R}$ generated by $\mathcal{B}$. For any $k \in \mathbb{N}$ and $j \in \{0, 1, \ldots, N\}$, there exists a $j' \in \{0, 1, \ldots, N\}$ with $j' \neq j$ such that $\lambda^k(b_j - b'_j) \in \mathbb{Z}(\mathcal{B})$.*

Proof. Fix $k \in \mathbb{N}$. Iterating (3.1) k times yields

$$G(\lambda^k\xi) = G(\xi)Q(\xi),$$

where $Q(\xi) = P(\xi)P(\lambda\xi)\cdots P(\lambda^{k-1}\xi)$. Write $Q(\xi) = \sum_{j=0}^{L} u_j e^{-2\pi i v_j\xi}$, where each $v_j \in \mathbb{R}$, $v_j \neq 0$ and $v_0 < v_1 < \cdots < v_L$. Define an equivalence relation $\sim$ on $\{0, 1, \ldots, L\}$ by setting $j \sim j'$ if $v_j - v_{j'} \in \mathbb{Z}(\mathcal{B})$. For any $\ell \in \{0, 1, \ldots, L\}$ denote by $[\![\ell]\!]$ the equivalent class containing ℓ. Let $\mathcal{U}$ be the set of all equivalent classes. Then for different equivalent classes $[\![\ell_1]\!]$ and $[\![\ell_2]\!]$ the quasi-trigonometric polynomials $\sum_{m=0}^{N} a_m e^{-2\pi i b_m\xi} \sum_{j\in[\![\ell_\epsilon]\!]} u_j e^{-2\pi i v_j\xi}$ ($\epsilon = 1, 2$) have no terms with the same exponent. Since

$$\sum_{j=0}^{N} a_j e^{-2\pi i\lambda^k b_j\xi} = \sum_{[\![\ell]\!]\in\mathcal{U}} \sum_{m=0}^{N} a_m e^{-2\pi i b_m\xi} \sum_{j\in[\![\ell]\!]} u_j e^{-2\pi i v_j\xi},$$

we deduce that for each $[\![\ell]\!] \in \mathcal{U}$, $\sum_{m=0}^{N} a_m e^{-2\pi i b_m\xi} \sum_{j\in[\![\ell]\!]} u_j e^{-2\pi i v_j\xi}$ is a quasi-trigonometric polynomial of form $\sum_{j\in\Lambda} a_j e^{-2\pi i\lambda^k b_j\xi}$, where Λ, depending on $[\![\ell]\!]$, is a subset of $\{0, 1, \ldots, N\}$ with cardinality larger than 1. For $j, j' \in \Lambda$ with $j \neq j'$, there exist $s_1, s_2 \in \{0, 1, \ldots, N\}$ and $t_1, t_2 \in [\![\ell]\!]$ such that $\lambda^k b_j = b_{s_1} + t_1$ and $\lambda^k b_{j'} = b_{s_2} + t_2$. Thus $\lambda^k b_j - \lambda^k b_{j'} = b_{s_1} - b_{s_2} + (t_1 - t_2) \in \mathbb{Z}(\mathcal{B})$. ∎

Corollary 3.2. *There exists a finite set $\mathcal{A} \subset \mathbb{R}$ such that $\lambda^k \in \mathbb{Z}(\mathcal{A})$ for every $k \in \mathbb{N}$, where $\mathbb{Z}(\mathcal{A})$ is the additive subgroup of $\mathbb{R}$ generated by $\mathcal{A}$.*

Proof. By the above lemma, we may take $\mathcal{A} = \bigcup_{j>j'} \frac{1}{b_j - b'_j}\mathcal{B}$. ∎

Proof of Theorem 1.3 (C) and (D): We first prove part (C), namely the dilation factor λ for a compactly supported refinable spline must be an algebraic integer. By Corollary 3.2 all $\lambda^k \in \mathbb{Z}(\mathcal{A})$ for some finite set $\mathcal{A}$ in $\mathbb{R}$. Consider the subgroup F of $\mathbb{Z}(\mathcal{A})$ generated by $\{\lambda^k : \ k \geq 0\}$. Since $\mathbb{Z}(\mathcal{A})$ is a finitely generated Abelian group, F must be finitely generated. In particular $F = \mathbb{Z}(1, \lambda, \ldots, \lambda^{M-1})$ for some $M > 0$. Hence $\lambda^M = \sum_{j=0}^{M-1} m_j \lambda^j$ as $\lambda^M \in F$. Thus λ is an algebraic integer, proving (C).

To prove part (D), let $\{\alpha(m + \xi_0) : \ 0 \leq m \leq N\}$ be zeros of $H(\xi)$, where $\xi_0, \alpha \in \mathbb{R}$ by part (B) of the theorem (we prove part (B) later, independently of part (D)). Since we can rescale the translations of the refinement equation without affecting the hypotheses of the theorem, we may without loss of generality assume that $\alpha = \lambda^{-1}$. It follows from $G(\lambda\xi) = \lambda^{d+1}G(\xi)H(\xi)$ that $W_k := \{\lambda^k m + \lambda^k \xi_0 :\ 0 \leq m \leq N\}$ are all zeros of $G(\xi)$ for $k \geq 0$.

Let $G(\xi) = \sum_{j=0}^r e^{-2\pi i s_j \xi} G_j(\xi)$ be the standard decomposition of $G(\xi)$. By considering the zeros $W_0 = \{m + \xi_0 :\ 0 \leq m \leq N\}$ the identical Vandermonde matrix arguments used to prove Proposition 2.3 now yields $G_j(\xi_0) = 0$ for all j. Hence each $G_j(\xi)$ contains at least two terms. It follows that for each b_j there exists a $j' \neq j$ such that $b_j - b_{j'} \in \mathbb{Z}$.

Now consider the zeros W_k of $G(\xi)$. Set $F(\xi) = G(\lambda^k \xi)$. Then $F(\xi)$ is a quasi-trigonometric polynomial with coefficients for the exponents $\{\lambda^k b_j\}$, and $W_0 = \{m+\xi_0 :\ 0 \leq m \leq N\}$ is a set of zeros for $F(\xi)$. The above result now states that for any j there exists a $j' \neq j$ such that $\lambda^k b_j - \lambda^k b_{j'} \in \mathbb{Z}$. Since there are only finitely many pairs (j, j') but there are infinitely many $k \geq 0$, we may find $j_0 \neq j_0'$ such that $\lambda^{k_1}(b_{j_0} - b_{j_0'}) \in \mathbb{Z}$ and $\lambda^{k_2}(b_{j_0} - b_{j_0'}) \in \mathbb{Z}$, $k_1 < k_2$. Thus $\lambda^{k_2-k_1} \in \mathbb{Q}$. However, λ is an algebraic integer, and so is $\lambda^{k_2-k_1}$. This yields $\lambda^{k_2-k_1} \in \mathbb{Z}$, proving part (D) of the theorem. ∎

4. Structure of Refinable Splines

We first complete the proof of Theorem 1.3. To do so we recall some fundamental results on entire functions, particularly the Weierstrass Factorization Theorem and Hadamard's Theorem. Let $f(z)$ be an entire function on $\mathbb{C}$ with nonzero roots $\{z_n\}$, where a root of multiplicity k is listed k times. Suppose that $\sum_n |z_n|^{-p-1} < \infty$, with $p \geq 0$ being the smallest such integer. Then p is called the *rank* of $f(z)$. The Weierstrass Factorization Theorem states that

$$f(z) = z^m e^{g(z)} \prod_n E_p(z/z_n),$$

where m is a nonnegative integer, g is an entire function and $E_0(z) := 1 - z$, $E_p(z) := (1-z)\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p})$ for $p > 0$. Hadamard's Theorem states that if $|f(z)| < \exp(|z|^a)$ for some $a \geq 0$ and all z with $|z|$ sufficiently large then $p \leq a$ and $g(z)$ is a polynomial of degree $\leq a$, cf. [3].

Proof of Theorem 1.3. We have already proved much of the theorem. The only remaining parts are $f(x)$ is symmetric and $\mathcal{Z}_G \setminus \{0\} \neq \emptyset$, $\mathcal{Z}_G \subset \mathbb{R}$.

We first prove $\mathcal{Z}_G \subset \mathbb{R}$. Consider $G(\xi) = \sum_{j=0}^{N} a_j e^{-2\pi i b_j \xi}$ with $b_0 < b_1 < \ldots < b_N$. Write $\xi = s + it$ we have

$$G(s+it) = \sum_{j=0}^{N} a_j e^{-2\pi i b_j s} \cdot e^{2\pi b_j t}.$$

Now $G(s+it) \neq 0$ as $t \to +\infty$ because the term $e^{2\pi b_N t}$ dominates all other terms. Similarly $G(s+it) \neq 0$ as $t \to -\infty$. It follows that $T := \sup\{|\mathrm{Im}|(\xi) : \ \xi \in \mathcal{Z}_G\} < \infty$. Assume that $T > 0$. The relation $G(\xi) = \lambda^{d+1} H(\lambda^{-1}\xi) G(\lambda^{-1}\xi)$ implies $\mathcal{Z}_G = \lambda \mathcal{Z}_H \cup \lambda \mathcal{Z}_G$. In particular

$$T \geq \sup\{|\mathrm{Im}|(\xi) : \ \xi \in \lambda \mathcal{Z}_G\} = |\lambda| T.$$

This is a contradiction, and hence $\mathcal{Z}_G \subset \mathbb{R}$. Of course, this also implies $\mathcal{Z}_H \subset \mathbb{R}$.

We next prove that $f(x)$ is symmetric and $\mathcal{Z}_G \setminus \{0\}$, $\mathcal{Z}_H \setminus \{0\}$ are both nonempty. Observe that $|H(\xi)| \leq C e^{D|\xi|}$ where $D = \max\{|d_j|\}$ and $C > 0$ is a constant. Hence $|H(\xi)| < \exp(|\xi|^a)$ for any $a > 1$ and all ξ with $|\xi|$ sufficiently large. It follows from Hadamard's Theorem that

$$H(\xi) = e^{g(\xi)} \prod_n E_p(\xi / z_n)$$

where $p \leq 1$, $g(\xi)$ is a polynomial of degree at most 1, and $\mathcal{Z}_H = \{z_n\}$. (We don't get the ξ^m term because $H(0) = 1$.) Assume that $\mathcal{Z}_H \setminus \{0\} = \emptyset$. Then $H(\xi) = e^{g(\xi)} = e^{\alpha\xi + \beta}$. So $n = 0$ and $\alpha = d_0$. This is impossible. Hence $\mathcal{Z}_H \setminus \{0\}$, and therefore $\mathcal{Z}_G \setminus \{0\}$, are nonempty.

To go further we note that if $\xi \in \mathcal{Z}_H$ then so does $-\xi$ by taking the conjugate of $H(\xi)$. So the zeros of H can be listed as $x_1, -x_1, x_2, -x_2, \ldots$. Since $E_p(\xi/x_n) E_p(-\xi/x_n) = (1 - \xi^2/x_n^2)$ for both $p = 0$ and $p = 1$, and $\prod_n (1 - \xi^2/x_n^2)$ converges absolutely and uniformly on compact sets because $\sum_n |x_n|^{-2} < \infty$, as a result of $\sum_n |x_n|^{-p-1} < \infty$, a standard argument now yields $\prod_n E_1(\xi/z_n) = \prod_n (1 - \xi^2/x_n^2)$. Therefore

$$H(\xi) = e^{\alpha\xi + \beta} \prod_n (1 - \xi^2/x_n^2).$$

Now, $H(0) = 1 = e^{\beta}$ and $H'(0) = \alpha e^{\beta} = -2\pi i (\sum_{j=0}^{n} d_j c_j)$. It follows that $\beta = 0$ and $\alpha = -2\pi i \omega$ for some $\omega \in \mathbb{R}$. In particular $\tilde{H}(\xi) := e^{2\pi i \omega \xi} H(\xi)$ satisfies $\tilde{H}(\xi) = \tilde{H}(-\xi)$. This means the mask for the refinement equation is symmetric, and therefore $f(x)$ is symmetric, see Belogay and Wang [1]. ∎

As a corollary of the symmetry of a refinable spline we have

Proposition 4.1. *Let $f(x)$ be a compactly supported λ-refinable spline. Then $f(x)$ is also $(-\lambda)$-refinable.*

Proof. Suppose that $f(x)$ satisfies

$$f(x) = \sum_{j=0}^{n} c_j f(\lambda x - d_j), \qquad \sum_j c_j = |\lambda|.$$

Then there exists an $\omega \in \mathbb{R}$ such that $\tilde{H}(\xi) = e^{2\pi i \omega \xi} H(\xi)$ has $\tilde{H}(\xi) = \tilde{H}(-\xi)$, as proved in a moment ago. Thus the refinable function given by the translated refinement equation

$$g(x) = \sum_{j=0}^{n} c_j g(\lambda x - d_j + \omega)$$

is symmetric about the origin, i.e. $g(-x) = g(x)$. Now for $\mu = -\lambda$ we have $g(\lambda x - d_j + \omega) = g(\mu x + d_j - \omega)$. Hence

$$g(x) = \sum_{j=0}^{n} c_j g(\mu x + d_j - \omega).$$

But $g(x) = f(x - \frac{\omega}{\lambda - 1})$. Thus $f(x)$ is also μ-refinable.

We point out that if the translation set $\{d_j\}$ is on a lattice then for the λ-refinement equation then so is the translation set for the new $(-\lambda)$-refinement equation. ∎

In the rest of this section we prove Theorem 1.2. The proof is presented in such a way that we shall introduce several lemmas in the middle of the proof. These lemmas will help making the proof more readable.

Proof of Theorem 1.2: We first prove the simple parts, and leave the most difficult part (B) for the last. Part (A) of the theorem is really just a corollary of Theorem 1.3 (D). Note that the mask $H(\xi)$ for the refinement equation (1.3) is a trigonometric polynomial. Therefore for any zero ξ_0 of H all $\xi_0 + m$ with $m \in \mathbb{Z}$ are zeros of H. Hence $\lambda^k \in \mathbb{Z}$ for some $k > 0$.

Assume (B), which we prove later, part (C) is easily established. To see it, let $M = \lambda^K$ where K is the smallest intger with $\lambda^K \in \mathbb{Z}$. Note that $H(\xi)$ is in fact also the mask for the M-refinement equation (1.4), so

$$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} H(M^{-j}\xi)\, \widehat{\phi}(0), \qquad \widehat{\phi}(0) \neq 0.$$

It follows that

$$\widehat{f}(\xi) = \prod_{j=1}^{\infty} H(\lambda^{-j}\xi)\,\widehat{f}(0) = c\prod_{j=0}^{K-1}\widehat{\phi}(\lambda^j\xi) \tag{4.1}$$

for some $c \neq 0$. This yields $f(x) = \alpha\phi(x) * \phi(\lambda^{-1}x) * \cdots * \phi(\lambda^{-(K-1)}x)$ for some constant α.

We now prove part (B). Without loss of generality we assume that the translation set $\{d_j\}$ of the refinement equation is normalized so that $d_0 = 0$ and $d_j > 0$ for $j > 0$. We have already shown in Theorem 1.3 that $\widehat{f}(\xi) = \xi^{-d-1}G(\xi)$ for some quasi-trigonometric polynomial $G(\xi) = \sum_{j=0}^N a_j e^{-2\pi i b_j\xi}$ where $b_0 < b_1 < \cdots < b_N$. Furthermore, $G(\lambda\xi) = \lambda^{d+1}H(\xi)G(\xi)$. We shall denote $\mathcal{B} := \{b_j\}$ in the remainder of the proof.

Lemma 4.2. (1) *If $\lambda > 1$ then $b_0 = 0$ and $b_N = \frac{d_n}{\lambda-1}$. Otherwise if $\lambda < -1$ then $b_0 = \frac{\lambda d_n}{\lambda^2-1}$ and $b_N = \frac{d_n}{\lambda^2-1}$.*

(2) *For any $b \in \mathcal{B}$ there exist two integers ℓ_1, ℓ_2 such that $\ell_1 \geq 0$, $\ell_2 \leq -d_n$ and $b - \ell_\epsilon \in \lambda\mathcal{B}$ ($\epsilon = 1, 2$).*

Proof. It is easy to obtain part (1) by comparing the largest and smallest coefficients for the exponents of $G(\lambda\xi)$ and $\lambda^{d+1}G(\xi)H(\xi)$.

To prove part (2), we define an equivalence relation $\sim$ on $\{0, 1, \ldots, N\}$ by setting $j \sim j'$ if $b_j - b_{j'} \in \mathbb{Z}$. For any $\ell \in \{0, 1, \ldots, L\}$ denote by $[\ell]$ the equivalent class containing ℓ. Let $\mathcal{E}$ be the set of all equivalent classes. Then for different equivalent classes $[\ell_1]$ and $[\ell_2]$ the quasi-trigonometric polynomials $\sum_{j=0}^N a_j e^{-2\pi i b_j\xi}H(\xi)$ ($\epsilon = 1, 2$) have no terms with the same exponent. Since

$$\sum_{j=0}^N a_j e^{-2\pi i\lambda b_j\xi} = \lambda^{d+1}\sum_{[\ell]\in\mathcal{E}}\sum_{j\in[\ell]} a_j e^{-2\pi i b_j\xi}H(\xi),$$

we deduce that for each $[\ell] \in \mathcal{E}$, $\lambda^{d+1}\sum_{j\in[\ell]} a_j e^{-2\pi i b_j\xi}H(\xi)$ is a quasi-trigonometric polynomial of form $\sum_{j\in\Lambda} a_j e^{-2\pi i\lambda b_j\xi}$, where Λ, depending on $[\ell]$, is a subset of $\{0, 1, \ldots, N\}$ with cardinality larger than 1. Let u_1, u_2 be the smallest and largest elememts in $[\ell]$. Considering the smallest and largest coefficients for the exponents of $\sum_{j\in[\ell]} a_j e^{-2\pi i b_j\xi}H(\xi)$, we have $u_1 \in \lambda\mathcal{B}$ and $u_2 + d_n \in \lambda\mathcal{B}$, and this implies part (2). ∎

Lemma 4.3. $\mathcal{B} - \mathcal{B} := \{b_j - b_{j'}\} \subset W := \left\{\sum_{j=0}^{K-1} n_j\lambda^j : \; n_j \in \mathbb{Z}\right\}$.

Proof. We first assume $\lambda > 1$. In this case $b_0 = 0$. Let p be a large integer such that $b_N < \lambda^p b_1$. For any $b \in \mathcal{B}$, by part (2) of Lemma 4.2 there exist nonnegative integers

$u_1, u_2, \ldots, u_p$ and $b' \in \mathcal{B}$ such that $b = u_1 + \lambda u_2 + \ldots + \lambda^{p-1}u_p + \lambda^p b'$. Thus $b \geq \lambda^p b'$, from which we deduce that $b' = 0$. Therefore $b = u_1 + \lambda u_2 + \ldots + \lambda^{p-1}u_p \in W$ (using the fact $\lambda^K \in \mathbb{Z}$).

Now we assume $\lambda < -1$. In this case $b_0 = \frac{\lambda}{\lambda^2-1}$. Let p be a large integer such that

$$(|\lambda| + |\lambda|^3 + \ldots + |\lambda|^{2p-1})d_n + \lambda^{2p}b_1 > b_N.$$

For any $b \in \mathcal{B}$, by part (2) of Lemma 4.2 there exist integers $u_1, u_2, \ldots, u_{2p}$ and $b' \in \mathcal{B}$ such that $b = u_1 + \lambda u_2 + \ldots + \lambda^{2p-1}u_{2p} + \lambda^{2p}b'$, where $u_j \geq 0$ for odd j and $u_j \leq -d_n$ for even j. Thus $b \geq (|\lambda| + |\lambda|^3 + \ldots + |\lambda|^{2p-1})d_n + \lambda^{2p}b'$, from which we deduce that $b' = b_0$. Therefore $b - b_0 = u_1 + \lambda u_2 + \ldots + \lambda^{2p-1}u_{2p} \in W$. This completes the proof of the lemma. ∎

Corollary 4.4. *Let L be a nonzero integer and let $F(\xi) = G(L\xi)$. Then $\mathbf{A}_F(\xi) = \mathbf{A}_G(L\xi)$.*

Proof. Let $G(\xi) = \sum_{j=0}^r e^{-2\pi i s_j \xi}G_j(\xi)$ be the standard decomposition of G. By Lemma 4.3, $s_j - s_{j'} \in W$ for any j, j'. Since $1, \lambda, \ldots, \lambda^{K-1}$ are linearly independent over $\mathbb{Q}$, $s_j - s_{j'}$ with $j \neq j'$ is either in $\mathbb{Z}$ or irrational. Hence $s_j - s_{j'}$ with $j \neq j'$ must be irrational, and thus $Ls_j - Ls_{j'}$ with $j \neq j'$ is not in $\mathbb{Z}$. Now the standard decomposition of $F(\xi)$ has

$$F(\xi) = \sum_{j=0}^r e^{-2\pi i t_j \xi}F_j(\xi) = \sum_{j=0}^r e^{-2\pi i L s_j \xi}G_j(L\xi)$$

where $t_j = \{Ls_j\}$ is the fractional part of Ls_j and $F_j(\xi) = e^{-2\pi i \lfloor Ls_j \rfloor \xi}G_j(L\xi)$. Since the extra factor $e^{-2\pi i \lfloor Ls_j \rfloor \xi}$ does not affect the greatest common divisor of $\{F_j\}$, the lemma now follows. ∎

Lemma 4.5. *Let $g(x)$ be a compactly supported distribution such that $\widehat{g}(\xi) = \xi^{-l-1}F(\xi)$ for some quasi-trigonometric polynomial $F(\xi)$ and nonnegative integer l. Then g is a spline of degree l.*

Proof. Let $F(\xi) = \sum_{j=0}^N a_j e^{-2\pi i b_j \xi}$. We have

$$\widehat{g}^{(l+1)}(\xi) = (-2\pi i \xi)^{l+1}\widehat{g}(\xi) = (-2\pi i)^{l+1}F(\xi).$$

Thus $g^{(l+1)}(x) = (-2\pi i)^{l+1}\sum_{j=0}^N a_j \delta(x - b_j)$ where δ is the Dirac function. This implies that g is piecewise polynomial. Since we already assumed that g is compactly supported, g is a spline. ∎

We now return to the proof of part (B). By Lemma 4.5 it suffices to prove that $\widehat{\phi}(\xi) = \xi^{-l-1}F(\xi)$ for some $l \geq 0$ and quasi-trigonometric polynomial $F(\xi)$. We in fact prove that

$F(\xi)$ is a trigonometric polynomial. By (4.1) we have

$$\widehat{f}(\xi) = c \prod_{j=0}^{K-1} \widehat{\phi}(\lambda^j \xi).$$

Hence

$$\xi^{-d-1} G(\xi) = c \prod_{j=0}^{K-1} \widehat{\phi}(\lambda^j \xi). \tag{4.2}$$

For simplicity and without loss of generality we assume that $\widehat{f}(0) = \widehat{\phi}(0) = 1$. This forces $c = 1$.

Lemma 4.6. *Counting multiplicity the set of nonzero zeros of* $\mathbf{A}_G$ *is precisely the set of zeros of* $\widehat{\phi}$.

Proof. We first prove that any zero ξ_0 of $\widehat{\phi}$ of multiplicity k is a zero of $\mathbf{A}_G$ of multiplicity at least k. Note that $\widehat{\phi}(0) = 1$ so $\xi_0 \neq 0$. Since $\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} H(M^{-j}\xi)$, there exists a $q > 0$ such that ξ_0 is a zero of multiplicity k for $F(\xi) := \prod_{j=1}^{q} H(M^{-j}\xi)$. Now $F(\xi)$ is M^q-periodic, so $\xi_0 + M^q m$ is a zero of F of multiplicity k for any integer m. It follows that $\xi_0 + M^q m$ is a zero of G of multiplicity at least k for any integer m, and thus $M^{-q}\xi_0 + m$ is a zero of $G_q(\xi) := G(M^q \xi)$ of multiplicity at least k for any integer m. By Proposition 2.3 $M^{-q}\xi_0$ is a zero of $\mathbf{A}_{G_q}$ of multiplicity at least k. However $\mathbf{A}_{G_q}(\xi) = \mathbf{A}_G(M^q \xi)$ by Corollary 4.4. It follows that ξ_0 is a zero of $\mathbf{A}_G$ of multiplicity at least k.

Next we show that $\mathbf{A}_G$ does not have any nonzero zeros that are not already accounted for by $\widehat{\phi}(\xi)$. The key observation is that all zeros of $\widehat{\phi}(\xi)$ must be rational, for if not we consider $\mathcal{Z}_{\widehat{\phi}} = \bigcup_{j\geq 1} M^j \mathcal{Z}_H$. If $\xi_0 \in \mathcal{Z}_H \setminus \mathbb{Q}$ for some ξ_0 then $\{M^j \xi_0\}_{j>0}$ are all in $\mathcal{Z}_{\widehat{\phi}}$, and hence are all zeros of $\mathbf{A}_G$. $\mathbf{A}_G$ is a trigonometric polynomial so that its zero set has the form $\mathcal{A} + \mathbb{Z}$ for some finite set $\mathcal{A}$. But the difference of any two elements in the infinite set $\{M^j \xi_0\}_{j>0}$ is non-integer. This is a contradiction. Thus $\mathcal{Z}_{\widehat{\phi}(\xi)} \subset \mathbb{Q}$. A consequence is that the zeros of $\widehat{\phi}(\lambda^j \xi)$ are all irrational for any $1 \leq j \leq K-1$.

To complete the proof, if $\mathbf{A}_G$ has an extra zero $\xi^* \neq 0$ then it is a zero of $\widehat{\phi}(\lambda^j \xi)$ for some $1 \leq j < K$, and it is irrational. This means $\xi^* + 1$ is a zero of $\widehat{\phi}(\lambda^{j'} \xi)$ for some $1 \leq j' < K$ since it is also a zero of $\mathbf{A}_G$ and is irrational. Hence $\xi^* = \lambda^j a$ and $\xi^* + 1 = \lambda^{j'} b$ for some rationals a, b. But this is not possible because $\lambda^{j'} b - \lambda^j a = 1$. Therefore the nonzero zeros of $\mathbf{A}_G$ and $\widehat{\phi}$, counting multiplicity, are identical. ∎.

We are ready to finish proving part (B) of the theorem. The Weierstrass Factorization Theorem implies that $\mathbf{A}_G(\xi) = \xi^m e^{g(\xi)} \prod_j E_p(\xi/z_j)$ and $\widehat{\phi}(\xi) = e^{h(\xi)} \prod_j E_p(\xi/z_j)$ for some $p \geq 0$, where g, h are entire functions and $\{z_j\}$ are the zeros of and $\widehat{\phi}$ (counting multiplicity), $m \geq 0$. Clearly $|\mathbf{A}_G(\xi)| \leq C_1 e^{D_1|\xi|}$ for some $C_1, D_1 > 0$. It is well known that $|\widehat{\phi}(\xi)| \leq C_2 e^{D_2|\xi|}$ for some $C_2, D_2 > 0$, see Daubechies [4]. As it did in the beginning of the section to prove the symmetry of a refinable spline, Hadamard's Theorem yields $g(\xi) = \alpha_1\xi + \beta_1$ and $h(\xi) = \alpha_2\xi + \beta_2$ for some constants α_j, β_j. Thus

$$\widehat{\phi}(\xi) = \xi^{-m} e^{\alpha\xi+\beta} \mathbf{A}_G(\xi) = c\xi^{-m} e^{\alpha\xi} \mathbf{A}_G(\xi). \tag{4.3}$$

Combining (4.3) with (4.2) we have

$$\xi^{-d-1} G(\xi) = c_1 \xi^{-mK} e^{\gamma\xi} \prod_{j=0}^{K-1} \mathbf{A}_G(\lambda^j \xi)$$

for some constant c_1 and $\gamma = \alpha(1+\lambda+\cdots+\lambda^{K-1})$. Thus $d+1 = mK$ and $e^{\gamma\xi} \prod_{j=0}^{K-1} \mathbf{A}_G(\lambda^j\xi)$ is a quasi-trigonometric polynomial. Hence γ is pure imaginary, and so is α. Let $\alpha = 2\pi i\omega$. Now, $\widehat{\phi}(\xi) = c\xi^{-m} e^{2\pi i\omega\xi} \mathbf{A}_G(\xi)$. So ϕ is a spline by Lemma 4.5. ∎

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