CLASSIFICATION OF REFINABLE SPLINES IN $\mathbb{R}^d$

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ABSTRACT. A refinable spline in $\mathbb{R}^d$ is a compactly supported refinable function whose support can be decomposed into simplices such that the function is a polynomial on each simplex. The best known refinable splines in $\mathbb{R}^d$ are the box splines. Refinable splines play a key role in many applications, such as numerical computation, approximation theory and computer aided geometric design. Such functions have been classified in one dimension in [6, 14]. In higher dimensions Sun [17] characterized those splines when the dilation matrices are of the form $A = mI$ where $m \in \mathbb{Z}$ and I is the identity matrix. For more general dilation matrices the problem becomes more complex. In this paper we give a complete classification of refinable splines in $\mathbb{R}^d$ for arbitrary dilation matrices $A \in M_d(\mathbb{Z})$.

1. INTRODUCTION

A compactly supported function $f(\mathbf{x})$ in $\mathbb{R}^d$ with $\mathrm{supp}\,(f) = \Omega$ is called a *spline* if there exists a partition $\Omega = \bigcup_{j=1}^N R_j$ of Ω into simplices $\{R_j\}$ in $\mathbb{R}^d$ such that f is a polynomial on each R_j. Notice that we do not assume that a spline is continuous. In this paper we study the structure of splines in $\mathbb{R}^d$ that are also refinable. Refinable functions and splines are among the most important functions, used extensively in applications such as numerical solutions to differential and integral equations, digital signal processing, image compression, and many others. Refinable splines such as the B-splines in $\mathbb{R}$ or the box splines in $\mathbb{R}^d$ play a key role in approximation theory and in computer aided geometric design. We aim to characterize compactly supported refinable splines in $\mathbb{R}^d$. Let $f(\mathbf{x})$ be a compactly supported function on $\mathbb{R}^d$ and $A \in M_d(\mathbb{Z})$ be an expanding matrix, i.e. all eigenvalues of A have $|\lambda_j| > 1$. We say $f(\mathbf{x})$ is *refinable* if it satisfies a *refinement equation*

$$f(\mathbf{x}) = \sum_{j=0}^{n} c_j f(A\mathbf{x} - \mathbf{d}_j), \tag{1.1}$$

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where all $c_j \in \mathbb{R}$ and $\mathbf{d}_j \in \mathbb{R}^d$. The matrix A is called the *dilation matrix* or simply the *dilation* for the refinable function f, and $\{\mathbf{d}_j\}$ the *translations* or *translates* of f. It is well known that if a given refinement equation has a compactly supported distribution solution then it is unique up to a constant multiple. Throughout this paper we shall simply say f is *A-refinable with translations* $\{\mathbf{d}_j\}$. It should be noted that a refinable function has neither a unique dilation factor, nor a unique set of translations. A simple but important fact one observes is that if $f(\mathbf{x})$ satisfies (1.1) then $g(\mathbf{x}) = f(\mathbf{x} - (A-I)^{-1}\mathbf{b})$ satisfies the refinement equation

$$g(\mathbf{x}) = \sum_{j=0}^{n} c_j g(A\mathbf{x} - \mathbf{d}_j + \mathbf{b}), \tag{1.2}$$

which has the same dilation but a new translation set $\{\mathbf{d}_j - \mathbf{b}\}$. In practical applications the translations $\{\mathbf{d}_j\}$ are required to be in $\mathbb{Z}^d$. In this case we shall say that the refinement equation is *integral* and the corresponding refinable function an *integral refinable function.* In this paper we shall focus mostly on integral refinable splines. Occasionally for simplicity we shall relax the condition to allow the translates $\{\mathbf{d}_j\}$ to be in $\mathbb{Q}^d$. Such a refinement equation is said to be *rational.* Clearly, classification of rational refinable splines leads to classification of integral refinable splines and vice versa by a simple rescaling. We should point out that in most of the literature the additional condition that $\sum_{j=1}^n c_j = |\det(A)|$ is imposed on the refinement equation (1.1). These are the refinement equations that are most useful in applications such as the construction of orthonormal wavelets and wavelet frames as well as subdivision algorithms. It is well known that the condition $\sum_{j=1}^n c_j = |\det(A)|$ is equivalent to $\widehat{f}(0) \neq 0$, see Daubechies and Lagarias [8].

Refinable functions form the foundation for the theory of compactly supported wavelets and the theory of subdivision schemes. There is a vast literature on both subjects. We refer the readers to Daubechies [7] and Cavaretta, Dahmen and Micchelli [2] as well as other sources for more details. Other areas refinable functions play important roles are fractal geometry and self-affine tilings, cf. Falconer [10] and Lagarias and Wang [13]. In the case of refinable splines we have an almost complete classification in one dimension, see [14, 5] and the references therein. In particular Dai, Feng and Wang [5] classified refinable splines with non-integer dilations. In higher dimension Sun [17] classified all integral refinable splines where the dilation matrices have the form $A = mI$. Refinable splines with general dilations in higher dimensions have not been classified, even in the integral case. It is often the case in the study of refinable functions and compactly supported wavelets that problems become much more complex when the dilation matrices are not of the form $A = mI$. The objective of this paper is to classify integral (rational) refinable splines in $\mathbb{R}^d$ for all dilation matrices $A \in M_d(\mathbb{Z})$.

The best known refinable splines in $\mathbb{R}^d$ are the *box splines*. Let $E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of vectors in $\mathbb{Z}^d$ that span $\mathbb{R}^d$. Here the vectors $\mathbf{v}_j$ do not have to be distinct so we count multiplicity in E. The box spline $B_E(\mathbf{x})$ with respect to E is given by

$$\widehat{B_E}(\boldsymbol{\xi}) = \prod_{j=1}^{n} \frac{1 - e^{-2\pi i \langle \mathbf{v}_j, \boldsymbol{\xi} \rangle}}{2\pi i \langle \mathbf{v}_j, \boldsymbol{\xi} \rangle}.$$

It is known that $B_E(\mathbf{x})$ is compactly supported and that it is refinable for any integer dilation matrix $A = mI$, where $|m| \geq 2$. A way to generate more refinable splines is to consider combinations of translates of a box spline. Let $L(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^d} q_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}}$ where only finitely many $q_{\boldsymbol{\alpha}} \neq 0$. Here we adopt the standard notation $\mathbf{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$ and $\mathbf{z}^k = z_1^k \cdots z_d^k$ for $\boldsymbol{\alpha} \in \mathbb{Z}^d$ and $k \in \mathbb{Z}$. $L(\mathbf{z})$ is a Laurent polynomial. Under suitable conditions $g(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^d} q_{\boldsymbol{\alpha}} B_E(\mathbf{x} - \boldsymbol{\alpha})$ is a refinable spline. Another way to construct more refinable splines is to differentiate a refinable spline. Let $\mathbf{D} = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$. For any homogeneous polynomial $P(\mathbf{z})$ of d variables the function $P(\mathbf{D})B_E$ is also a refinable spline for any dilation matrix $A = mI$, provided that the derivatives exist. Sun [17] proved the following result:

Theorem 1.1 (Sun [17])**.** *Let $A = mI_d$ with $m \in \mathbb{Z}, m > 1$ and $E = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ (counting multiplicity) be vectors in $\mathbb{Z}^d$ that span $\mathbb{R}^d$. Assume that $L(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^d} q_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}}$ is a Laurent polynomial such that $Q(\mathbf{z}) = L(\mathbf{z}) \prod_{j=1}^{n} (\mathbf{z}^{\mathbf{v}_j} - 1)$ satisfies $Q(\mathbf{z}) | Q(\mathbf{z}^m)$. Then for any homogeneous polynomial $P(\mathbf{z})$ and $\mathbf{v} = \frac{1}{m-1}\boldsymbol{\alpha}_0$ with $\boldsymbol{\alpha}_0 \in \mathbb{Z}^d$ the function*

$$f(\mathbf{x}) = P(\mathbf{D}) \Big(\sum_{\boldsymbol{\alpha} \in \mathbb{Z}^d} q_{\boldsymbol{\alpha}} B_E(\mathbf{x} - \boldsymbol{\alpha} - \mathbf{v}) \Big) \tag{1.3}$$

is an A-refinable spline with translates in $\mathbb{Z}^d$, provided that the derivatives are well defined. Conversely, any compactly supported A-refinable spline $f(\mathbf{x})$ with integer translates must be of the above form.

It should be pointed out that if we require that the refinable spline has $\widehat{f}(0) \neq 0$ then the polynomial $P(\mathbf{z})$ is a constant, for otherwise it is clear that $\widehat{f}(0) = \int f = 0$.

With a more general dilation matrix A it is possible that there exists no A-refinable splines. Key to this observation is a result of Strichartz and Wang [16] on the convex hull of a self-affine set. Note that the support of a refinable function is a self-affine set with contraction A^{-1}. For a spline it is a union of polytopes. In particular its convex hull is a polytope in $\mathbb{R}^d$. The results in [16] in fact classify all self-affine sets whose convex hulls are polytopes. Using their results we obtain

Theorem 1.2. *An expanding matrix $A \in M_d(\mathbb{Z})$ is the dilation matrix of a refinable spline $f(\mathbf{x})$ in $\mathbb{R}^d$ with integer (or rational) translates if and only if A is diagonalizable in $M_d(\mathbb{C})$ and there exists an $N > 0$ such that the eigenvalues of A^N are integers, or equivalently there exist $N > 0$ and $P \in M_d(\mathbb{Z})$ such that $P^{-1}A^NP = \operatorname{diag}(m_1, \ldots, m_d)$ where all $m_j \in \mathbb{Z}$.*

Theorem 1.2 allows us to study the structure of refinable splines in $\mathbb{R}^d$ for all admissible dilation matrices. We shall state our main results on the structure of refinable splines with integer (and rational) translates in Section 2. Sections 3 and 4 contain the proof of the main theorems.

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2. Statement of Main Results

For any $A \in M_d(\mathbb{Z})$ it is known that we can find a unimodular $P \in M_d(\mathbb{Z})$ (i.e. $\det(A) = \pm 1$) such that $P^{-1}AP$ is block upper triangular with diagonal blocks $A_1, \ldots, A_m$ such that each $A_j \in M_{r_j}(\mathbb{Z})$ and its characteristic polynomial is irreducible. Unfortunately, eigenvalues alone cannot decide integral similarity, see Newman [15]. Nevertheless, when we assume two integer matrices are diagonalizable (in $M_d(\mathbb{C})$) then they are *rationally* similar if and only if they have the same eigenvalues.

Now as a result, assume that $A \in M_d(\mathbb{Z})$ is expanding and satisfies the conditions in Theorem 1.2. We can then always find a $Q \in M_d(\mathbb{Z})$ (not necessarily unimodular) such that

$$Q^{-1}AQ = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix} \tag{2.1}$$

where $A_i \in M_{r_i}(\mathbb{Z})$ and $A_i^N = \lambda_i I_{r_i}$, $\lambda_i \in \mathbb{Z}$. Furthermore we may assume that $|\lambda_1| < |\lambda_2| < \cdots < |\lambda_m|$. For simplicity we shall refer to this block diagonal form $Q^{-1}AQ = \mathrm{diag}\,(A_1, \ldots, A_m)$ as a *standard block diagonal form.*

Theorem 2.1. *Let $f(\mathbf{x})$ be a compactly supported refinable function in $\mathbb{R}^d$ with dilation matrix $A = \mathrm{diag}\,(A_1, \ldots, A_m)$ in the standard block diagonal form.*

(A) *$f(\mathbf{x})$ is an A-refinable spline with integer (resp. rational) translates and $\widehat{f}(0) \neq 0$ if and only if for each $1 \leq i \leq m$ there exists a refinable splines $g_i(\mathbf{x}_i)$ with dilation A_i and integer (resp. rational) translates, $\widehat{g}_i(0) \neq 0$, such that $f(\mathbf{x}) = \prod_{i=1}^m g_i(\mathbf{x}_i)$.*

(B) *$f(\mathbf{x})$ is an A-refinable spline with integer (resp. rational) translates and $\widehat{f}(0) = 0$ if and only if there exists a non-constant real polynomial $P(\boldsymbol{\xi})$ with the property that $P(A^T\boldsymbol{\xi}) = CP(\boldsymbol{\xi})$ for some $C \in \mathbb{R}$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, such that $f(\mathbf{x}) = P(\mathbf{D})f^*(\mathbf{x})$ for some $f^*(\mathbf{x}) = \prod_{i=1}^m g_i(\mathbf{x}_i)$, where each $g_i(\mathbf{x}_i)$ is an A_i-refinable spline with integer (resp. rational) translates, and the derivatives are well-defined.*

(C) *$f(\mathbf{x})$ is an A-refinable spline with rational translates if and only if there exists a $K \in \mathbb{N}$ such that $g(\mathbf{x}) := f(K^{-1}\mathbf{x})$ is an A-refinable spline with integer translates.*

Remark 2.1. We clarify what we mean by the well-definedness of $f(\mathbf{x}) = P(\mathbf{D})f^*(\mathbf{x})$ in (B). The simplest way to define it is that $P(\boldsymbol{\xi})\widehat{f^*}(\boldsymbol{\xi}) \in L^2(\mathbb{R}^d)$.

Remark 2.2. Note that if $A = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_d)$ where all $\lambda_i \in \mathbb{Z}$ and $|\lambda_1| < |\lambda_2| < \cdots < |\lambda_m|$ then an A-refinable function $f(\mathbf{x})$ with $\widehat{f}(0) \neq 0$ is a spline with integer (resp. rational) translates if and only if $f(\mathbf{x}) = \prod_{i=1}^d g(x_i)$ where each g_i is a refinable spline in $\mathbb{R}$ with integer (resp. rational) translates with $\widehat{g}_i(0) \neq 0$. The structure of such splines have been completely characterized in [6]. Thus we shall refer readers to this paper as well as [14] and [5] for details about the structure of these splines.

Remark 2.3. In (B) the refinable spline f^* may not satisfy $\widehat{f^*}(0) \neq 0$. In theory for $d > 1$ it might be possible to obtain a refinable spline $f(\mathbf{x})$ such that $\widehat{f}(0) = 0$ and yet f is not a derivative of another spline. We have not been able to construct such an example.

Proposition 2.2. *Let $f(\mathbf{x})$ be a compactly supported refinable function in $\mathbb{R}^d$ with dilation matrix A and let $Q^{-1}AQ = B$ where $Q \in M_d(\mathbb{Z})$. Then*

(A) *$f(\mathbf{x})$ is an A-refinable spline with rational translates if and only if $g(\mathbf{x}) := f(Q\mathbf{x})$ is a B-refinable spline with rational translates.*

(B) *$f(\mathbf{x})$ is an A-refinable spline with integer translates if and only if $g(\mathbf{x}) := f(Q\mathbf{x})$ is a B-refinable spline with translates in $Q^{-1}\mathbb{Z}^d$.*

Theorem 1.2 and Proposition 2.2 allow us to focus on dilation matrices that are block diagonal in the standard block diagonal form. By doing so we can apply Theorem 2.1 and the next theorm to classify all refinable splines with integer and rational translates.

To complete our classification of refinable splines we now consider the case where the dilation matrix $A \in M_r(\mathbb{Z})$ has $A^K = mI$ for some $K, m \in \mathbb{Z}$, as stated in Theorem 1.1. A set $E \subset \mathbb{R}^r$ is a *basic A-cycle* if $E = \{\mathbf{a}, A\mathbf{a}, \dots, A^{p-1}\mathbf{a}\}$ for some $\mathbf{a} \in \mathbb{R}^r$ with the property that no two vectors in E are parallel and $A^p\mathbf{a} = \lambda\mathbf{a}$ for some $\lambda \in \mathbb{R}$. We say E is *A-cyclic* if E is a finite union of basic A-cycles (counting multiplicity). All A-refinable splines with integer translates are characterized by the next Theorem:

Theorem 2.3. *Let $f(\mathbf{x})$ be a compactly supported refinable function in $\mathbb{R}^r$ with dilation matrix $A \in M_r(\mathbb{Z})$ such that $A^K = mI$ for some $K \geq 1$ and $m \in \mathbb{Z}$.*

(A) *$f(\mathbf{x})$ is an A-refinable spline with integer translates.*

(B) *There exists an A-cyclic set $E = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ (counting multiplicity) with $\mathbf{a}_j \in \mathbb{Z}^r$, a homogeneous polynomial $P(\boldsymbol{\xi})$, and $\boldsymbol{\alpha}_0, \mathbf{d}_1, \dots, \mathbf{d}_k \in \mathbb{Z}^r$ such that*

$$f(\mathbf{x}) = P(\mathbf{D})\Big(\sum_{j=1}^{k} q_j B_E(\mathbf{x} - \mathbf{d}_j - (A-I)^{-1}\boldsymbol{\alpha}_0)\Big), \tag{2.2}$$

where all $q_j \in \mathbb{C}$ and the derivatives are well defined. Furthermore, the trigonometric polynomial $G(\boldsymbol{\xi}) := (\sum_{j=1}^{k} q_j e^{-2\pi i\langle \mathbf{a}_j, \boldsymbol{\xi}\rangle}) \prod_{j=1}^{n}(e^{-2\pi i\langle \mathbf{a}_j, \boldsymbol{\xi}\rangle} - 1)$ has $G(\boldsymbol{\xi}) | G(A^T\boldsymbol{\xi})$ and the polynomial $P(\boldsymbol{\xi})$ has $P(A^T\boldsymbol{\xi}) = CP(\boldsymbol{\xi})$ for some constant C.

3. Proof of Theorem 1.2

In this section we apply results from iterated function system (IFS) to prove Theorem 1.2. Given a refinable function $f(\mathbf{x})$ in $\mathbb{R}^d$ satisfying the refinement equation (1.1), let $\phi_j(\mathbf{x}) = A^{-1}(\mathbf{x} + \mathbf{d}_j)$. By a well known result of Hutchinson [12] there is a unique compact set T satisfying $T = \bigcup_{j=1}^{n} \phi_j(T)$. The set T is called the *attractor* of the IFS $\{\phi_j\}$. Let

$\Phi(S) = \bigcup_{j=1}^n \phi_j(S)$ for any compact $S \subset \mathbb{R}^d$. Then $T = \lim_{k\to\infty} \Phi^k(S_0)$ in the Hausdorff metric for any nonempty compact S_0. Now let Ω be the support of $f(\mathbf{x})$. It follows from the refinement equation (1.1) that $\Omega \subseteq \Phi(\Omega)$. Iterate it we obtain $\Omega \subset T$, where T is the attractor of the IFS $\{\phi_j\}$.

Our focus is on the convex hulls of Ω and T in the refinable spline case. Assume that $f(\mathbf{x})$ is a refinable spline, then $\mathrm{Conv}(\Omega)$ is a polytope.

Lemma 3.1. $\mathrm{Conv}(T) = \mathrm{Conv}(\Omega)$.

Proof. Let $E = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be the vertices of $\mathrm{Conv}(\Omega)$. Since $\Omega \subseteq \Phi(\Omega)$ we must have $\mathrm{Conv}(\Omega) \subseteq \mathrm{Conv}(\Phi(\mathrm{Conv}(\Omega))$. It follows that $E \subset \Phi(E)$. We show that $\Phi(E) \subseteq \mathrm{Conv}(E)$. Assume otherwise, there exists a $\mathbf{u} \in \Phi(E)$ such that $\mathbf{u}$ is outside $\mathrm{Conv}(E)$ and is a vertex of $\mathrm{Conv}(\Phi(E))$. Thus there exists an $\mathbf{a} \in \mathbb{R}^d$ such that $\langle \mathbf{a}, \mathbf{u}\rangle > \langle \mathbf{a}, \mathbf{v}\rangle$ for any $\mathbf{v} \in \Phi(E)$. Now $\mathbf{u} = \phi_i(\mathbf{u}^*) \in A^{-1}(E) + A^{-1}\mathbf{d}_i$ for some $\mathbf{u}^* \in E$ and i. It follows that $\langle \mathbf{a}, A^{-1}\mathbf{d}_i\rangle > \langle \mathbf{a}, A^{-1}\mathbf{d}_j\rangle$ for all $j \neq i$ and $\langle \mathbf{a}, A^{-1}\mathbf{u}^*\rangle > \langle \mathbf{a}, A^{-1}\mathbf{v}\rangle$ for all $\mathbf{v} \neq \mathbf{u}^*$ in E. Going back to the refinement equation (1.1) we have

$$f(\mathbf{x}) = \sum_{j=1}^n c_j f(A\mathbf{x} - \mathbf{d}_j),$$

we infer that the term $c_j f(A\mathbf{x} - \mathbf{d}_i)$, which is supported on $A^{-1}(\Omega) + \mathbf{d}_i$, cannot be cancelled out by any of the other terms in a neighborhood of $\mathbf{x} = \mathbf{u}^*$. Thus $\mathbf{u}^* \in \Omega \subseteq \mathrm{Conv}(E)$. This is a contradiction. Therefore we must have $\Phi(E) \subseteq \mathrm{Conv}(E)$. Since $E \subseteq \Phi(E)$, it follows that the vertices of $\mathrm{Conv}(\Phi(E))$ are precisely E. Iterations of E now yields $\Phi^m(E) \subseteq \mathrm{Conv}(E)$. However, $\Phi^m(E) \to T$ as $m \to \infty$ and $\Omega \subseteq T$. Thus $\mathrm{Conv}(E) = \mathrm{Conv}(\Omega) = \mathrm{Conv}(T)$. ∎

Lemma 3.2. *Assume that the translates $\{\mathbf{d}_j\}$ of the A-refinable spline $f(\mathbf{x})$ satisfying (1.1) are rational. Then the vertices of* $\mathrm{Conv}(T) = \mathrm{Conv}(\Omega)$ *are rational.*

Proof. Let $E = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be the vertices of $\mathrm{Conv}(T)$. Since $T = \Phi(T)$ we must have $\mathbf{u}_0 := \mathbf{v}_1 = \phi_{k_1}(\mathbf{u}_1)$ for some ϕ_{k_1} and some $\mathbf{u}_1 \in E$. By the same argument $\mathbf{u}_1 = \phi_{k_2}(\mathbf{u}_2)$ for some ϕ_{k_2} and $\mathbf{u}_2 \in E$. Repeating this process we obtain a sequence $\mathbf{u}_i = \phi_{k_{i+1}}(\mathbf{u}_{i+1})$ in E. Thus we have a cycle $\mathbf{u}_m = \mathbf{u}_{m+p}$. This implies that

$$\mathbf{u}_m = A^{-p}\mathbf{u}_m + \sum_{j=1}^p A^{-p+j}\mathbf{d}_{k_{m+j}}.$$

Since all $\mathbf{d}_j$ are rational, it follows that $\mathbf{u}_m$ is rational. Going backward we infer that $\mathbf{u}_{m-1}$ is also rational. It follows that $\mathbf{u}_0 = \mathbf{v}_1$ must be rational. Hence all $\mathbf{v}_j$ are rational. ■

Proof of Theorem 1.2: By Lemma 3.1 the attractor of the IFS $\{\phi_j(\mathbf{x}) := A^{-1}(\mathbf{x} + \mathbf{d}_j)\}$ has a polygonal convex hull. Furthermore by Lemma 3.2 the vertices of the convex hull are all rational. Thus the normal vectors of the faces of the convex hull can be chosen to be rational. Now by a theorem of Strichartz and Wang [16], these normal vectors must be the eigenvectors of A^k for some $k \in \mathbb{N}$. Let K the the least common multiple of these k's. Then the normal vectors of the faces of the convex hull are all eigenvectors of A^K. Now the convext hull is a d-dimensional polytope so of among the normal vectors there are d linearly independent ones. Thus A^K is diagonalizable. Furthermore, A^K is an integer matrix so the rationality of the eigenvectors implies that all eigenvalues of A^K are integers. Note the eigenvalues of A are nonzero. Hence A itself must be diagonalizable in $M_d(\mathbb{C})$.

To prove the converse we only need to construct refinable splines for each matrix satisfying the hypothesis. By Proposition 2.2 we only need to show the existence for A in the block diagonal standard form (2.1). In this case, the existence follows from Theorem 2.1 and Theorem 2.3, which give explicit structure for A-refinable splines. These results will be proved in the next section. ■

4. Proof of Main Theorems

We first introduce notation and terminology. An important class of functions in this study is the so-called *quasi-trigonometric polynomials*, which are functions of the form

$$G(\boldsymbol{\xi}) = \sum_{j=1}^{n} c_j e^{2\pi i \langle \mathbf{a}_j, \boldsymbol{\xi} \rangle},$$

where $\mathbf{a}_j \in \mathbb{R}^d$. Let F be a subset of $\mathbb{R}^d$. We use $\mathcal{T}(F)$ to denote the set of quasi-trigonometric polynomials with all $\mathbf{a}_j \in F$. Thus $\mathcal{T}(\mathbb{Z}^d)$ is the set of all trigonometric polynomials in d variables and $\mathcal{T}(\mathbb{R}^d)$ is the set of all quasi-trigonometric polynomials in d variables. A polynomial $P(\boldsymbol{\xi})$ where $\boldsymbol{\xi} = [\xi_1, \ldots, \xi_d]^T$ is a *principal homogeneous polynomial* of degree k if

$$P(\boldsymbol{\xi}) = \prod_{j=1}^{k} \langle \mathbf{a}_j, \boldsymbol{\xi} \rangle$$

for some nonzero $\mathbf{a}_1, \ldots, \mathbf{a}_k \in \mathbb{R}^d$.

One of the most important properties of the refinable function satisfying (1.1) is that the Fourier transform $\widehat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-2\pi i \langle \mathbf{x}, \boldsymbol{\xi}\rangle}\, d\mathbf{x}$ of $f(\mathbf{x})$ satisfies

$$\widehat{f}(A^T\boldsymbol{\xi}) = H(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}) \tag{4.1}$$

where $H(\boldsymbol{\xi}) = \sum_{j=1}^k c_j e^{-2\pi i \langle \mathbf{d}_j, \boldsymbol{\xi}\rangle}$. $H(\boldsymbol{\xi})$ is called the *mask* of the refinement equation (1.1). Conversely, if $f(\mathbf{x})$ satisfies (4.1) then $f(\mathbf{x})$ satisfies the refinement equation (1.1). We shall use (4.1) to classify all refinable splines in $\mathbb{R}^d$, which have rather structured Fourier transforms.

Lemma 4.1. *Let $f(\mathbf{x})$ be a compactly supported spline in $\mathbb{R}^d$. Then $\widehat{f}$ has the form*

$$\widehat{f}(\boldsymbol{\xi}) = \sum_{j=1}^n \frac{T_j(\boldsymbol{\xi})}{P_j(\boldsymbol{\xi})}, \tag{4.2}$$

where each $T_j(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{R}^d)$ and each $P_j(\boldsymbol{\xi})$ is a principal homogeneous polynomial.

Proof. See Sun [17]. ∎

Lemma 4.2. *Assume that $\widehat{f}(\boldsymbol{\xi})$ is of the form (4.2). Then there exist unique (up to scalar multiplication) polynomials $q_j(\boldsymbol{\xi})$ with $\gcd(q_j(\boldsymbol{\xi})) = 1$ and a principal homogeneous polynomial $Q(\boldsymbol{\xi})$, as well as a unique $E = \{\mathbf{a}_j\} \subset \mathbb{R}^d$ such that*

$$\widehat{f}(\boldsymbol{\xi}) = \sum_{j=1}^m \frac{q_j(\boldsymbol{\xi})}{Q(\boldsymbol{\xi})}\, e^{2\pi i \langle \mathbf{a}_j, \boldsymbol{\xi}\rangle}. \tag{4.3}$$

Furthermore, if f is A-refinable then $Q(A^T\boldsymbol{\xi}) = CQ(\boldsymbol{\xi})$ for some constant C.

Proof. We first prove that if $g(\boldsymbol{\xi}) := \sum_{j=1}^k h_j(\boldsymbol{\xi}) e^{2\pi i \langle \mathbf{b}_j, \boldsymbol{\xi}\rangle} = 0$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, where each h_j is a polynomial and $\{\mathbf{b}_j\}$ are distinct in $\mathbb{R}^d$, then all $h_j = 0$. We may assume that all h_j are nonzero. Without loss of generality we may assume that $\mathbf{b}_1$ is a vertex on the convex hull of $\{\mathbf{b}_j\}$. Thus there is an $\mathbf{a}^* \in \mathbb{R}^d$ such that $\langle \mathbf{a}^*, \mathbf{b}_1\rangle > \langle \mathbf{a}^*, \mathbf{b}_j\rangle$ for all $j > 1$. Thus for any $\mathbf{a} \in \mathbb{R}^d$ in a small neighborhood of $\mathbf{a}^*$ the inequality still holds. We can find such an $\mathbf{a} \in \mathbb{R}^d$ such that $\langle \mathbf{a}, \mathbf{b}_1\rangle > \langle \mathbf{a}, \mathbf{b}_j\rangle$ for all $j > 1$ as well as $h_j(-i\mathbf{a}) \neq 0$ for all j.

Now $g(\boldsymbol{\xi}) = 0$ in fact for all $\boldsymbol{\xi} \in \mathbb{C}^d$ because it is analytic. Take $\boldsymbol{\xi} = -it\mathbf{a}$ where $t > 0$. Then

$$e^{-2\pi t \langle \mathbf{b}_1, \mathbf{a}\rangle} g(\boldsymbol{\xi}) := h_1(-it\mathbf{a}) + \sum_{j=2}^k h_j(-it\mathbf{a}) e^{-2\pi t (\langle \mathbf{b}_1, \mathbf{a}\rangle - \langle \mathbf{b}_j, \mathbf{a}\rangle)} = 0.$$

Since each $h_j(-it\mathbf{a})$ is a polynomial of t and at $t = 1$ it is nonzero, $|h_1(-it\mathbf{a})|\to\infty$ as $t\to\infty$. On the other hand, each $h_j(-it\mathbf{a})e^{-2\pi t(\langle \mathbf{b}_1,\mathbf{a}\rangle - \langle \mathbf{b}_j,\mathbf{a}\rangle)}$ goes to 0 for $j \geq 2$. This is a contradiction.

Clearly in the above we may replace the polynomials h_j with rational functions and the conclusion still holds, because we can multiply everything by the common denominator. Thus we can write $\widehat{f}(\boldsymbol{\xi})$ uniquely as $\widehat{f}(\boldsymbol{\xi}) = \sum_{j=1}^m g_j(\boldsymbol{\xi})e^{2\pi i\langle \mathbf{a}_j,\boldsymbol{\xi}\rangle}$ where each $g_j(\boldsymbol{\xi})$ is a rational function that is a sum of reciprocals of principal homogeneous polynomials. This means that $g_j(\boldsymbol{\xi}) = h_j(\boldsymbol{\xi})/p_j(\boldsymbol{\xi})$ where $p_j(\boldsymbol{\xi})$ is a principal homogeneous polynomial. Let $Q(\boldsymbol{\xi})$ be the least common multiple of $\{p_j(\boldsymbol{\xi})\}$. Then $g_j(\boldsymbol{\xi}) = q_j(\boldsymbol{\xi})/Q(\boldsymbol{\xi})$ with $\gcd(q_j(\boldsymbol{\xi})) = 1$. The uniqueness is obvious.

Finally, if f is A-refinable there exists some $H(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{R}^d)$ such that $\widehat{f}(A^T\boldsymbol{\xi}) = H(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi})$. Thus

$$\widehat{f}(A^T\boldsymbol{\xi}) = \sum_{j=1}^m \frac{q_j(A^T\boldsymbol{\xi})}{Q(A^T\boldsymbol{\xi})}\, e^{2\pi i\langle \mathbf{a}_j, A^T\boldsymbol{\xi}\rangle}.$$

On the other hand,

$$H(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}) = \sum_{j=1}^m \frac{q_j(\boldsymbol{\xi})}{Q(\boldsymbol{\xi})}\, H(\boldsymbol{\xi})e^{2\pi i\langle \mathbf{a}_j,\boldsymbol{\xi}\rangle}.$$

The uniqueness of $Q(\boldsymbol{\xi})$ and $q_j(\boldsymbol{\xi})$ now implies that $Q(A^T\boldsymbol{\xi}) = CQ(\boldsymbol{\xi})$. ∎

We shall refer to (4.3) as the *standard representation* of $\widehat{f}$.

Lemma 4.3. *Let $Q(\boldsymbol{\xi}) = \prod_{j=1}^n \langle \mathbf{v}_j, \boldsymbol{\xi}\rangle$ be a principal homogeneous polynomial where $\mathbf{v}_j \in \mathbb{R}^d$. Let A be an expanding matrix in $M_d(\mathbb{R})$ such that $Q(A^T\boldsymbol{\xi}) = CQ(\boldsymbol{\xi})$. Then each $\mathbf{v}_j$ is an eigenvector of A^k for some $k \in \mathbb{N}$.*

Proof. Assume that some $\mathbf{v}_j$, say $\mathbf{v}_1$, is not an eigenvector of A^k for some k. Note that

$$Q(A^T\boldsymbol{\xi}) = \prod_{j=1}^n \langle A\mathbf{v}_j, \boldsymbol{\xi}\rangle = C\prod_{j=1}^n \langle \mathbf{v}_j, \boldsymbol{\xi}\rangle. \tag{4.4}$$

Thus $\langle A\mathbf{v}_1, \boldsymbol{\xi}\rangle = c_1\langle \mathbf{v}_i, \boldsymbol{\xi}\rangle$ for some $\mathbf{v}_i$ and hence $\mathbf{v}_i = c_1 A\mathbf{v}_1$. By the same token another $\mathbf{v}_{i'}$ is parallel to $\mathbf{v}_i$ and hence $A^2\mathbf{v}_1$. It follows that there is a vector in $\{\mathbf{v}_j\}$ that is parallel to $A^k\mathbf{v}_1$ for each k. But all $A^k\mathbf{v}_1$ are pairwise non-parallel. This is a contracdition. ∎

Going back to the standard representation (4.3), note that as stated in the proof of Lemma 4.2 each $q_j(\boldsymbol{\xi})/Q(\boldsymbol{\xi})$ can be written as sum of reciprocals of principal homogeneous polynomials. Clearly these principal homogeneous polynomials are factors of $Q(\boldsymbol{\xi})$. Thus we can rewrite (4.3) as

$$\widehat{f}(\boldsymbol{\xi}) = \sum_{j=1}^{n} \frac{T_j(\boldsymbol{\xi})}{Q_j(\boldsymbol{\xi})}, \tag{4.5}$$

where $T_j(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{R}^d)$ and $Q_j(\boldsymbol{\xi})$ is a principal homogeneous polynomial. Furthermore we may assume that $\{Q_1^{-1}(\boldsymbol{\xi})\}$ are linearly independent, for otherwise we can regroup to use only those $Q_j^{-1}(\boldsymbol{\xi})$ that are linearly independent from the others.

Lemma 4.4. *Let $A = \operatorname{diag}(A_1, \dots, A_m)$ be expanding such that $A_i \in M_{r_i}(\mathbb{R})$ satisfies $A^K = \lambda_i I_{r_i}$, where all $|\lambda_i|$ are distinct. Let $f(\mathbf{x})$ be an A-refinable spline (not necessarily with integer translates) with $\widehat{f}$ given by (4.5) such that $\{Q_j^{-1}(\boldsymbol{\xi})\}$ are linearly independent. Then there exist constants c_j, C_j and $H^*(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{R}^d)$ such that $Q_j((A^T)^K\boldsymbol{\xi}) = c_j\, Q_j(\boldsymbol{\xi})$ and $T_j((A^T)^K\boldsymbol{\xi}) = C_j\, H^*(\boldsymbol{\xi})T_j(\boldsymbol{\xi})$. Furthermore each $Q_j(\boldsymbol{\xi}) = Q_j(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_m)$ has the form $Q_j(\boldsymbol{\xi}) = \prod_{i=1}^m q_{ji}(\boldsymbol{\xi}_i)$ where $\boldsymbol{\xi}_i \in \mathbb{R}^{r_i}$.*

Proof. By Lemma 4.3 $Q(\boldsymbol{\xi})$ is a principal homogeneous polynomial $Q(\boldsymbol{\xi}) = \prod_{j=1}^n \langle \mathbf{v}_j, \boldsymbol{\xi}\rangle$ where each $\mathbf{v}_j$ is an eigenvector of A^k for some k. It follows immediately that $Q(\boldsymbol{\xi}) = \prod_{i=1}^k R_i(\boldsymbol{\xi}_i)$ for some principal homogeneous polynomials $R_i(\boldsymbol{\xi}_i)$, $\boldsymbol{\xi}_i \in \mathbb{R}^{r_i}$. Now each $Q_j(\boldsymbol{\xi})|Q(\boldsymbol{\xi})$. Thus each $Q_j(\boldsymbol{\xi}) = Q_j(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_m)$ has the form $Q_j(\boldsymbol{\xi}) = \prod_{i=1}^m q_{ji}(\boldsymbol{\xi}_i)$, with $\boldsymbol{\xi}_i \in \mathbb{R}^{r_i}$. It also follows that $Q_j((A^T)^K\boldsymbol{\xi}) = c_j\, Q_j(\boldsymbol{\xi})$ for some c_j.

To complete the proof, observe that given the linear independence of $\{Q_j^{-1}\}$ the representation (4.5) is unique. This can be seen from a term by term comparison. We only need to show that if $\sum_{j=1}^n S_j(\boldsymbol{\xi})Q_j^{-1}(\boldsymbol{\xi}) \equiv 0$ where $S_j \in \mathcal{T}(\mathbb{R}^d)$ then all $S_j(\boldsymbol{\xi}) \equiv 0$. First, by Lemma 4.2 we must have $\sum_{j=1}^n a_j Q_j^{-1}(\boldsymbol{\xi}) \equiv 0$, where a_j is the constant term of S_j. The linear independence now yields all $a_j = 0$. The same argument applies to all terms to obtain $S_j \equiv 0$. The uniqueness now follows.

Now we have $\widehat{f}((A^T)^K\boldsymbol{\xi}) = H^*(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi})$ where $H^*(\boldsymbol{\xi}) = \prod_{j=0}^{K-1} H((A^T)^j\boldsymbol{\xi})$. It follows that

$$\sum_{j=1}^{n} \frac{T_j((A^T)^K\boldsymbol{\xi})}{Q_j((A^T)^K\boldsymbol{\xi})} = \sum_{j=1}^{n} \frac{T_j((A^T)^K\boldsymbol{\xi})}{c_j Q_j(\boldsymbol{\xi})} = \sum_{j=1}^{n} \frac{H^*(\boldsymbol{\xi})T_j(\boldsymbol{\xi})}{Q_j(\boldsymbol{\xi})}.$$

The uniqueness now yields $T_j((A^T)^K\boldsymbol{\xi}) = C_j\, H^*(\boldsymbol{\xi})T_j(\boldsymbol{\xi})$, with $C_j = 1/c_j$. ∎

Lemma 4.5. *Let $H(\boldsymbol{\xi}), G_1(\boldsymbol{\xi}), G_2(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{R}^d)$ and not identically 0. Let $A \in M_d(\mathbb{R})$ be expanding such that $G_i(A^T\boldsymbol{\xi}) = C_i H(\boldsymbol{\xi}) G_i(\boldsymbol{\xi})$ for $i = 1, 2$, where $C_i \in \mathbb{C}$. Then $C_1 = C_2$ and $G_1(\boldsymbol{\xi}) = CG_2(\boldsymbol{\xi})$ for some $C \in \mathbb{C}$.*

Proof. Let $G_1(\boldsymbol{\xi}) = \sum_{j=1}^n c_j e^{2\pi i\langle \mathbf{a}_j, \boldsymbol{\xi}\rangle}$ and $G_2(\boldsymbol{\xi}) = \sum_{j=1}^m d_j e^{2\pi i\langle \mathbf{b}_j, \boldsymbol{\xi}\rangle}$, where $c_j, d_j \neq 0$. Set $K = \max_{j,i}(|\mathbf{a}_j - \mathbf{b}_i|)$. We first prove $\{\mathbf{a}_j\} = \{\mathbf{b}_j\}$. If not, we may assume without loss of generality that $\mathbf{a}_1 \notin \{\mathbf{b}_j\}$. Since A is expanding, we may choose k sufficiently large such that $|A^k\mathbf{a}_1 - A^k\mathbf{b}_i| > 2K$ for all i and $|A^k\mathbf{a}_1 - A^k\mathbf{a}_j| > 2K$ for all $j > 1$. Note that $G_1((A^T)^k\boldsymbol{\xi})/G_2((A^T)^k\boldsymbol{\xi}) = C'G_1(\boldsymbol{\xi})/G_2(\boldsymbol{\xi})$, where $C' = C_1/C_2$. Hence $G_1((A^T)^k\boldsymbol{\xi})G_2(\boldsymbol{\xi}) = C'G_2((A^T)^k\boldsymbol{\xi})G_1(\boldsymbol{\xi})$. It follows that

$$\sum_{j=1}^n c_j e^{2\pi i\langle A^k\mathbf{a}_j, \boldsymbol{\xi}\rangle} \sum_{j=1}^m d_j e^{2\pi i\langle \mathbf{b}_j, \boldsymbol{\xi}\rangle} = C' \sum_{j=1}^m d_j e^{2\pi i\langle A^k\mathbf{b}_j, \boldsymbol{\xi}\rangle} \sum_{j=1}^n c_j e^{2\pi i\langle \mathbf{a}_j, \boldsymbol{\xi}\rangle}.$$

Note that the term $c_1 d_1 e^{2\pi i\langle A^k\mathbf{a}_1, \boldsymbol{\xi}\rangle} e^{2\pi i\langle \mathbf{b}_1, \boldsymbol{\xi}\rangle}$ cannot be cancelled out by any other terms. This is a contradiction. Thus $\{\mathbf{a}_j\} = \{\mathbf{b}_j\}$. Without loss of generality we may assume that $\mathbf{a}_j = \mathbf{b}_j$. The same argument now implies that by taking k sufficiently large we get $c_i d_j = C' d_i c_j$ for all i, j. Thus $C' = 1$ and $c_j = Cd_j$ with $C = c_1/d_1$. ∎

Lemma 4.6. *Let $H(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{Z}^d)$ and $T(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{R}^d)$ such that $T(A^T\boldsymbol{\xi}) = cH(\boldsymbol{\xi})T(\boldsymbol{\xi})$ for some $c \in \mathbb{C}$, where $A \in M_d(\mathbb{Z})$ is expanding. Then there exists some $\boldsymbol{\alpha}_0 \in \mathbb{Z}^d$ and $\mathbf{v} = (A - I)^{-1}\boldsymbol{\alpha}_0$ such that $e^{-2\pi i\langle \mathbf{v}, \boldsymbol{\xi}\rangle}T(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{Z}^d)$.*

Proof. There exist unique $\mathbf{a}_1, \dots, \mathbf{a}_n \in [0, 1)^d$ such that $T(\boldsymbol{\xi}) = \sum_{j=1}^n c_j e^{2\pi i\langle \mathbf{a}_j, \boldsymbol{\xi}\rangle} T_j(\boldsymbol{\xi})$, where each $T_j(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{Z}^d)$. It follows from $T(A^T\boldsymbol{\xi}) = cH(\boldsymbol{\xi})T(\boldsymbol{\xi})$ that

$$\sum_{j=1}^n c_j e^{2\pi i\langle A\mathbf{a}_j, \boldsymbol{\xi}\rangle} T_j(A^T\boldsymbol{\xi}) = c \sum_{j=1}^n c_j e^{2\pi i\langle \mathbf{a}_j, \boldsymbol{\xi}\rangle} H(\boldsymbol{\xi}) T_j(\boldsymbol{\xi}).$$

Observe that $T_j(A^T\boldsymbol{\xi})$ and $H(\boldsymbol{\xi})T_j(\boldsymbol{\xi})$ are all trigonometric polynomials. So $\{A\mathbf{a}_j \pmod 1\} = \{\mathbf{a}_j\}$. In other words $A\mathbf{a}_1 \pmod 1, \dots, A\mathbf{a}_n \pmod 1$ is a permutation of $\mathbf{a}_1, \dots, \mathbf{a}_n$. Let K be the order of this permutation. Then $A^K\mathbf{a}_j \pmod 1 = \mathbf{a}_j$ for all j, i.e. $A^K\mathbf{a}_j = \mathbf{a}_j + \boldsymbol{\alpha}_j$ for some $\boldsymbol{\alpha}_j \in \mathbb{Z}^d$. Set $H_K(\boldsymbol{\xi}) = c^K \prod_{j=0}^{K-1} H((A^T)^j\boldsymbol{\xi})$. Then $H_K \in \mathcal{T}(\mathbb{Z}^d)$ and $T(B^T\boldsymbol{\xi}) = H_K(\boldsymbol{\xi})T(\boldsymbol{\xi})$ where $B = A^K$. This yields

$$\sum_{j=1}^n c_j e^{2\pi i\langle \mathbf{a}_j, \boldsymbol{\xi}\rangle} H_K(\boldsymbol{\xi}) T_j(\boldsymbol{\xi}) = \sum_{j=1}^n c_j e^{2\pi i\langle B\mathbf{a}_j, \boldsymbol{\xi}\rangle} T_j(B^T\boldsymbol{\xi}) = \sum_{j=1}^n c_j e^{2\pi i\langle \mathbf{a}_j, \boldsymbol{\xi}\rangle} e^{2\pi i\langle \boldsymbol{\alpha}_j, \boldsymbol{\xi}\rangle} T_j(B^T\boldsymbol{\xi}).$$

Hence we must have $T_j(B^T\boldsymbol{\xi}) = e^{-2\pi i\langle \boldsymbol{\alpha}_j, \boldsymbol{\xi}\rangle} H_k(\boldsymbol{\xi}) T_j(\boldsymbol{\xi})$.

Set $T_j^*(\boldsymbol{\xi}) = e^{-2\pi i\langle \mathbf{u}_j,\boldsymbol{\xi}\rangle}T_j(\boldsymbol{\xi})$ where $\mathbf{u}_j = (B-I)^{-1}\boldsymbol{\alpha}_j$. It is easy to check that $T_j^*(B^T\boldsymbol{\xi}) = H_K(\boldsymbol{\xi})T_j^*(\boldsymbol{\xi})$. However, $T(\boldsymbol{\xi})$ satisfies the same equation. It follows from Lemma 4.5 that $T(\boldsymbol{\xi}) = C_jT_j^*(\boldsymbol{\xi}) = C_je^{-2\pi i\langle \mathbf{u}_j,\boldsymbol{\xi}\rangle}T_j(\boldsymbol{\xi})$. This immediately implies that $n = 1$ and $T(\boldsymbol{\xi}) = C_1e^{-2\pi i\langle \mathbf{u}_1,\boldsymbol{\xi}\rangle}T_1(\boldsymbol{\xi})$, where $T_1(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{Z}^d)$. Now

$$T(A^T\boldsymbol{\xi}) = C_1e^{-2\pi i\langle A\mathbf{u}_1,\boldsymbol{\xi}\rangle}T_1(A^T\boldsymbol{\xi}) = cH(\boldsymbol{\xi})T(\boldsymbol{\xi}) = cC_1H(\boldsymbol{\xi})e^{-2\pi i\langle \mathbf{u}_1,\boldsymbol{\xi}\rangle}T_1(\boldsymbol{\xi}).$$

Since both $T(\boldsymbol{\xi})$ and $T_1(A^T\boldsymbol{\xi})$ are trigonometric polynomials we must have $A\mathbf{u}_1 = \mathbf{u}_1 - \boldsymbol{\alpha}_0$ for some $\boldsymbol{\alpha}_0 \in \mathbb{Z}^d$. Setting $\mathbf{v} = -\mathbf{u}_1$ now proves the lemma. ∎

Lemma 4.7. *Let $\lambda_i \in \mathbb{N}$ such that $1 < \lambda_1 < \cdots < \lambda_m$. Let $P(\mathbf{z}) = P(\mathbf{z}_1,\dots,\mathbf{z}_m)$ be a polynomial, $\mathbf{z}_i \in \mathbb{R}^{r_i}$, such that $P(\mathbf{z})$ divides $P(\mathbf{z}_1^{\lambda_1},\dots,\mathbf{z}_m^{\lambda_m})$. Then there exist polynomials $P_1(\mathbf{z}_1),\dots,P_m(\mathbf{z}_m)$ such that $P(\mathbf{z}_1,\dots,\mathbf{z}_m) = \prod_{j=1}^m P_j(\mathbf{z}_m)$.*

Proof. We shall prove that $P(\mathbf{z}_1,\dots,\mathbf{z}_m) = P_m(\mathbf{z}_m)\tilde{P}(\mathbf{z}_1,\dots,\mathbf{z}_{m-1})$. We prove it first for the case where $\mathbf{z}_m$ is a scalar variable, i.e. $r_m = 1$. We shall use z_m for $\mathbf{z}_m$.

Write $P(\mathbf{z}) = \sum_{k=1}^N c_kp_k(z_m)\mathbf{w}^{\boldsymbol{\beta}_k}$ where $\mathbf{w} := (\mathbf{z}_1,\dots,\mathbf{z}_{m-1})$, $\boldsymbol{\beta}_k \in \mathbb{Z}^{d-r_m}$, $c_k \neq 0$ and $p_k(z_m)$ are monic polynomials. Let $L = \max_k \|\boldsymbol{\beta}_k\|_1$ where $\|.\|_1$ denotes the l^1-norm. Let $Q_n(\mathbf{w},z_m) = P(\mathbf{z}_1^{\lambda_1^n},\dots,\mathbf{z}_{m-1}^{\lambda_{m-1}^n},z_m^{\lambda_m^n})$. Then $P|Q_n$ and $Q_n(\mathbf{w},1)$ is a polynomial of $\mathbf{w}$ of degree no more than $L\lambda_{m-1}^n$. Now for sufficiently large n let ω be a primitive λ_d^n-th root of unity. Then $Q_n(\mathbf{w},\omega^j) = Q_n(\mathbf{w},1)$ for all $0 \le j < \lambda_m^n$. Thus $P(\mathbf{w},\omega^j)$ is a factor of $Q_n(\mathbf{w},1)$. But up to constant multiples $Q_n(\mathbf{w},1)$ cannot have more than $L\lambda_{m-1}^n + 1$ factors. Thus there exist $j_1,\dots,j_K$ with $K = \lfloor \lambda_m^n/(L\lambda_{m-1}^n+1)\rfloor$ such that up to constant multiples all $P(w,\omega^{j_k})$ are the same. Thus the vectors $\{[p_1(\omega^{j_k}),\dots,p_N(\omega^{j_k})]^T\}$ are pairwise parallel. However, by taking n sufficiently large K can be arbitrarily large (greater than $2\max_{1\le j\le N}(\deg p_j)$). This immediately yields that the vectors $\{[p_1(z_m),\dots,p_N(z_m)]^T : z_m \in \mathbb{C}\}$ are all parallel. This is only possible if all $p_j(z_m)$ are scalar multiples of one another. Since all p_k are monic, it follows that all of them are equal, $p_j(z_m) = P_m(z_m)$ for some P_m. Hence $P(\mathbf{w},z_m) = \tilde{P}(\mathbf{w})P_m(z_m)$.

In the case $r_m > 1$ let $M = \deg(P) + 1$. For $\mathbf{z}_m = (s_1,\dots,s_{r_m})$ we make the substitution $\mathbf{z}_m \mapsto (t^{M^0},t^{M^1},\dots,t^{M^{r_m-1}})$. With this substitution we obtain the polynomial $Q(\mathbf{w},t) := P(\mathbf{w},\mathbf{z}_m)$ with $\mathbf{z}_m = (t^{M^0},t^{M^1},\dots,t^{M^{r_m-1}})$. Observe that after the substitution a term $\mathbf{w}^{\boldsymbol{\alpha}}\mathbf{z}_m^{\boldsymbol{\beta}}$ is mapped to some $\mathbf{w}_{\boldsymbol{\alpha}}t^{n_{\boldsymbol{\beta}}}$, and the map is one-to-one. Now clearly Q divides $Q(\mathbf{z}_1^{\lambda_1},\dots,\mathbf{z}_{m-1}^{\lambda_{m-1}},t^{\lambda_m})$. Thus $Q(\mathbf{w},t) = \tilde{P}(\mathbf{w})Q_m(t)$. Now all powers of t come from $Q_m(t)$, and by the fact that the map of $\mathbf{w}^{\boldsymbol{\alpha}}\mathbf{z}_m^{\boldsymbol{\beta}}$ to some $\mathbf{w}^{\boldsymbol{\alpha}}t^{n_\beta}$ is one-to-one we can reconstruct the polynomial P from Q. It follows that $P(\mathbf{z}_1,\dots,\mathbf{z}_m)=P_m(\mathbf{z}_m)\tilde{P}(\mathbf{z}_1,\dots,\mathbf{z}_{m-1})$.

The rest of the lemma follows from induction on m. ∎

We now come to our most important lemma, which summarizes what we have proved in previous lemmas.

Lemma 4.8. *Let $A=\operatorname{diag}(A_1,\dots,A_m)\in M_d(\mathbb{Z})$ such that $A_i^K=\lambda_i I_{r_i}$ and all $|\lambda_i|\in\mathbb{Z}$ distinct. Let $f(\mathbf{x})$ be an A-refinable spline with integer translates. Then*

$$\widehat{f}(\boldsymbol{\xi})=\frac{T(\boldsymbol{\xi})P(\boldsymbol{\xi})}{Q(\boldsymbol{\xi})}, \tag{4.6}$$

where $T(\boldsymbol{\xi})\in\mathcal{T}(\mathbb{R}^d)$, $Q(\boldsymbol{\xi})$ is a principal polynomial and $P(\boldsymbol{\xi})$ is a polynomial. Furthermore they satisfy the following properties:

(A) *There exist $T_i(\boldsymbol{\xi}_i),H_i(\boldsymbol{\xi})\in\mathcal{T}(\mathbb{Z}^{r_i})$ for $1\le i\le m$ and $\mathbf{v}=(A-I)^{-1}\boldsymbol{\alpha}_0$ with $\boldsymbol{\alpha}_0\in\mathbb{Z}^d$ such that $T_i(A_i^T\boldsymbol{\xi}_i)=H_i(\boldsymbol{\xi})T_i(\boldsymbol{\xi})$ for all i and*

$$T(\boldsymbol{\xi})=T(\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_m)=e^{2\pi i\langle\mathbf{v},\boldsymbol{\xi}\rangle}\prod_{i=1}^m T_i(\boldsymbol{\xi}_i). \tag{4.7}$$

(B) *There exist constants c_1,c_2 such that $P(A^T\boldsymbol{\xi})=c_1P(\boldsymbol{\xi})$ and $Q(A^T\boldsymbol{\xi})=c_2Q(\boldsymbol{\xi})$. Furthermore P and Q have no common factor, and there exist principal homogeneous polynomials $Q_i(\boldsymbol{\xi}_i)$, $\boldsymbol{\xi}_i\in\mathbb{R}^{r_i}$, such that*

$$Q(\boldsymbol{\xi})=Q(\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_m)=\prod_{i=1}^m Q_i(\boldsymbol{\xi}_i). \tag{4.8}$$

Proof. It is clear that (4.6) follows directly from (4.5) and the subsequent Lemma 4.4 and Lemma 4.5. Since $f(\mathbf{x})$ is A-refinable with integer translates there exists $H(\boldsymbol{\xi})\in\mathcal{T}(\mathbb{Z}^d)$ such that $\widehat{f}(A^T\boldsymbol{\xi})=H(\boldsymbol{\xi})f(\boldsymbol{\xi})$. It now follows from (4.6) that $T(\boldsymbol{\xi})=aH(\boldsymbol{\xi})T(\boldsymbol{\xi})$, and from Lemma 4.6 we have $T(\boldsymbol{\xi})=e^{2\pi i\langle\mathbf{v},\boldsymbol{\xi}\rangle}\tilde{T}(\boldsymbol{\xi})$ where $\tilde{T}(\boldsymbol{\xi})\in\mathcal{T}(\mathbb{Z}^d)$. Using the property that $\mathbf{v}=(A-I)^{-1}\boldsymbol{\alpha}_0$ for some $\boldsymbol{\alpha}_0\in\mathbb{Z}^d$ we see that $\tilde{T}(A\boldsymbol{\xi})=e^{2\pi i\langle\boldsymbol{\alpha}_0,\boldsymbol{\xi}\rangle}H(\boldsymbol{\xi})\tilde{T}(\boldsymbol{\xi})$. This implies that $\tilde{T}(\boldsymbol{\xi})|\tilde{T}(A^T\boldsymbol{\xi})$ and hence $\tilde{T}(\boldsymbol{\xi})|\tilde{T}((A^T)^K\boldsymbol{\xi})=\tilde{(}\lambda_1\boldsymbol{\xi}_1,\dots,\lambda_m\boldsymbol{\xi}_m)$.

Without loss of generality we may assume all $\lambda_j>0$, for otherwise we can iterate the equation once to use A^{2K} for A^K. Lemma 4.7 now applies to the trigonometric polynomial $\tilde{T}$ to yield $\tilde{T}(\boldsymbol{\xi})=\prod_{i=1}^m T_i(\boldsymbol{\xi}_i)$ where each $T_i(\boldsymbol{\xi}_i)\in\mathcal{T}(\mathbb{Z}^{r_i})$. Clearly $T(\boldsymbol{\xi}_i)|T_i(A_i^T\boldsymbol{\xi}_i)$. Setting $H_i(\boldsymbol{\xi}_i)=T_i(A^T\boldsymbol{\xi}_i)/T_i(\boldsymbol{\xi}_i)$ proves (A).

To prove (B) note that we can always assume $P(\boldsymbol{\xi})$ and $Q(\boldsymbol{\xi})$ have no common factors. Furthermore clearly we have $P(A^T\boldsymbol{\xi})Q^{-1}(A^T\boldsymbol{\xi}) = CP(\boldsymbol{\xi})Q^{-1}(\boldsymbol{\xi})$ for some constant C. Since $Q(\xi)$ and $Q(A^T\xi)$ have the same degree, it follows that $Q(A^T\boldsymbol{\xi}) = c_2Q(\boldsymbol{\xi})$ and $P(A^T\boldsymbol{\xi}) = c_1P(\boldsymbol{\xi})$. The property (4.8) follows from Lemma 4.4. ∎

Proof of Theorem 2.1. Part (C) of the theorem is obvious. So we shall focus on (A) and (B).

(A) Clearly if $f(\mathbf{x}) = \prod_{i=1}^m g_i(\mathbf{x}_i)$ where each $g_i(\mathbf{x}_i)$ is an A_i-refinable spline with integer (resp. rational) translates, $\mathbf{x}_i \in \mathbb{R}^{r_i}$, then $f(\mathbf{x})$ is an A-refinable spline with integer (resp. rational) translates. We need to prove the converse. Since the case of rational translates is a rescaling of the case of integer translates, we only need to prove it for the latter case.

Now assume $f(\mathbf{x})$ is an A-refinable spline with integer translates. By Lemma 4.8 we have $\widehat{f}(\boldsymbol{\xi}) = T(\boldsymbol{\xi})P(\boldsymbol{\xi})Q^{-1}(\boldsymbol{\xi})$ where P, Q have no common factor and Q is a principal homogeneous polynomial. Assume that

$$Q(\boldsymbol{\xi}) = \prod_{j=1}^n \langle \mathbf{a}_j, \boldsymbol{\xi}\rangle.$$

It is shown in Sun [17] that there exist nonzero $t_1, \dots, t_n \in \mathbb{R}$ such that $t_j\mathbf{a}_j \in \mathbb{Z}^d$ and

$$T(\boldsymbol{\xi}) = S(\boldsymbol{\xi})\prod_{j=1}^n (1 - e^{2\pi i\langle t_j\mathbf{a}_j, \boldsymbol{\xi}\rangle})$$

for some $S(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{R}^d)$. Now since rescaling $\mathbf{a}_j$ yield a principal homogeneos polynomial that differs from Q only by a constant multiple, we may without loss of generality assume that all $t_j = 1$ and $\mathbf{a}_j \in \mathbb{Z}^d$.

We now apply (4.8) in Lemma 4.8, which states that $Q(\boldsymbol{\xi}) = \prod_{i=1}^m Q_i(\boldsymbol{\xi}_i)$. Write $Q_i(\boldsymbol{\xi}) = \prod_{j=1}^{n_i}\langle \mathbf{b}_{ij}, \boldsymbol{\xi}_i\rangle$ where $\mathbf{b}_{ij} \in \mathbb{Z}^{r_i}$. Since $T(\boldsymbol{\xi}) = e^{2\pi i\langle \mathbf{v}, \boldsymbol{\xi}\rangle}\prod_{i=1}^m T_i(\boldsymbol{\xi})$ by (4.7) we have $T_i(\boldsymbol{\xi}) = S_i(\boldsymbol{\xi}_i)\prod_{j=1}^{n_i}(1 - e^{2\pi i\langle \mathbf{b}_{ij}, \boldsymbol{\xi}_i\rangle})$ for some $S_i(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{Z}^{r_i})$. Let

$$G_i(\boldsymbol{\xi}_i) = \frac{T_i(\boldsymbol{\xi}_i)}{Q_i(\boldsymbol{\xi}_i)} = (2\pi i)^{n_i}S_i(\boldsymbol{\xi}_i)\prod_{j=1}^{n_i}\frac{1 - e^{2\pi i\langle \mathbf{b}_{ij}, \boldsymbol{\xi}_i\rangle}}{2\pi i\langle \mathbf{b}_{ij}, \boldsymbol{\xi}_i\rangle}.$$

Then by Lemma 4.8 we have $G_i(A_i^T\boldsymbol{\xi}_i) = c_iH_i(\boldsymbol{\xi})G_i(\boldsymbol{\xi}_i)$. Since $H_i \in \mathcal{T}(\mathbb{Z}^{r_i})$ it follows that the inverse Fourier transform $\tilde{g}(\mathbf{x}_i)$ of G_i is A_i-refinable. But $S_i(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{Z}^{r_i})$, so $\tilde{g}_i(\mathbf{x}_i)$ is a linear combination of some integer translates of the box spline $B_{E_i}(\mathbf{x}_i)$, where $E_i = \{\mathbf{b}_{ij}\}$ (counting multiplicity). Thus $\tilde{g}_i(\mathbf{x}_i)$ is an A_i-refinable spline with integer translates.

Finally, set $g_i(\mathbf{x}_i) = \tilde{g}_i(\mathbf{x}_i + \mathbf{v}_i)$ where $\mathbf{v} = [\mathbf{v}_1, \dots, \mathbf{v}_m]$ with $\mathbf{v}_i = (A_i - I)^{-1}\boldsymbol{\alpha}_i$ for some $\alpha_i \in \mathbb{Z}^{r_i}$. By (1.2) g_i is a refinable spline in $\mathbb{R}^{r_i}$ with integer translates. Hence $g(\mathbf{x}) = \prod_{i=1}^m g_i(\mathbf{x}_i)$ is an A-refinable spline with integer translates. Furthermore,

$$\widehat{f}(\boldsymbol{\xi}) = \widehat{g}(\boldsymbol{\xi})P(\boldsymbol{\xi}). \tag{4.9}$$

Now $P(A^T\boldsymbol{\xi}) = CP(\boldsymbol{\xi})$ for some C. Since $\widehat{f}(0) \neq 0$ we must have $P(0) \neq 0$. So $C = 1$. It is easy to see that the only polynomial with this property is a constant. Thus $f(\mathbf{x}) = C_0 g(\mathbf{x})$. This proves (A).

(B) Note that (4.9) is derived without the assumption that $\widehat{f}(0) = 0$. Assume that $f(\mathbf{x})$ is an A-refinable spline with integer translates. Thus it follows from (4.9) that $f(\mathbf{x}) = P(\mathbf{D})g(\mathbf{x})$, where $g(\mathbf{x}) = \prod_{i=1}^m g_i(\mathbf{x}_i)$ and each $g_i(\mathbf{x}_i)$ is an A_i-refinable spline with integer translates. Now $\widehat{f}(\boldsymbol{\xi}) = P(\boldsymbol{\xi})\widehat{g}(\boldsymbol{\xi})$. It follows that $f(\mathbf{x}) = P(\mathbf{D})g(\mathbf{x})$. The property $P(A^T\boldsymbol{\xi}) = CP(\boldsymbol{\xi})$ is already established in Lemma 4.8. Conversely, if $g(\mathbf{x})$ is an A-refinable spline and $P(\boldsymbol{\xi})$ is a polynomial such that $\widehat{g}(\boldsymbol{\xi})P(\boldsymbol{\xi}) \in L^2(\mathbb{R}^d)$ then $f(\mathbf{x})$ given by $\widehat{f}(\boldsymbol{\xi}) = \widehat{g}(\boldsymbol{\xi})P(\boldsymbol{\xi})$ is in $L^2(\mathbb{R}^d)$. It is A-refinable with integer translates as

$$\widehat{f}(A^T\boldsymbol{\xi}) = \widehat{g}(A^T\boldsymbol{\xi})P(A^T\boldsymbol{\xi}) = CH_1(\boldsymbol{\xi})\widehat{g}(\boldsymbol{\xi})P(\boldsymbol{\xi}),$$

where $H_1(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{Z}^d)$. Hence $f(\mathbf{x}) = P(\mathbf{D})g(\mathbf{x})$ is an A-refinable spline. This completes the proof of (B). ∎

Proof of Proposition 2.2 Assume $f(\mathbf{x})$ is A-refinable with translates $\{\mathbf{d}_j\}_{j=1}^n$,

$$f(\mathbf{x}) = \sum_{j=1}^n c_j f(A\mathbf{x} - \mathbf{d}_j).$$

Then $g(\mathbf{x}) = f(Q\mathbf{x})$ satisfies the refinement equation

$$g(\mathbf{x}) = f(Q\mathbf{x}) = \sum_{j=1}^n c_j f\big(Q(Q^{-1}AQ\mathbf{x} - Q^{-1}\mathbf{d}_j)\big) = \sum_{j=1}^n c_j g(B\mathbf{x} - Q^{-1}\mathbf{d}_j).$$

Thus $g(\mathbf{x})$ is B-refinable with translates $\{Q^{-1}\mathbf{d}_j\}$. Both (A) and (B) of the proposition follows immediately. ∎

Proof of Theorem 2.3. We first prove that (B) implies (A). Clearly $G(\boldsymbol{\xi})|G(m\boldsymbol{\xi})$ as $(A^T)^K = mI$. By the result of Sun [17] $f(\mathbf{x})$ is an mI-refinable spline with integer translates. In particular it is a spline. We only need to show that f is A-refinable. To achieve this, observe that

$$\widehat{f}(\boldsymbol{\xi}) = e^{2\pi i\langle \mathbf{v},\boldsymbol{\xi}\rangle} G(\boldsymbol{\xi})P(\boldsymbol{\xi})R_E^{-1}(\boldsymbol{\xi}) \tag{4.10}$$

where $R_E(\boldsymbol{\xi}) = \prod_{j=1}^n \langle \mathbf{a}_j, \boldsymbol{\xi}\rangle$. Now E is A-cyclic so $R_E(A^T\boldsymbol{\xi}) = C_1 R_E(\boldsymbol{\xi})$. By assumption we also have $G(A^T\boldsymbol{\xi}) = H_1(\boldsymbol{\xi})G(\boldsymbol{\xi})$ for some $H_1 \in \mathcal{T}(\mathbb{Z}^r)$ and $P(A^T\boldsymbol{\xi}) = CP(\boldsymbol{\xi})$. Finally, $\langle \mathbf{v}, A^T\boldsymbol{\xi}\rangle = \langle A\mathbf{v}, \boldsymbol{\xi}\rangle = \langle \boldsymbol{\alpha}_0 + \mathbf{v}, \boldsymbol{\xi}\rangle$. Thus $e^{2\pi i\langle \mathbf{v},A^T\boldsymbol{\xi}\rangle} = e^{2\pi i\langle \boldsymbol{\alpha}_0,\boldsymbol{\xi}\rangle} e^{2\pi i\langle \mathbf{v},\boldsymbol{\xi}\rangle}$, which yields

$$\widehat{f}(A^T\boldsymbol{\xi}) = H(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}), \quad \text{where} \quad H(\boldsymbol{\xi}) := CC_1^{-1} e^{2\pi i\langle \boldsymbol{\alpha}_0,\boldsymbol{\xi}\rangle} H_1(\boldsymbol{\xi}).$$

Since $H(\boldsymbol{\xi}) \in \mathcal{T}(\mathbb{Z}^r)$ it follows that f is A-refinable with integer translates.

To prove the converse (A) implies (B), notice that since f is A-refinable with integer translates it must also be mI-refinable with integer translates. Thus $f(\mathbf{x})$ must be in the form (2.2) by [17]. Hence $\widehat{f}(\boldsymbol{\xi})$ is given by (4.10). By Lemma 4.8 part (B) we have $P(A^T\boldsymbol{\xi}) = CP(\boldsymbol{\xi})$ and $R_E(A^T\boldsymbol{\xi}) = C_1 R_E(\boldsymbol{\xi})$. We now prove that E can be rescaled to become an A-cyclic set. Let $\mathbf{a} \in E$. By $R_E(A^T\boldsymbol{\xi}) = C_1 R_E(\boldsymbol{\xi})$, $\langle \mathbf{a}, A^T\boldsymbol{\xi}\rangle = \langle A\mathbf{a}, \boldsymbol{\xi}\rangle$ must be a factor of $R_E(\boldsymbol{\xi})$. Thus there exists an $\mathbf{a}' \in E$ such that $\mathbf{a}'$ is parallel to $A\mathbf{a}$. Thus we may replace $\mathbf{a}'$ with $A\mathbf{a}$ so the new R_E differ from the old by only a constant multiple. This procedure can be repeated until we make E an A-cyclic set. ∎

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