# A CLASS OF SELF-AFFINE AND SELF-AFFINE MEASURES

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## 1. Introduction

Let $\mathcal{I} = \{\phi_j\}_{j=1}^m$ be an iterated function system (IFS) consisting of a family of contractive affine maps on $\mathbb{R}^d$. Hutchinson [8] proved that there exists a unique compact set $K = K(\mathcal{I})$, called the *attractor* of the IFS $\mathcal{I}$, such that $K = \bigcup_{j=1}^m \phi_j(K)$. Moreover, for any given probability vector $\mathbf{p} = (p_1, \dots, p_m)$, i.e. $p_j > 0$ for all $j$ and $\sum_{j=1}^m p_j = 1$, there exists a unique compactly supported probability measure $\nu = \nu_{\mathcal{I},\mathbf{p}}$ such that

$$\nu = \sum_{j=1}^m p_j \, \nu \circ \phi_j^{-1}. \tag{1.1}$$

This paper is devoted to the study of fundamental properties of a class of self-affine sets and measures, such as the $L^q$ spectrum, the Hausdorff dimension and the entropy dimension.

It is well known that problems concerning self-affine sets and measures are typically difficult. Questions that may be trivial in the self-similar setting are often intractable in the self-affine setting. A telling example is calculating the Hausdorff and box dimensions of the attractor of an IFS $\mathcal{I} = \{\phi_j\}_{j=1}^m$. If all $\phi_j$ are similitudes and $\mathcal{I}$ satisfies the so-called *open set condition* (OSC) the Haudorff dimension and the box dimension of the attractor $K(\mathcal{I})$ agree, and they are easily computable by the formula

$$\sum_{j=1}^m \rho_j^{\dim_H(K)} = 1$$

where $\rho_i$ denotes the contraction ratio of $\phi_j$, see e.g. Falconer [3]. Even without the open set condition the dimension of $K(\mathcal{I})$ can often be computed if $\mathcal{I}$ belongs to a more general class called the *finite type* IFS, see e.g. Lalley [11] and Ngai and Wang [18] and the references therein. However this is no longer the case when $\phi_i$ are affine maps. Even under the open

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set condition we know how to compute the Hausdorff dimension of $K(\mathcal{I})$ only for very special $\mathcal{I}$'s, and for which the solutions are quite nontrivial. McMullen [16] and Bedford [1] independently computed the Hausdorff and box dimensions of $K(\mathcal{I})$ for $\mathcal{I}=\{\phi_j\}_{j=1}^m$ in which all $\phi_j$ have the form

$$\phi_j(x)=\begin{bmatrix} n^{-1} & 0 \\ 0 & k^{-1}\end{bmatrix} x+\begin{bmatrix} a_j/n \\ b_j/k\end{bmatrix} \tag{1.2}$$

where all $a_j, b_j$ are integers, $0 \leq a_j < n$ and $0 \leq b_j < m$. They found that the Hausdorff dimension and the box dimension are not the same in general. Lalley and Gatzouras [12], in a highly technical paper along the same spirit of [16], computed the Hausdorff and box dimensions for a broader class of IFS $\mathcal{I}=\{\phi_j\}_{j=1}^m$, in which $\phi_j$ map the unit square $(0,1)^2$ into disjoint rectangles having certain geometric arrangement inside the unit square. More precisely, in the Lalley-Gatzouras class all rectangles $\phi_j((0,1)^2)$ are parallel to the axes and have longer sides parallel to the $x$-axis. Furthermore once projected onto the $x$-axis these rectangles are either identical or disjoint. Aside from a few other special cases such as the graph-directed McMullen class studied by Kenyon and Peres [9], the Lalley-Gatzouras class (which inculdes the McMullen class) remains the only substantial class of true self-affine sets whose Hausdorff and box dimensions are known.

We focus on the $L^q$ spectrum and the Hausdorff and entropy dimensions of a self-affine measure in this paper. These quantities are important basic ingredients in the study of fractal geometry, particularly in the study of multifractal phenomena. As a by-product we also obtain results on the dimension of the self-affine sets. Let $\nu$ be a compactly supported measure in $\mathbb{R}^d$ and $q \in \mathbb{R}$. For each $n \geq 1$ let $\mathbf{D}_n$ be the set of cubes $\{[0,2^{-n})^d+\alpha:\ \alpha \in 2^{-n}\mathbb{Z}^d\}$. The $L^q$ *spectrum of* $\nu$ is defined as

$$\tau(\nu,q)=\lim_{n\to\infty}\frac{\log \tau_n(\nu,q)}{-n\log 2}, \qquad \text{where } \tau_n(\nu,q)=\sum_{Q\in\mathbf{D}_n}\nu^q(Q), \tag{1.3}$$

if the limit exists. Related to $\tau(\nu,q)$ are the $L^q$ *dimension* $D(\nu,q)$ and the *entropy dimension* $h(\nu)$ of $\nu$, defined respectively by

$$D(\nu,q):=\frac{\tau(\nu,q)}{q-1} \tag{1.4}$$

and

$$h(\nu):=\lim_{n\to\infty}\frac{\sum_{Q\in\mathbf{D}_n}\nu(Q)\log(1/\nu(Q))}{n\log 2} \tag{1.5}$$

if the limit exists. For a similarity IFS $\mathcal{I} = \{\phi_j\}_{j=1}^m$ with the open set condition and any probability vector $\mathbf{p}$ the $L^q$ spectrum of $\nu = \nu_{\mathcal{I},\mathbf{p}}$ is known to be analytic for $q \in \mathbb{R}$, given by the equation

$$\sum_{j=1}^m p_j^q \, \rho_j^{-\tau(\nu,q)} = 1, \tag{1.6}$$

where $\rho_j$ denotes the contraction ratio of $\phi_j$, see Cawley and Mauldin [2] and Olsen [19]. Moreover, the Legender transform $\tau^*(\nu,\alpha)$ of $\tau(\nu,q)$ given by

$$\tau^*(\nu,\alpha) := \inf\Big\{q\alpha - \tau(\nu,q) : \; q \in \mathbb{R}\Big\} \tag{1.7}$$

equals the Hausdorff dimension of the set

$$K(\alpha) := \Big\{x \in \operatorname{supp}(\nu) : \; \lim_{r\to 0^+} \frac{\log \nu\big(B_r(x)\big)}{\log r} = \alpha\Big\}.$$

For a self-similar measure without the open set condition, however, the $L^q$ spectrum is generally difficult to obtain and is calculated for only a few special cases, see [13, 4, 5, 14]. One important such special case is the class of finite type IFS's ([18]), a substantially larger class than the class with the OSC. For a finite type IFS in $\mathbb{R}$, Feng [6] expressed $\tau(\nu,q)$ via products of certain nonnegative matrices, and proved that $\tau(\nu,q)$ is differentiable for $q \in (0,\infty)$.

As one would expect, even less is known about the $L^q$ spectrum and the Hausdorff and entropy dimensions of a self-affine measure. King [10] calculated $\tau(\nu,q)$ for $\nu = \nu_{\mathcal{I},\mathbf{p}}$ where the IFS $\mathcal{I}$ is in the McMullen class (1.2). He gave a detailed multifractal analysis for such measures. Olsen [20] generalized King's results to dimensions $d \geq 3$. Peres and Solomyak [21] proved the existence of $\tau(\nu,q)$ and $h(\nu)$ for the class of self-conformal measures, and asked whether they also exist for all self-affine measures.

In this paper we calculate the $L^q$ spectrum and the entropy dimension for a class of self-affine measures in $\mathbb{R}^2$. This class of self-affine measures $\nu_{\mathcal{I},\mathbf{p}}$ requires only that the underlying IFS's $\mathcal{I} = \{\phi_j\}_{j=1}^m$ satisfy the *rectangular open set condition (ROSC).* It is a much larger class than the McMullen class studied in [10] and the Lalley-Gatzouras class. Simply speaking, $\mathcal{I} = \{\phi_j\}_{j=1}^m$ in $\mathbb{R}^2$ satisfies the ROSC if there is an open rectangle $T$ such that the maps $\phi_i$ map $T$ into disjoint rectangles parallel to the axes inside $T$. As an application we obtain the formula for the box dimension of $K(\mathcal{I})$ under the ROSC as well as the Hausdorff dimension of $\nu_{\mathcal{I},\mathbf{p}}$ under some additional assumptions. Our results on the box dimension can be viewed as an extension of the box dimension results by Lalley and Gatzouras [12]. The techniques we employ here are different from those in any of the previous studies mentioned in the paper.

## 2. Statement of Main Results

We first introduce some definitions and notations. The dimension in the rest of the paper will be set to $d = 2$, although most of the definitions extend easily to higher dimensions. Let $\mathcal{I} = \{\phi_j\}_{j=1}^m$ be an affine IFS in $\mathbb{R}^2$. Throughout this paper we shall always assume that $\phi_j(x) = (a_j x + c_j, b_j y + d_j)$ with $0 < a_j,\, b_j < 1$ for all $j$. Thus each $\phi_j$ maps any square $(0,R)^2 + v$ to a rectangle parallel to the axes.

**Definition 2.1.** We say that $\mathcal{I} = \{\phi_j\}_{j=1}^m$ satisfies the *rectangular open set condition (ROSC)* if there exists an open rectangle $T = (0,R_1) \times (0,R_2) + v$ such that $\{\phi_j(T)\}_{j=1}^m$ are disjoint rectangles parallel to the axes inside $T$.

For a self-affine measure $\nu = \nu_{\mathcal{I},\mathbf{p}}$ associated with $\mathcal{I}$ and probability vector $\mathbf{p}$ we shall define the projections $\nu^x$ and $\nu^y$ of $\nu$ onto the $x$- and $y$-axes, which we rely on heavily in this paper. Let $\mathcal{I}^x := \{\pi_x \circ \phi_j \circ \pi_x^{-1}\}$ and $\mathcal{I}^y := \{\pi_y \circ \phi_j \circ \pi_y^{-1}\}$ be the projections of $\mathcal{I}$, where $\pi_x$ and $\pi_y$ are the canonical projections of $\mathbb{R}^2$ onto the $x$- and $y$-axes, respectively. It is also easy to check that in our setting both $\mathcal{I}^x$ and $\mathcal{I}^y$ are well-defined. In fact $\pi_x \circ \phi_j \circ \pi_x^{-1}(x) = a_j x + c_j$ and $\pi_y \circ \phi_j \circ \pi_y^{-1}(y) = b_j y + d_j$. We define $\nu^x = \nu_{\mathcal{I}^x,\mathbf{p}}$ and $\nu^y = \nu_{\mathcal{I}^y,\mathbf{p}}$. It is easy to check that $\nu^x = \nu \circ \pi_x^{-1}$ and $\nu^y = \nu \circ \pi_y^{-1}$. For any $\mathbf{d} = (d_1, d_2, \ldots, d_m)$ we use $\Gamma(\mathbf{d})$ to denote

$$\Gamma(\mathbf{d}) := \Big\{ \mathbf{t} = (t_1, t_2, \ldots, t_m) :\ t_j \ge 0,\ \textstyle\sum_{j=1}^m t_j = 1,\ \sum_{j=1}^m d_j t_j \ge 0 \Big\}. \tag{2.1}$$

Our main theorem concerning the $L^q$ spectrum of $\nu$ is:

**Theorem 2.1.** *Let $\mathcal{I} = \{\phi_j\}_{j=1}^m$ be an affine IFS in $\mathbb{R}^2$ satisfying the ROSC, with $\phi_j(x,y) = (a_j x + c_j, b_j y + d_j)$ and $0 < a_j, b_j < 1$ for all $j$. Let $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ be a probability vector. Then the $L^q$ spectrum of $\nu = \nu_{\mathcal{I},\mathbf{p}}$ is $\tau(\nu,q) = \min(\theta_a, \theta_b)$, where*

$$\begin{aligned}
\theta_a &= \inf_{\mathbf{t} \in \Gamma(\mathbf{e}^a)} \frac{\sum_{j=1}^m t_j \Big( -\log t_j - \tau(\nu^y, q)(\log b_j - \log a_j) + q \log p_j \Big)}{\sum_{j=1}^m t_j \log a_j}, \\
\theta_b &= \inf_{\mathbf{t} \in \Gamma(\mathbf{e}^b)} \frac{\sum_{j=1}^m t_j \Big( -\log t_j - \tau(\nu^x, q)(\log a_j - \log b_j) + q \log p_j \Big)}{\sum_{j=1}^m t_j \log b_j},
\end{aligned}$$

*with* $\mathbf{e}^a = (\log(b_1/a_1), \log(b_2/a_2), \ldots, \log(b_m/a_m))$ *and* $\mathbf{e}^b = -\mathbf{e}^a$.

We point out that if $a_j \leq b_j$ for all $i$ then the set $\Gamma(\mathbf{e}^b)$ is empty, and if so we have $\tau(\nu, q) = \theta_a$. In fact we prove:

**Theorem 2.2.** *Under the hypotheses of Theorem 2.1, assume furthermore that* $a_j \leq b_j$ *for all* $j$. *Then* $\tau(\nu, q)$ *satisfies*

$$\sum_{j=1}^{m} a_j^{\tau(\nu^y,q)-\tau(\nu,q)} b_j^{-\tau(\nu^y,q)} p_j^q = 1. \tag{2.2}$$

Theorem 2.2 allows us to easily calculate the $L^q$ spectrum if $\tau(\nu^x, q)$ is known, which is the case if $\mathcal{I}$ is in the McMullen or Lalley-Gatzouras class. Moreoever, Theorem 2.1 allows us to calculate $\tau(\nu, q)$, at least in theory, if the projections of $\mathcal{I}$ onto the two axes are of finite type by the result of Feng [6], making the $L^q$ spectrum computable at least in theory for a considerably larger class of IFS's than the Lalley-Gatzouras class. We shall give examples in these settings later on.

One of the applications of the above two theorems is calculating the box dimension of $K = K(\mathcal{I})$. It is easy to see that $K = \mathrm{supp}\,(\nu)$ and by definition $\dim_B(K)$ is simply $-\tau(\nu, 0)$. Therefore we also obtain as a bonus of Theorems 2.1 and 2.2 a formula for $\dim_B(K)$. Other than the Lalley-Gatzouras class, $\dim_B(K)$ is easily computable when all $a_j \geq b_j$ and the projection $\pi_x(K)$ has dimension 1; it is given by

$$\sum_{j=1}^{m} b_j^{\dim_B(K)-1} a_j = 1. \tag{2.3}$$

Another application of the theorems is computing the Hausdorff dimension of a self-affine measure. Let $\nu$ be a finite Borel measure in $\mathbb{R}^d$. It is said to be *exactly dimensional* if there exists a constant $c$ such that

$$\lim_{r \to 0} \frac{\log \nu(B_r(x))}{\log r} = c \qquad \nu - \text{a.e. } x \in \mathbb{R}^d.$$

Ngai [17] proved that if $\tau(\nu, q)$ is differentiable at $q = 1$ then $\nu$ is exactly dimensional, and $\dim_H(\nu) = c = \frac{d}{dq}\tau(\nu, 1)$. As a corollary of Theorem 2.2 we obtain a Ledrappier-Young type formula (see [15]) for $\dim_H(\nu)$:

**Theorem 2.3.** *Under the hypotheses of Theorem 2.2, if $\tau(\nu^x, q)$ is differentiable at $q = 1$ then so is $\tau(\nu, q)$, and*

$$\dim_H(\nu) = \frac{\sum_{j=1}^m p_j \log p_j + \dim_H(\nu^x) \sum_{j=1}^m p_j (\log b_j - \log a_j)}{\sum_{j=1}^m p_j \log b_j}.$$

*In particular if $a_j = a$ and $b_j = b$ for all $j$ then*

$$\dim_H(\nu) = \frac{\sum_{j=1}^m p_j \log p_j}{\log b} + \dim_H(\nu^x) \frac{\log b - \log a}{\log b}.$$

Our technique can also be used to study the entropy dimension, which for a Borel measure $\nu$ is defined in (1.5). It is known [21] that the entropy dimension exists for all self-similar (in fact self-conformal) measures. We determine the entropy dimension for the self-affine measures with ROSC:

**Theorem 2.4.** *Let $\nu$ be the self-affine measure in Theorem 2.1. Then*

$$h(\nu) = \begin{cases} \dfrac{h(\nu^x) \sum_{j=1}^m p_j (\log b_j - \log a_j) + \sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \log b_j}, & \text{if } \sum_{j=1}^m p_j (\log a_j - \log b_j) \geq 0 \\[2ex] \dfrac{h(\nu^y) \sum_{j=1}^m p_j (\log a_j - \log b_j) + \sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \log a_j}, & \text{otherwise.} \end{cases}$$

## 3. Some Combinatorial Results

We establish two combinatorial results that will be needed to prove our main theorems in this paper.

First let us introduce some notations on symbolic spaces. These notations are mostly standard. We use $\Sigma = \Sigma(m)$ to denote the alphabet $\{1, 2, \ldots, m\}$. Whenever there is no ambiguity we shall use $\Sigma$ rather than $\Sigma(m)$, as $m$ is usually fixed in this paper. The set of all words in $\Sigma$ of length $n$ is denoted by $\Sigma^n$, with $\Sigma^* := \bigcup_{n \geq 0} \Sigma^n$ and $\Sigma^{\mathbb{N}}$ being the set of all one-sided infinite words. Here we adopt the convention that $\Sigma^0$ contains only the empty word $\emptyset$. Associated with $\Sigma^*$ are two actions: The *left shift* action $\sigma$ and and the *right shift* action $\delta$, defined respectively by $\sigma(\emptyset) = \delta(\emptyset) = \emptyset$ and

$$\sigma(i_1 i_2 \cdots i_k) = i_2 \cdots i_k, \qquad \delta(i_1 i_2 \cdots i_k) = i_1 \cdots i_{k-1}$$

for each $i_1 i_2 \cdots i_k \in \Sigma^*$ with $k \geq 1$.

We shall use boldface letters $\mathbf{i}, \mathbf{j}, \mathbf{l}$ to denote elements in $\Sigma^*$ or $\Sigma^{\mathbb{N}}$. For each sequence $\mathbf{a} = (a_j)_{j=1}^m$ we may extend it to a function $f_{\mathbf{a}} : \Sigma^* \longrightarrow \mathbb{R}$ by $f_{\mathbf{a}}(\emptyset) = 1$ and $f_{\mathbf{a}}(\mathbf{i}) = a_{i_1} \cdots a_{i_k}$ for $\mathbf{i} = i_1 \cdots i_k$. Most of the time, because there is no ambiguity, we shall use the simplified notation $a_{\mathbf{i}}$ in place of $f_{\mathbf{a}}(\mathbf{i})$.

The above are general purpose notations. Now we introduce some that are specific to this paper. Suppose that $\mathbf{a} = (a_j)_{j=1}^m$ and $\mathbf{b} = (b_j)_{j=1}^m$ are two sequences with $0 < a_j,\, b_j < 1$ for all $j$. For any $0 < r < 1$ let

$$\mathcal{A}_r := \mathcal{A}_r(\mathbf{a}, \mathbf{b}) = \Big\{ \mathbf{i} \in \Sigma^* : \ a_{\delta(\mathbf{i})} \geq r,\, b_{\delta(\mathbf{i})} \geq r,\, \min(a_{\mathbf{i}}, b_{\mathbf{i}}) < r \Big\}$$

and

$$\mathcal{A}_r^a := \{\mathbf{i} \in \mathcal{A}_r : \ a_{\mathbf{i}} \leq b_{\mathbf{i}}\}, \qquad \mathcal{A}_r^b := \{\mathbf{i} \in \mathcal{A}_r : \ a_{\mathbf{i}} > b_{\mathbf{i}}\}.$$

Suppose that $\mathbf{c} = (c_j)_{j=1}^m$ is another sequence of positive real numbers. The objective of this section is to evaluate several limits. Set

$$\begin{aligned} \Theta^a &= \Theta^a(\mathbf{c}) := \lim_{r \to 0^+} \frac{\log\big(\sum_{\mathbf{i} \in \mathcal{A}_r^a} c_{\mathbf{i}}\big)}{\log r}, \qquad \text{and} \\ \Theta^b &= \Theta^b(\mathbf{c}) := \lim_{r \to 0^+} \frac{\log\big(\sum_{\mathbf{i} \in \mathcal{A}_r^b} c_{\mathbf{i}}\big)}{\log r}. \end{aligned}$$

Similarly, for any probability vector $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ set

$$\begin{aligned} \Omega^a &= \Omega^a(\mathbf{c}, \mathbf{p}) := \lim_{r \to 0^+} \frac{\sum_{\mathbf{i} \in \mathcal{A}_r^a} p_{\mathbf{i}} \log c_{\mathbf{i}}}{\log r}, \qquad \text{and} \\ \Omega^b &= \Omega^a(\mathbf{c}, \mathbf{p}) := \lim_{r \to 0^+} \frac{\sum_{\mathbf{i} \in \mathcal{A}_r^b} p_{\mathbf{i}} \log c_{\mathbf{i}}}{\log r}. \end{aligned}$$

We prove the following results:

**Proposition 3.1.** *Given sequences* $\mathbf{a} = (a_j)_{j=1}^m$ *and* $\mathbf{b} = (b_j)_{j=1}^m$ *with all* $a_j,\, b_j$ *in* $(0, 1)$ *let* $\mathbf{e}^a = (\log(b_1/a_1), \log(b_2/a_2), \ldots, \log(b_m/a_m))$ *and* $\mathbf{e}^b = -\mathbf{e}^a$.

(i) *If* $a_j \leq b_j$ *for some* $1 \leq j \leq m$ *then* $\Theta^a(\mathbf{c})$ *exists, and*

$$\Theta^a(\mathbf{c}) = \inf_{\mathbf{t} \in \Gamma(\mathbf{e}^a)} \frac{\sum_{j=1}^m t_j\big(-\log t_j + \log c_j\big)}{\sum_{j=1}^m t_j \log a_j},$$

*where* $\Gamma(\mathbf{e}^a)$ *is defined in (2.1).*

(ii) *If* $a_j > b_j$ *for some* $1 \leq j \leq m$ *then* $\Theta^b(\mathbf{c})$ *exists, and*

$$\Theta^b(\mathbf{c}) = \inf_{\mathbf{t} \in \Gamma(\mathbf{e}^b)} \frac{\sum_{j=1}^m t_j\big(-\log t_j + \log c_j\big)}{\sum_{j=1}^m t_j \log b_j}.$$

(iii) $\Theta^a(\mathbf{c})$ *(resp.* $\Theta^b(\mathbf{c})$*) is continuous with respect to* $\mathbf{c}$ *if it exists.*

**Proposition 3.2.** *Under the assumptions of Proposition 3.1, and let* $\mathbf{p}$ *be a probability vector.*

(i) *If* $\sum_{j=1}^m p_j(\log b_j - \log a_j) > 0$ *then*
$$\Omega^a(\mathbf{c},\mathbf{p}) = \frac{\sum_{j=1}^m p_j \log c_j}{\sum_{j=1}^m p_j \log a_j}, \qquad \text{and} \qquad \Omega^b(\mathbf{c},\mathbf{p}) = 0.$$
(ii) *If* $\sum_{j=1}^m p_j(\log b_j - \log a_j) < 0$ *then*
$$\Omega^b(\mathbf{c},\mathbf{p}) = \frac{\sum_{j=1}^m p_j \log c_j}{\sum_{j=1}^m p_j \log b_j}, \qquad \text{and} \qquad \Omega^a(\mathbf{c},\mathbf{p}) = 0.$$
(iii) *If* $\sum_{j=1}^m p_j(\log b_j - \log a_j) = 0$ *then*
$$\begin{aligned} \frac{\sum_{\mathbf{i}\in\mathcal{A}_r^a} p_{\mathbf{i}} \log(b_{\mathbf{i}}/a_{\mathbf{i}})}{\log r} &= 0 = \frac{\sum_{\mathbf{i}\in\mathcal{A}_r^b} p_{\mathbf{i}} \log(a_{\mathbf{i}}/b_{\mathbf{i}})}{\log r} \qquad \text{and} \\ \frac{\sum_{\mathbf{i}\in\mathcal{A}_r} p_{\mathbf{i}} \log c_{\mathbf{i}}}{\log r} &= \frac{\sum_{j=1}^m p_j \log c_j}{\sum_{j=1}^m p_j \log a_j}. \end{aligned}$$

We need to first prove some lemmas. For any $\mathbf{i} = i_1 i_2 \cdots i_n \in \Sigma^*$ let $[\mathbf{i}] \subset \Sigma^{\mathbb{N}}$ denote the $\mathbf{i}$-*cylinder*
$$[\mathbf{i}] := \{j_1 j_2 j_3 \cdots \in \Sigma^{\mathbb{N}} : \ j_k = i_k \text{ for } 1 \le k \le n\}.$$

**Lemma 3.3.** *For any* $0 < r < 1$, $\{[\mathbf{i}] : \ \mathbf{i} \in \mathcal{A}_r\}$ *is a partition of* $\Sigma^{\mathbb{N}}$.

**Proof.** It is clear that $\{[\mathbf{i}] : \ \mathbf{i} \in \mathcal{A}_r\}$ are distinct subsets in $\Sigma^{\mathbb{N}}$. Furthermore, for any $\mathbf{j} = j_1 j_2 j_3 \cdots \in \Sigma^{\mathbb{N}}$ there exists a smallest $n$ such that $\min(a_{j_1} \cdots a_{j_n}, \ b_{j_1} \cdots b_{j_n}) < r$. Therefore $\mathbf{j} \in [j_1 \cdots j_n]$ and $[j_1 \cdots j_n] \in \mathcal{A}_r$. This proves the lemma. ∎

**Lemma 3.4.** *Let* $n = n_1 + n_2 + \cdots + n_m$ *with each* $n_j \in \mathbb{N}$. *Then*
$$\frac{1}{n} \log\Big(\frac{n!}{n_1! n_2! \cdots n_m!}\Big) = -\sum_{j=1}^m t_j \log t_j + O\Big(\frac{\log n}{n}\Big),$$
*where* $t_j = \frac{n_j}{n}$.

**Proof.** We apply the Stirling's formula $\log(q!) = q\log q - q + \frac{1}{2}\log q + O(1)$. Thus $\log(n!) = n\log n - n + O(\log n)$ and
$$\log(\prod_{j=1}^m n_j!) = \sum_{j=1}^m (n_j \log n_j - n_j + O(\log n_j)) = \sum_{j=1}^m n_j \log n_j - n + O(\log n).$$

Now $\log n_j = \log(t_j n) = \log t_j + \log n$. It follows that

$$\begin{aligned} \frac{1}{n}\log\Big(\frac{n!}{n_1!n_2!\cdots n_m!}\Big) &= \log n - \sum_{j=1}^{m}\frac{n_j \log n_j}{n} + O\Big(\frac{\log n}{n}\Big) \\ &= -\sum_{j=1}^{m} t_j \log t_j + O\Big(\frac{\log n}{n}\Big). \end{aligned}$$

∎

For each $\mathbf{i} = i_1 i_2 \cdots i_n \in \Sigma^*$ we use $|\mathbf{i}| = n$ to denote the length of $\mathbf{i}$ and $|\mathbf{i}|_j = \#\{k : \ i_k = j\}$ to denote the number of occurences of the letter $j$ in $\mathbf{i}$.

**Lemma 3.5.** *There exists a constant $C > 1$ such that $C^{-1}\log r^{-1} \le |\mathbf{i}| \le C\log r^{-1}$ for any $0 < r < \frac{1}{2}$ and $\mathbf{i} \in \mathcal{A}_r$.*

**Proof.** Let $s_+ = \max\{a_j, b_j : \ 1 \le j \le m\}$ and $s_- = \min\{a_j, b_j : \ 1 \le j \le m\}$. Then we have $s_-^{|\mathbf{i}|} \le a_{\mathbf{i}}, b_{\mathbf{i}} \le s_+^{|\mathbf{i}|}$ for any $\mathbf{i} \in \Sigma^*$. The lemma follows by setting $C = \max(|\log s_-|, |\log s_+|^{-1} + |\log 2|^{-1})$. Note that the condition $0 < r < \frac{1}{2}$ can be replaced with $0 < r < r_0$ for any fixed $r_0 < 1$. ∎

**Proof of Proposition 3.1.** We shall prove part (i) of the proposition only, as part (ii) follows from an identical argument and part (iii) is rather obvious. To prove (i) we estimate the sum $\sum_{\mathbf{i}\in\mathcal{A}^a} c_{\mathbf{i}}$.

For any $\mathbf{i} = i_1 i_2 \cdots i_n \in \mathcal{A}^a$ we observe that $\mathbf{i}' = \mathbf{j} i_n$ is also in $\mathcal{A}^a$, where $\mathbf{j}$ is any permutation of $\delta(\mathbf{i}) = i_1 \cdots i_{n-1}$, which gives $c_{\mathbf{i}} = c_{\mathbf{i}'}$. The number of distinct such $\mathbf{i}'$ is precisely $(n-1)!/\prod_{j=1}^{m} n_j!$ where $n_j := |\delta(\mathbf{i})|_j$. Let

$$T(\mathbf{i}) := \frac{(n-1)!}{\prod_{j=1}^{m} n_j!}\prod_{j=1}^{m} c_j^{n_j} = \frac{1}{c_{i_n}}\sum_{\mathbf{i}'=\mathbf{j}i_n} c_{\mathbf{i}'}$$

where $\mathbf{j}$ runs through all permutations of $\delta(\mathbf{i})$. We prove that for sufficiently small $r$ we have

$$\frac{1}{\max\{c_j\}}\sup_{\mathbf{i}\in\mathcal{A}_r^a} T(\mathbf{i}) \le \sum_{\mathbf{i}\in\mathcal{A}_r^a} c_{\mathbf{i}} \le O(\log^m r^{-1})\sup_{\mathbf{i}\in\mathcal{A}_r^a} T(\mathbf{i}). \tag{3.1}$$

The left inequality is clear. To see the right inequality we have from Lemma 3.5 that $|\mathbf{i}| \le C\log r^{-1}$ for any $\mathbf{i} \in \mathcal{A}_r^a$. When $\mathbf{i}$ runs through $\mathcal{A}^a$ the number of distinct vectors $(|\delta(\mathbf{i})|_1, |\delta(\mathbf{i})|_2, \cdots, |\delta(\mathbf{i})|_m)$ is bounded by $(C\log r^{-1})^m = O(\log^m r^{-1})$. Also there are at most $m$ choices for the last letter of $\mathbf{i}$. The right inequality in (3.1) then follows.

Now for any $\mathbf{i} = i_1 \cdots i_n i_{n+1} \in \mathcal{A}^a$ set $t_j = \frac{n_j}{n}$ where $n_j = |\delta(\mathbf{i})|_j$ and $n = |\delta(\mathbf{i})|$. By Lemma 3.4,

$$\frac{\log T(\mathbf{i})}{n} = \sum_{j=1}^{m} \big(-t_j \log t_j + t_j \log c_j\big) + O\Big(\frac{\log n}{n}\Big).$$

On the other hand we have $a_{\mathbf{i}} = \prod_{j=1}^{m} a_j^{n_j} a_{i_n} < r \le \prod_{j=1}^{m} a_j^{n_j}$. Hence

$$\frac{-\log r}{n} = -\sum_{j=1}^{m} t_j \log a_j + O\Big(\frac{\log n}{n}\Big).$$

Combining the two estimates yields

$$\frac{\log T(\mathbf{i})}{\log r} = \frac{\sum_{j=1}^{m}\big(-t_j \log t_j + t_j \log c_j\big)}{\sum_{j=1}^{m} t_j \log a_j} + O\Big(\frac{\log n}{n}\Big). \tag{3.2}$$

The condition $a_{\mathbf{i}} \le b_{\mathbf{i}}$ is equivalent to

$$\sum_{j=1}^{m} t_j \log b_j + \frac{b_{i_{n+1}}}{n} \ge \sum_{j=1}^{m} t_j \log a_j + \frac{a_{i_{n+1}}}{n}. \tag{3.3}$$

The proposition follows from (3.2) and (3.3), by letting $n$ tends to $\infty$. ∎

We now prove Proposition 3.2. We will need to invoke the following Large Deviation Principle:

**Lemma 3.6** (Large Deviation Principle)**.** *Let $\mathbf{p} = (p_1, \ldots, p_m)$ be a probability vector. For any $\varepsilon > 0$ there exists an $N = N(\varepsilon) > 0$ such that $\sum_{\mathbf{i} \in \mathcal{B}_n(\varepsilon)} p_{\mathbf{i}} < e^{-nN}$ for all sufficiently large $n$, where*

$$\mathcal{B}_n(\varepsilon) := \Big\{\mathbf{i} \in \Sigma^n : \ \sum_{j=1}^{m} \Big|\frac{|\mathbf{i}|_j}{n} - p_j\Big| > \varepsilon\Big\} \tag{3.4}$$

*with $\Sigma = \Sigma(m)$.*

**Proof.** Standard. ∎

**Proof of Proposition 3.2.** As with Proposition 3.1, we prove (i) only. The others are proved using identical arguments.

Assume that $\sum_{j=1}^{m} p_j(\log b_j - \log a_j) = \delta_0 > 0$. For any $\eta > 0$ let $\varepsilon = \varepsilon(\eta) = \eta/M$ where

$$M = 2m \sum_{j=1}^{m} (|\log c_j| + |\log a_j| + |\log b_j|).$$

Then it is easily verified that for any $\mathbf{i} \in \Sigma^n \setminus \mathcal{B}_n(\varepsilon)$ we have

$$\Big|\frac{1}{n}\log c_{\mathbf{i}} - \sum_{j=1}^m p_j \log c_j\Big| < \eta,$$

as well as

$$\Big|\frac{1}{n}\log a_{\mathbf{i}} - \sum_{j=1}^m p_j \log a_j\Big| < \eta, \quad \Big|\frac{1}{n}\log b_{\mathbf{i}} - \sum_{j=1}^m p_j \log b_j\Big| < \eta.$$

Therefore

$$\frac{\log c_{\mathbf{i}}}{\log a_{\mathbf{i}}} = \frac{\sum_{j=1}^m p_j \log c_j}{\sum_{j=1}^m p_j \log a_j} + O(\eta). \tag{3.5}$$

Note that for this $\varepsilon > 0$ there is an $N(\varepsilon) > 0$ such that $\sum_{\mathbf{i}\in\mathcal{B}_n(\varepsilon)} p_{\mathbf{i}} < e^{-nN(\varepsilon)}$ for all $n \geq n_0$.

By Lemma 3.4, for any $\mathbf{i} \in \mathcal{A}_r$ we have $C^{-1}\log r^{-1} \leq |\mathbf{i}| \leq C\log r^{-1}$. Let $r > 0$ be sufficiently small so that $C^{-1}\log r^{-1} \geq n_0$. We now decompose $\mathcal{A}_r$ into $\mathcal{A}_{r,1}$ and $\mathcal{A}_{r,2}$ with

$$\mathcal{A}_{r,1} = \mathcal{A}_r \setminus \mathcal{A}_{r,2}, \qquad \text{and} \qquad \mathcal{A}_{r,2} = \mathcal{A}_r \cap \big(\bigcup_{n\geq 1} \mathcal{B}_n(\varepsilon)\big).$$

By observing that $\frac{\log c_{\mathbf{i}}}{-\log r} \leq C_0 := C\max_{1\leq j\leq m}\{|\log c_j|\}$ we obtain

$$\Big|\sum_{\mathbf{i}\in\mathcal{A}_{r,2}} \frac{p_{\mathbf{i}}\log c_{\mathbf{i}}}{-\log r}\Big| \leq C_0 \sum_{\mathbf{i}\in\mathcal{A}_{r,2}} p_{\mathbf{i}} \leq C_0 \sum_{C^{-1}\log r^{-1}\leq k\leq C\log r^{-1}} e^{-nN(\varepsilon)}.$$

Hence $|\sum_{\mathbf{i}\in\mathcal{A}_{r,2}} \frac{p_{\mathbf{i}}\log c_{\mathbf{i}}}{-\log r}|$ tends to 0 as $r \longrightarrow 0$. On the other hand, because $a_{\mathbf{i}} < r \leq a_{\delta(\mathbf{i})}$ we have $\log r = \log a_{\mathbf{i}} + O(1)$. If $\mathbf{i} \in A_{r,1}$ then by (3.5)

$$\begin{aligned}\sum_{\mathbf{i}\in\mathcal{A}_{r,1}} \frac{p_{\mathbf{i}}\log c_{\mathbf{i}}}{-\log r} &= -\sum_{\mathbf{i}\in\mathcal{A}_{r,1}} p_{\mathbf{i}}\Big(\frac{\sum_{j=1}^m p_j\log c_j}{\sum_{j=1}^m p_j \log a_j} + O(\eta)\Big) \\ &= -\frac{\sum_{j=1}^m p_j\log c_j}{\sum_{j=1}^m p_j \log a_j}\sum_{\mathbf{i}\in\mathcal{A}_{r,1}} p_{\mathbf{i}} + O(\eta).\end{aligned}$$

Since $\sum_{\mathbf{i}\in\mathcal{A}_r} p_{\mathbf{i}} = 1$ because $\{[\mathbf{i}] : \ \mathbf{i}\in\mathcal{A}_r\}$ is a partition of $\Sigma^{\mathbb{N}}$ and

$$\sum_{\mathbf{i}\in\mathcal{A}_{r,2}} p_{\mathbf{i}} \leq \sum_{C^{-1}\log r^{-1}\leq k\leq C\log r^{-1}} e^{-nN(\varepsilon)} \longrightarrow 0$$

as $r \longrightarrow 0$, we must have $\lim_{r\to 0}\sum_{\mathbf{i}\in\mathcal{A}_{r,1}} p_{\mathbf{i}} = 1$. Now because $\sum_{j=1}^m p_j(\log b_j - \log a_j) = \delta_0 > 0$, $\mathcal{A}_{r,1} \subseteq \mathcal{A}_r^a$ whenever $\eta$ (and hence $\varepsilon$) is sufficiently small. It follows that

$$\sum_{\mathbf{i}\in\mathcal{A}_{r,1}} \frac{p_{\mathbf{i}}\log c_{\mathbf{i}}}{-\log r} \leq \sum_{\mathbf{i}\in\mathcal{A}_r^a} \frac{p_{\mathbf{i}}\log c_{\mathbf{i}}}{-\log r} \leq \sum_{\mathbf{i}\in\mathcal{A}_r} \frac{p_{\mathbf{i}}\log c_{\mathbf{i}}}{-\log r}. \tag{3.6}$$

Taking limit $r\longrightarrow 0$ yields $\Omega^a = \frac{\sum_{j=1}^m p_j \log c_j}{\sum_{j=1}^m p_j \log a_j}$. To see that $\Omega^b = 0$ we only need to observe that by (3.6),

$$\Omega^a + \Omega^b = \lim_{r\to 0} \sum_{\mathbf{i}\in\mathcal{A}_r} \frac{p_{\mathbf{i}} \log c_{\mathbf{i}}}{\log r} = \Omega^a.$$

∎

## 4. Proof of Theorem 2.1.

We adopt the following definition from [21]:

**Definition 4.1.** Let $K$ be a compact set in $\mathbb{R}^d$. Fix $M, \varepsilon > 0$ and $N \in \mathbb{N}$. A covering $\{G_i\}_{i=1}^n$ of $K$ by Borel sets is said to be $(M, \varepsilon, N)$–*good* if diam $(G_i) \le M\varepsilon$ for all $i$, and any $\varepsilon$-cube in $\mathbb{R}^d$ intersects at most $N$ elements in the covering.

**Lemma 4.1.** *Let $M, q > 0$ and $N, d \in \mathbb{N}$. There exists a constant $C_1 = C_1(M, N, d, q)$ such that for any compactly supported probability measure $\nu$ on $\mathbb{R}^d$ and any $(M, 2^{-n}, N)$-good Borel covering $\{G_i\}$ of* supp $(\nu)$ *we have*

$$C_1^{-1} \tau_n(\nu, q) \le \sum_i \nu^q(G_i) \le C_1 \tau_n(\nu, q)$$

*where $\tau_n(\nu, q)$ is defined in (1.3).*

**Proof.** See [21], Lemma 2.2. ∎

Let $\{\phi_j\}_{j=1}^m$ be an IFS in $\mathbb{R}^d$. For any $\mathbf{i} = i_1 i_2 \cdots i_n \in \Sigma^n$, $\Sigma = \{1, 2, \ldots, m\}$ we let $\phi_{\mathbf{i}}$ denote $\phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_n}$.

**Lemma 4.2.** *Let $\mathcal{I} = \{\phi_j\}_{j=1}^m$ be an IFS in $\mathbb{R}^d$ and $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ be a probability vector. Then for any compact set $F$ we have*

$$\nu_{\mathcal{I},\mathbf{p}}(F) = \lim_{n\to\infty} \sum_{\mathbf{i}\in\mathcal{B}_n} p_{\mathbf{i}}$$

*where $\mathcal{B}_n = \{\mathbf{i} \in \Sigma^n : \ \phi_{\mathbf{i}}(K) \cap F \neq \emptyset\}$ and $K = K(\mathcal{I})$.*

**Proof.** Standard. ∎

**Lemma 4.3.** *Under the assumptions of Theorem 2.1, for any $\mathbf{i} \in \Sigma^n$ we have $\nu(\phi_{\mathbf{i}}(K)) = p_{\mathbf{i}}$, where $\nu := \nu_{\mathcal{I},\mathbf{p}}$ and $K = K(\mathcal{I})$.*

**Proof.** Let $T$ be the open rectangle for the ROSC, so $\{\phi_j(T)\}_{j=1}^m$ are disjoint open rectangles in $T$. First we consider the case in which $\phi_j(\overline{T}) \subset T$ for some $1 \le j \le m$. Without loss of generality we assume that $\phi_1(\overline{T}) \subset T$.

Let $\mathbf{l} \in \Sigma^n$. By Lemma 4.2 we have

$$\mathcal{B}_k = \Big\{\mathbf{i} \in \Sigma^k : \ \phi_{\mathbf{i}}(K) \cap \phi_{\mathbf{l}}(K) \neq \emptyset\Big\} \supseteq \{\mathbf{l}\} \times \Sigma^{k-n}.$$

Hence $\nu(\phi_{\mathbf{l}}(K)) = \lim_{k\to\infty} \sum_{\mathbf{i}\in\mathcal{B}_k} p_{\mathbf{i}} \ge p_{\mathbf{l}}$. We prove the converse. In fact we prove $\nu(\phi_{\mathbf{l}}(\overline{T})) \le p_{\mathbf{l}}$. Let

$$\mathcal{C}_k = \{\mathbf{i} \in \Sigma^k : \ \phi_{\mathbf{i}}(K) \cap \phi_{\mathbf{l}}(\overline{T}) \neq \emptyset\}$$

and set

$$\mathcal{C}_k^1 = \{\mathbf{i} \in \mathcal{C}_k : \ \phi_{\mathbf{i}}(K) \subseteq \phi_{\mathbf{l}}(\overline{T})\}, \qquad \mathcal{C}_k^2 = \mathcal{C}_k \setminus \mathcal{C}_k^1.$$

Note that $\mathcal{C}_k^1 = \{\mathbf{l}\} \times \Sigma^{k-n}$. Hence $\lim_{k\to\infty} \sum_{\mathbf{i}\in\mathcal{C}_k^1} p_{\mathbf{i}} = p_{\mathbf{l}}$. On the other hand,

$$\mathcal{C}_k^2 \subseteq \Big\{\mathbf{i} = 1_1 i_2 \cdots i_k : \ i_1 i_2 \cdots i_n \neq \mathbf{l}, \ l_j \neq 1 \text{ for } j > n\Big\}.$$

Hence $\sum_{\mathbf{i}\in\mathcal{C}_k^2} p_{\mathbf{i}} < (1-p_1)^{k-n} \longrightarrow 0$ as $k \longrightarrow \infty$. Thus $\lim_{k\to\infty} \sum_{\mathbf{i}\in\mathcal{C}_k} p_{\mathbf{i}} = p_{\mathbf{l}}$. It follows that

$$\nu(\phi_{\mathbf{l}}(K)) \le \nu(\phi_{\mathbf{l}}(\overline{T})) = p_{\mathbf{l}}. \tag{4.1}$$

By considering the iterations of the IFS $\mathcal{I}$ it is clear that the above proof extends to the case in which there exists an $\mathbf{i} \in \Sigma^*$ such that $\phi_{\mathbf{i}}(\overline{T}) \subset T$.

It remains to prove the lemma when $\phi_{\mathbf{i}}(\overline{T}) \cap \partial T \neq \emptyset$ for all nonempty $\mathbf{i} \in \Sigma^*$. In this case itis clear that $K$ is contained in a line parallel to one of the axes, say, the horizontal axis. Then $\nu$ is identical to its projection $\nu^x$ onto the $x$-axis up to a translation. Furthermore the projection IFS $\mathcal{I}^x$ must satisfy the OSC (and it is self-similar). Therefore the lemma still holds. ∎

**Proposition 4.4.** *Let $\nu$ be a self-similar probability measure in $\mathbb{R}$ with* $\operatorname{supp}(\nu) \subseteq [c,d]$. *For any $q, \delta > 0$ there exist constants $C_1, C_2 > 0$ depending on $\nu, q, \delta$ such that for any $n > 0$ we have*

$$C_1 n^{-\tau(\nu,q)-\delta} \le \sum_{i=1}^n \nu^q(I_i) \le C_2 n^{-\tau(\nu,q)+\delta}.$$

*where $I_i = [c + \frac{(i-1)(d-c)}{n}, \ c + \frac{i(d-c)}{n}]$.*

**Proof.** It is known that for a self-similar measure the $L^q$ spectrum exists, see Peres and Solomyak [21]. Set $\Delta_n = \frac{d-c}{n}$, which is the length of each interval $I_i$. By definition and Lemma 4.1 we have

$$\lim_{n\to\infty} \frac{\log \sum_{i=0}^n \nu^q(I_i)}{\log \Delta_n} = \tau(\nu, q).$$

and

$$\lim_{n\to\infty} \frac{\sum_{i=0}^n \nu^q(I_i)}{\log \Delta_n} = \tau(\nu, q).$$

Thus for any $\delta > 0$ there exists an $n_0$ such that for all $n > n_0$ we have

$$\Delta_n^{-\tau(\nu,q)-\delta} \leq \sum_{i=0}^n \nu^q(I_i) \leq \Delta_n^{-\tau(\nu,q)+\delta}.$$

Now for $1 \leq n \leq n_0$ we simply choose $C_1$ and $C_2$ to satisfy the inequalities of the proposition. ∎

**Proof of Theorem 2.1.** Let $r > 0$ be sufficiently small. We construct a covering $\{G_i\}$ of $\mathrm{supp}\,(\nu)$ as follows: Let $T$ be the open rectangle associated with the ROSC for the IFS $\mathcal{I}$. Without loss of generality we may assume that $T$ is a unit square. For $\mathbf{a} = (a_j)_{j=1}^m$ and $\mathbf{b} = (b_j)_{j=1}^m$ define the subsets of $\mathcal{A}_r$, $\mathcal{A}_r^a$ and $\mathcal{A}_r^b$ of $\Sigma^*$ with $\Sigma = \{1, 2, \ldots, m\}$ as in Section 3. For any $\mathbf{i} \in \mathcal{A}_r^a$ by definition $b_\mathbf{i} \geq a_\mathbf{i}$, and we set $w_a(\mathbf{i}) := [b_\mathbf{i}/a_\mathbf{i}]$. Note that $\phi_\mathbf{i}(\overline{T})$ is a closed rectangle of width $a_\mathbf{i}$ and height $b_\mathbf{i}$; so $w_a(\mathbf{i})$ is the aspect ratio of the rectangle rounded off to an integer. We now cut $\phi_\mathbf{i}(\overline{T})$ horizontally into $w_a(\mathbf{i})$ equal rectangles of width $a_\mathbf{i}$ and height $b_\mathbf{i}/w_a(\mathbf{i})$. Call these smaller rectangles $\{R_{\mathbf{i},k}^a\}_{k=1}^{w_a(\mathbf{i})}$. Similarly for any $\mathbf{i} \in \mathcal{A}_r^b$ by definition $a_\mathbf{i} > b_\mathbf{i}$, and we set $w_b(\mathbf{i}) := [a_\mathbf{i}/b_\mathbf{i}]$. $\phi_\mathbf{i}(\overline{T})$ is a closed rectangle of width $a_\mathbf{i}$ and height $b_\mathbf{i}$. We now cut $\phi_\mathbf{i}(\overline{T})$ vertically into $w_b(\mathbf{i})$ equal rectangles of width $a_\mathbf{i}/w_b(\mathbf{i})$ and height $b_\mathbf{i}$. Call these smaller rectangles $\{R_{\mathbf{i},k}^b\}_{k=1}^{w_b(\mathbf{i})}$.

Observe that if $s^- = \min\{a_j, b_j\}$ and $s^+ = \max\{a_j, b_j\}$ then each $R_{\mathbf{i},k}^a$ and $R_{\mathbf{i},k}^b$ has width and height between $s^- r$ and $r/s^+$. Furthermore

$$\mathbf{C}_r = \Big\{R_{\mathbf{i},k}^a : \ \mathbf{i} \in \mathcal{A}_r^a,\ 1 \leq k \leq w_a(\mathbf{i})\Big\} \cup \Big\{R_{\mathbf{i},k}^b : \ \mathbf{i} \in \mathcal{A}_r^b,\ 1 \leq k \leq w_b(\mathbf{i})\Big\}$$

is a covering of $\mathrm{supp}\,(\nu)$. It follows that $\mathbf{C}_r$ is an $(M, r, N)$-good covering of $\mathrm{supp}\,(\nu)$ with $M = s^-/2$ and $N = 4$.

The key is to estimate $\sum_{k=1}^{w_a(\mathbf{i})} \nu^q(R_{\mathbf{i},k}^a)$ for $\mathbf{i} \in \mathcal{A}_r^a$ and $\sum_{k=1}^{w_b(\mathbf{i})} \nu^q(R_{\mathbf{i},k}^b)$ for $\mathbf{i} \in \mathcal{A}_r^b$. We make the following claim:

*Claim: For any $\delta > 0$ there exist constants $C_1$ and $C_2$ independent of $\mathbf{i}$ and $r$ such that*

$$C_1 p_{\mathbf{i}}^q (w_a(\mathbf{i}))^{-\tau(\nu^y,q)-\delta} \leq \sum_{k=1}^{w_a(\mathbf{i})} \nu^q(R_{\mathbf{i},k}^a) \leq C_2 p_{\mathbf{i}}^q (w_a(\mathbf{i}))^{-\tau(\nu^y,q)+\delta}. \tag{4.2}$$

Proof of Claim: The combination of Lemma 4.2 and Lemma 4.3 implies that

$$\begin{aligned}
\nu(R_{\mathbf{i},k}^a) &= \lim_{n\to\infty} \Big\{ p_{\mathbf{j}} : \ \mathbf{j} \in \Sigma^n, \ \phi_{\mathbf{j}}(K) \subseteq \phi_{\mathbf{i}}(\overline{T}), \ \phi_{\mathbf{j}}(K) \cap R_{\mathbf{i},k}^a \neq \emptyset \Big\} \\
&= \lim_{n\to\infty} \Big\{ p_{\mathbf{j}} : \ \mathbf{j} \in \{\mathbf{i}\} \times \Sigma^{n-|\mathbf{i}|}, \ \phi_{\mathbf{j}}(K) \cap R_{\mathbf{i},k}^a \neq \emptyset \Big\} \\
&= p_{\mathbf{i}} \lim_{n\to\infty} \Big\{ p_{\mathbf{j}} : \ \mathbf{j} \in \Sigma^n, \ \phi_{\mathbf{i}} \circ \phi_{\mathbf{j}}(K) \cap R_{\mathbf{i},k}^a \neq \emptyset \Big\}.
\end{aligned}$$

But observe that the set $\{\mathbf{j} \in \Sigma^n : \ \phi_{\mathbf{i}} \circ \phi_{\mathbf{j}}(K) \cap R_{\mathbf{i},k}^a \neq \emptyset\}$ is precisely the set

$$\Big\{ \mathbf{j} \in \Sigma^n : \ \phi_{\mathbf{j}}^y(K_y) \cap I_{\mathbf{i},k} \neq \emptyset \Big\}, \tag{4.3}$$

where $\phi_{\mathbf{j}}^y := \pi_y \circ \phi_{\mathbf{j}} \circ \pi_y^{-1}$, $K_y = \pi_y(K)$ and $I_{\mathbf{i},k} := [\frac{c+(k-1)(d-c)}{w_a(\mathbf{i})}, \frac{c+k(d-c)}{w_a(\mathbf{i})}]$ with $[c,d] = \pi_y(\overline{T})$. (So by assumption actually $d-c=1$.) Proposition 4.4 now asserts that

$$C_1 p_{\mathbf{i}}^q (w_a(\mathbf{i}))^{-\tau(\nu^y,q)-\delta} \leq \sum_{k=1}^{w_a(\mathbf{i})} \nu^q(R_{\mathbf{i},k}^a) \leq C_2 p_{\mathbf{i}}^q (w_a(\mathbf{i}))^{-\tau(\nu^y,q)+\delta}$$

for some constants $C_1$ and $C_2$, proving the claim. ∎

By an identical argument we also have constants $C_1'$ and $C_2'$ such that

$$C_1' p_{\mathbf{i}}^q (w_b(\mathbf{i}))^{-\tau(\nu^x,q)-\delta} \leq \sum_{k=1}^{w_b(\mathbf{i})} \nu^q(R_{\mathbf{i},k}^b) \leq C_2' p_{\mathbf{i}}^q (w_b(\mathbf{i}))^{-\tau(\nu^x,q)+\delta} \tag{4.4}$$

for any $\mathbf{i} \in \mathcal{A}_r^b$.

To complete the proof of our theorem,

$$\sum_{D\in\mathbf{C}_r} \nu^q(D) = \sum_{\mathbf{i}\in\mathcal{A}_r^a} \sum_{k=1}^{w_a(\mathbf{i})} \nu^q(R_{\mathbf{i},k}^a) + \sum_{\mathbf{i}\in\mathcal{A}_r^b} \sum_{k=1}^{w_b(\mathbf{i})} \nu^q(R_{\mathbf{i},k}^b).$$

It follows from (4.2) and (4.4) that

$$C_1 \sum_{\mathbf{i}\in\mathcal{A}_r^a} p_{\mathbf{i}}^q (w_a(\mathbf{i}))^{-\tau(\nu^y,q)-\delta} \leq \sum_{\mathbf{i}\in\mathcal{A}_r^a} \sum_{k=1}^{w_a(\mathbf{i})} \nu^q(R_{\mathbf{i},k}^a) \leq \sum_{\mathbf{i}\in\mathcal{A}_r^a} C_2 p_{\mathbf{i}}^q (w_a(\mathbf{i}))^{-\tau(\nu^y,q)+\delta},$$

and similarly

$$C_1' \sum_{\mathbf{i}\in\mathcal{A}_r^b} p_{\mathbf{i}}^q (w_b(\mathbf{i}))^{-\tau(\nu^x,q)-\delta} \leq \sum_{\mathbf{i}\in\mathcal{A}_r^b} \sum_{k=1}^{w_b(\mathbf{i})} \nu^q(R_{\mathbf{i},k}^b) \leq \sum_{\mathbf{i}\in\mathcal{A}_r^b} C_2' p_{\mathbf{i}}^q (w_b(\mathbf{i}))^{-\tau(\nu^x,q)+\delta}.$$

Note that $\frac{b_{\mathbf{i}}}{2a_{\mathbf{i}}} \leq w_a(\mathbf{i}) \leq \frac{b_{\mathbf{i}}}{a_{\mathbf{i}}}$. Applying Proposition 3.1 twice with $\mathbf{c} = \{c_j\}_{j=1}^m$ set to be $c_j = p_j^q(\frac{b_j}{a_j})^{-\tau(\nu^y,q)-\delta}$ and $c_j = p_j^q(\frac{b_j}{a_j})^{-\tau(\nu^y,q)+\delta}$ respectively, and with $\delta \longrightarrow 0$, yields

$$\lim_{r\to 0} \frac{\log \sum_{\mathbf{i}\in\mathcal{A}_r^a} \sum_{k=1}^{w_a(\mathbf{i})} \nu^q(R_{\mathbf{i},k}^a)}{\log r} = \inf_{\mathbf{t}\in\Gamma(\mathbf{e}^a)} \frac{\sum_{j=1}^m t_j\Big(-\log t_j - \tau(\nu^y,q)(\log b_j - \log a_j) + q\log p_j\Big)}{\sum_{j=1}^m t_j \log a_j},$$

and similarly

$$\lim_{r\to 0} \frac{\log \sum_{\mathbf{i}\in\mathcal{A}_r^b} \sum_{k=1}^{w_b(\mathbf{i})} \nu^q(R_{\mathbf{i},k}^b)}{\log r} = \inf_{\mathbf{t}\in\Gamma(\mathbf{e}^b)} \frac{\sum_{j=1}^m t_j\Big(-\log t_j - \tau(\nu^x,q)(\log a_j - \log b_j) + q\log p_j\Big)}{\sum_{j=1}^m t_j \log b_j}.$$

The proof is finally complete by observing that for any $A \geq B > 0$ we have

$$\log A < \log(A+B) \leq \log A + \log 2.$$

∎

**Proof of Theorem 2.2.** It is clear from the proof of Theorem 2.1 that if all $a_j \leq b_j$ then $\tau(\nu,q) = \theta_a$ where $\theta_a$ is given in Theorem 2.1. In this case $\Gamma_0 := \Gamma(\mathbf{e}^a) = \{\mathbf{t} = (t_1,\ldots,t_m) : \ t_j \geq 0 \text{ and } \sum_{j=1}^m t_j = 1\}$. Hence

$$\tau(\nu,q) = \tau(\nu^y,q) + \inf_{\mathbf{t}\in\Gamma_0} \frac{\sum_{j=1}^m t_j\Big(-\log t_j - \tau(\nu^y,q)\log b_j + q\log p_j\Big)}{\sum_{j=1}^m t_j \log a_j}. \tag{4.5}$$

We first simplify the notation. Set $A_j = 1/a_j$ and $B_j = p_j^q b_j^{-\tau(\nu^y,q)}$. Then (4.5) becomes

$$\tau(\nu^x,q) - \tau(\nu,q) = \sup_{\mathbf{t}\in\Gamma_0} \frac{\sum_{j=1}^m t_j \log\frac{B_j}{t_j}}{\sum_{j=1}^m t_j \log A_j}. \tag{4.6}$$

Let $\theta$ be the unique real root of the equation $\sum_{j=1}^m (A_j)^{-\theta}B_j = 1$ (the existence of $\theta$ follows from the fact that $A_j > 1$ and $B_j > 0$ for all $j$). Then

$$\frac{\sum_{j=1}^m t_j \log\frac{B_j}{t_j}}{\sum_{j=1}^m t_j \log A_j} - \theta = \frac{\sum_{j=1}^m t_j \log\frac{(A_j)^{-\theta}B_j}{t_j}}{\sum_{j=1}^m t_j \log A_j}.$$

By the Jessen inequality,

$$\sum_{j=1}^m t_j \log\frac{(A_j)^{-\theta}B_j}{t_j} \leq \log\Big(\sum_{j=1}^m (A_j)^{-\theta}B_j\Big) = 0.$$

The "=" in the first inequality is achieved when $t_j = (A_j)^{-\theta}B_j$ for $1 \leq j \leq m$. It follows $\tau(\nu^x,q) - \tau(\nu,q) = \theta$. This proves the theorem. ∎

## 5. Proof of Theorem 2.4

Let $\nu$ be a compactly supported probability measure in $\mathbb{R}^d$. For any $\varepsilon > 0$ and $N \in \mathbb{N}$ a family of Borel sets $\{G_i\}$ is called an $(\varepsilon, N)$-*good partition* of $\operatorname{supp}(\nu)$ if the following conditions are met:

(i) $\bigcup_i G_i \supseteq \operatorname{supp}(\nu)$ and $\nu(G_i \cap G_j) = 0$ for all $i \neq j$.

(ii) Any cube of side $\varepsilon$ intersects at most $N$ elements of $\{G_i\}$ and $\operatorname{diam}(G_i) < \varepsilon$.

Set $f(x) = x\log(1/x) = -x\log x$ and define $h_n(\nu) = \sum_{Q\in\mathbf{D}_n} f(\nu(Q))$, where $\mathbf{D}_n$ is the standard partition of $\mathbb{R}^d$ by cubes of sides $2^{-n}$ defined in Section 1. We have

**Lemma 5.1.** *Let $\{G_i\}$ be a $(2^{-n}, N)$-good partition of* $\operatorname{supp}(\nu)$ *where $\nu$ is any compactly supported probability measure in $\mathbb{R}^d$. Then*

$$\Big|\sum_i f(\nu(G_i)) - h_n(\nu)\Big| \leq C$$

*where* $C = \max(\log N, \log 2^d)$.

**Proof.** It is easy to check that for all $x_1, \dots, x_k \geq 0$ we have

$$f\Big(\sum_{i=1}^k x_i\Big) \leq \sum_{i=1}^k f(x_i) \leq f\Big(\sum_{i=1}^k x_i\Big) + \Big(\sum_{i=1}^k x_i\Big)\log k. \tag{5.1}$$

Write $\mathbf{D}_n = \{Q_j\}$ and consider the refinement $\mathbf{G}^* = \{G_i \cap Q_j\}$ of the $(2^{-n}, N)$-good partition $\{G_i\}$. Note that $\operatorname{diam}(G_i) < 2^{-n}$ implies that $G_i$ intersects at most $2^d$ cubes in $\mathbf{D}_n$. It follows from (5.1) that

$$f(\nu(G_i)) \leq \sum_j f(\nu(G_i \cap Q_j)) \leq f(\nu(G_i)) + \log(2^d)\,\nu(G_i). \tag{5.2}$$

Conversely, also by (5.1) we have

$$f(\nu(Q_j)) \leq \sum_i f(\nu(G_i \cap Q_j)) \leq f(\nu(Q_j)) + \log(N)\,\nu(Q_j). \tag{5.3}$$

Summing up (5.2) over $i$ yields

$$\sum_i f(\nu(G_i)) \leq \sum_{i,j} f(\nu(G_i \cap Q_j)) \leq \sum_i f(\nu(G_i)) + \log(2^d), \tag{5.4}$$

and summing up (5.3) over $j$ yields

$$h_n(\nu) \leq \sum_{i,j} f(\nu(G_i \cap Q_j)) \leq h_n(\nu) + \log N. \tag{5.5}$$

The lemma now follows by combining (5.4) and (5.5). ∎

**Proposition 5.2.** *Let $\nu$ be a self-similar probability measure in $\mathbb{R}$ with* $\mathrm{supp}\,(\nu) \subseteq [c,d]$. *For any $q, \delta > 0$ there exist constants $C_1, C_2 > 0$ depending on $\nu$ and $\delta$ such that for any $n > 0$ we have*

$$(h(\nu) - \delta)\log n + C_1 \le \sum_{i=1}^{n} f(I_i) \le (h(\nu)+\delta)\log n + C_2,$$

*where $I_i = [c + \frac{(i-1)(d-c)}{n}, c + \frac{i(d-c)}{n})$.*

**Proof.** The proof is essentially identical to that of Proposition 4.4. By Lemma 5.1 we have $|\sum_{i=0}^{n} f(I_i) - h_k(\nu)| \le C$ for some constant $C$, where $k > 0$ is chosen so that $2^{-k} > \frac{d-c}{n} \ge 2^{-k-1}$, since $\{I_i\}$ is a $(2^{-k}, 2)$-good partition of $\mathrm{supp}\,(\nu)$. For any $\delta > 0$ there exists a $k_0 > 0$ such that for all $k > k_0$ there is

$$h(\nu) - \delta < \frac{h_k(\nu)}{k \log 2} < h(\nu) + \delta.$$

But clearly by assumption $|\log n - k\log 2| \le C'$ for some constant $C'$. Therefore for $k > k_0$ we have constants $C_1'$ and $C_2'$ such that

$$(h(\nu) - \delta)\log n + C_1' \le \sum_{i=0}^{n} f(I_i) \le (h(\nu)+\delta)\log n + C_2'.$$

Now for $k \le k_0$ there are only finitely many correpsonding $n$, so we may find $C_1$ and $C_2$ such that

$$(h(\nu) - \delta)\log n + C_1 \le \sum_{i=0}^{n} f(I_i) \le (h(\nu)+\delta)\log n + C_2.$$

This proves the proposition. ∎

**Proof of Theorem 2.4.** For $\nu = \nu_{\mathcal{I},\mathbf{p}}$ we may assume without loss of generality that $\nu(L) = 0$ for any line $L$ in $\mathbb{R}^2$, for otherwise $\nu$ is in essence a self-similar measure in the one dimension, leaving us with nothing to prove.

We adopt the same notations from the proof of Theorem 2.1. For any $r > 0$ sufficinetly small let $\mathbf{C}_r$ be the covering of $\mathrm{supp}\,(\nu)$ given by

$$\mathbf{C}_r = \left\{ R^a_{\mathbf{i},k} : \ \mathbf{i} \in \mathcal{A}^a_r, \ 1 \le k \le w_a(\mathbf{i}) \right\} \cup \left\{ R^b_{\mathbf{i},k} : \ \mathbf{i} \in \mathcal{A}^b_r, \ 1 \le k \le w_b(\mathbf{i}) \right\}$$

as in the proof of Theorem 2.1. It follows that $\mathbf{C}_r$ is a $(r, N)$-good covering of $K = \mathrm{supp}\,(\nu)$ for some suitable $N$ independent of $r$. We estimate $\sum_{Q \in \mathbf{C}_r} f(\nu(Q))$ for $f(x) = x\log(1/x) = -x\log x$.

We first estimate $\sum_{k=1}^{w_a(\mathbf{i})} f(\nu(R^a_{\mathbf{i},k}))$ for any $\mathbf{i} \in \mathcal{A}^a_r$. By (4.3) we have

$$\begin{aligned} \nu(R^a_{\mathbf{i},k}) &= p_{\mathbf{i}} \lim_{n\to\infty} \sum \left\{ \mathbf{j} \in \Sigma^n : \ \phi^y_{\mathbf{j}}(K_y) \cap I_{\mathbf{i},k} \neq \emptyset \right\} \\ &= p_{\mathbf{i}} \nu^y(I_{\mathbf{i},k}) \end{aligned}$$

where $K_y$ and $I_{\mathbf{i},k}$ are as in the proof of Theorem 2.1. Thus

$$\sum_{k=1}^{w_a(\mathbf{i})} f(\nu(R^a_{\mathbf{i},k})) = p_{\mathbf{i}} \sum_{k=1}^{w_a(\mathbf{i})} f(\nu^y(I_{\mathbf{i},k})) - p_{\mathbf{i}} \log p_{\mathbf{i}}.$$

It now follows from Proposition 5.2 that for any $\delta > 0$ there are $C_1$ and $C_2$ independent of $r$ such that

(5.6)

$$p_{\mathbf{i}}(h(\nu^y)-\delta) \log w_a(\mathbf{i}) - p_{\mathbf{i}} \log p_{\mathbf{i}} + C_1 p_{\mathbf{i}} \leq \sum_{k=1}^{w_a(\mathbf{i})} f(\nu(R^a_{\mathbf{i},k})) \leq p_{\mathbf{i}}(h(\nu^y)+\delta) \log w_a(\mathbf{i}) - p_{\mathbf{i}} \log p_{\mathbf{i}} + C_2 p_{\mathbf{i}}.$$

Similarly for any $\mathbf{i} \in \mathcal{A}^b_r$ there exist $C'_1$ and $C'_2$ independent of $r$ such that

(5.7)

$$p_{\mathbf{i}}(h(\nu^x)-\delta) \log w_b(\mathbf{i}) - p_{\mathbf{i}} \log p_{\mathbf{i}} + C'_1 p_{\mathbf{i}} \leq \sum_{k=1}^{w_b(\mathbf{i})} f(\nu(R^b_{\mathbf{i},k})) \leq p_{\mathbf{i}}(h(\nu^x)+\delta) \log w_b(\mathbf{i}) - p_{\mathbf{i}} \log p_{\mathbf{i}} + C'_2 p_{\mathbf{i}}.$$

Now, observe that $\frac{b_{\mathbf{i}}}{2a_{\mathbf{i}}} \leq w_a(\mathbf{i}) \leq \frac{b_{\mathbf{i}}}{a_{\mathbf{i}}}$. So $\log(b_{\mathbf{i}}/a_{\mathbf{i}}) - \log 2 \leq \log w_a(\mathbf{i}) \leq \log(b_{\mathbf{i}}/a_{\mathbf{i}})$. This means we may replace $w_a(\mathbf{i})$ in (5.6) with $b_{\mathbf{i}}/a_{\mathbf{i}}$, with only the modification of the constants $C_1$ and $C_2$. Now for (5.6) we apply Proposition 3.2 twice, with $c_j = p_j^{-1}(b_j/a_j)^{h(\nu^y)-\delta}$ and $c_j = p_j^{-1}(b_j/a_j)^{h(\nu^y)+\delta}$ respectively, and set $\delta \longrightarrow 0$. It follows that

$$\lim_{r\to 0} \frac{\sum_{\mathbf{i}\in\mathcal{A}^a_r} \sum_{k=1}^{w_a(\mathbf{i})} f(\nu(R^a_{\mathbf{i},k}))}{\log r} = \frac{h(\nu^y)\sum_{j=1}^m p_j(\log b_j - \log a_j) - \sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \log a_j} \tag{5.8}$$

if $\sum_{j=1}^m p_j(\log b_j - \log a_j) \geq 0$, and 0 if $\sum_{j=1}^m p_j(\log a_j - \log b_j) > 0$. Similarly,

$$\lim_{r\to 0} \frac{\sum_{\mathbf{i}\in\mathcal{A}^b_r} \sum_{k=1}^{w_b(\mathbf{i})} f(\nu(R^b_{\mathbf{i},k}))}{\log r} = \frac{h(\nu^x)\sum_{j=1}^m p_j(\log a_j - \log b_j) - \sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \log b_j} \tag{5.9}$$

if $\sum_{j=1}^m p_j(\log a_j - \log b_j) \geq 0$, and 0 if $\sum_{j=1}^m p_j(\log b_j - \log a_j) > 0$. Combining (5.8) and (5.9) yields

$$\lim_{r\to 0}\frac{\sum_{G\in\mathbf{C}_r} f(\nu(G))}{\log r} = \begin{cases} \frac{h(\nu^y)\sum_{j=1}^m p_j(\log b_j-\log a_j)-\sum_{j=1}^m p_j\log p_j}{\sum_{j=1}^m p_j\log a_j} \\ \quad (\text{ if } \sum_{j=1}^m p_j(\log b_j-\log a_j)>0), \\ \frac{h(\nu^x)\sum_{j=1}^m p_j(\log a_j-\log b_j)-\sum_{j=1}^m p_j\log p_j}{\sum_{j=1}^m p_j\log b_j} \\ \quad (\text{ if } \sum_{j=1}^m p_j(\log a_j-\log b_j)>0). \end{cases} \tag{5.10}$$

To estimate the left-hand side of the above equation whenever $\sum_{j=1}^m p_j(\log a_j-\log b_j)=0$, by summing (5.6) over $i\in\mathcal{A}_r^a$ and (5.7) over $i\in\mathcal{A}_r^b$ we have

$$Q_{r,1}\le\sum_{G\in\mathbf{C}_r} f(\nu(G))\le Q_{r,2}$$

with

$$Q_{r,1}:=C_3+\sum_{\mathbf{i}\in\mathcal{A}_r^a} p_{\mathbf{i}}(h(\nu^y)-\delta)\log(b_{\mathbf{i}}/a_{\mathbf{i}})+\sum_{\mathbf{i}\in\mathcal{A}_r^b} p_{\mathbf{i}}(h(\nu^x)-\delta)\log(a_{\mathbf{i}}/b_{\mathbf{i}})-\sum_{\mathbf{i}\in\mathcal{A}_r} p_{\mathbf{i}}\log p_{\mathbf{i}}$$

and

$$Q_{2,r}:=C_4+\sum_{\mathbf{i}\in\mathcal{A}_r^a} p_{\mathbf{i}}(h(\nu^y)+\delta)\log(b_{\mathbf{i}}/a_{\mathbf{i}})+\sum_{\mathbf{i}\in\mathcal{A}_r^b} p_{\mathbf{i}}(h(\nu^x)+\delta)\log(a_{\mathbf{i}}/b_{\mathbf{i}})-\sum_{\mathbf{i}\in\mathcal{A}_r} p_{\mathbf{i}}\log p_{\mathbf{i}},$$

where $C_3$ and $C_4$ are constants independent of $r$. Applying Proposition 3.2 (iii) we have

$$\lim_{r\to 0}\frac{\sum_{G\in\mathbf{C}_r} f(\nu(G))}{\log r}=\frac{-\sum_{j=1}^m p_j\log p_j}{\sum_{j=1}^m p_j\log a_j}\ \text{ when } \sum_{j=1}^m p_j(\log a_j-\log b_j)=0. \tag{5.11}$$

Note that $\mathbf{C}_r$ is a $(r,N)$-good covering of $\operatorname{supp}(\nu)$ for some constant $N$ independent of $r$. Taking $r=2^{-n}$ and applying Lemma 5.1 we have

$$h(\nu)=\lim_{n\to\infty}\frac{h_n(\nu)}{n\log 2}=\lim_{n\to\infty}\frac{\sum_{G\in\mathbf{C}_{2^{-n}}} f(\nu(G))}{n\log 2}=\lim_{r\to 0}\frac{\sum_{G\in\mathbf{C}_r} f(\nu(G)}{-\log r}.$$

The theorem now follows by combining (5.10) and (5.11). ∎

## References


[1] T. Bedford, *Crinkly curves, Markov partitions and box dimension in self-similar sets*, Ph.D. Thesis, University of Warwick, 1984.

[2] R. Cawley and R. D. Mauldin, Multifractal decompositions of Moran fractals. *Adv. Math.* **92** (1992), 196–236.

[3] K. J. Falconer, *Fractal geometry. Mathematical foundations and applications.* John Wiley & Sons, Ltd., Chichester, 1990.

[4] A. H. Fan, K. S. Lau and S. M. Ngai, Iterated function systems with overlaps. *Asian J. Math.* **4** (2000), 527–552.

[5] D. J. Feng, The limited Rademacher functions and Bernoulli convolutions associated with Pisot numbers. Preprint
[6] D. J. Feng, Smoothness of the $L^q$-spectrum of self-similar measures with overlaps. *J. London Math. Soc. (2)* **68** (2003), 102–118.
[7] H. Y. Hu, Box dimensions and topological pressure for some expanding maps. *Comm. Math. Phys.* **191** (1998), 397–407.
[8] J. E. Hutchinson, Fractals and self-similarity. *Indiana Univ. Math. J.* **30** (1981), 713–747.
[9] R. Kenyon and Y. Peres, Hausdorff dimensions of sofic affine-invariant sets. *Israel J. Math.* **94** (1996), 157–178.
[10] J. F. King, The singularity spectrum for general Sierpiński carpets. *Adv. Math.* **116** (1995), 1–11.
[11] S. P. Lalley, $\beta$-expansions with deleted digits for Pisot numbers $\beta$. *Trans. Amer. Math. Soc.* **349** (1997), 4355–4365.
[12] S. P. Lalley and D. Gatzouras, Hausdorff and box dimensions of certain self-affine fractals. *Indiana Univ. Math. J.* **41** (1992), 533–568.
[13] K. S. Lau and S. M. Ngai, $L^q$-spectrum of the Bernoulli convolution associated with the golden ratio. *Studia Math.* **131** (1998), no. 3, 225–251.
[14] K. S. Lau and X. Y. Wang, Some exceptional phenomena in multifractal formalism: Part I. Preprint
[15] F. Ledrappier and L. S. Young, The metric entropy of diffeomorphisms. II. Relations between entropy, exponents and dimension. *Ann. of Math. (2)* **122** (1985), 540–574.
[16] C. McMullen, The Hausdorff dimension of general Sierpiński carpets. *Nagoya Math. J.* **96** (1984), 1–9.
[17] S. M. Ngai, A dimension result arising from the $L^q$-spectrum of a measure. *Proc. Amer. Math. Soc.* **125** (1997), 2943–2951.
[18] S. M. Ngai and Y. Wang, Hausdorff dimension of self-similar sets with overlaps. *J. London Math. Soc. (2)* **63** (2001), 655–672.
[19] L. Olsen, A multifractal formalism. *Adv. Math.* **116** (1995), 82–196.
[20] L. Olsen, Self-affine multifractal Sierpinski sponges in $\mathbf{R}^d$. *Pacific J. Math.* **183** (1998), 143–199.
[21] Y. Peres and B. Solomyak, Existence of $L^q$ dimensions and entropy dimension for self-conformal measures. *Indiana Univ. Math. J.* **49** (2000), 1603–1621.

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