Existence and Regularity of Solutions to a Variational Problem of Mumford and Shah: a Constructive Approach

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Abstract

We study a variational problem arising in the approach of Mumford and Shah to the image segmentation problem of computer vision. Given $f \in L^\infty(D)$ for a domain D in $\mathbf{R}^2$ the simplified Mumford–Shah energy associated to a decomposition $D = \Omega_1 \cup \cdots \cup \Omega_N$ is

$$\mathbf{E_0}[\Gamma, \alpha] = \sum_{i=1}^{N} \int_{\Omega_i} (f(x) - c_{\Omega_i})^2 \, dx + \alpha |\Gamma|,$$

where $\alpha > 0$ is a constant, c_{Ω_i} is the average of $f(x)$ on Ω_i and where $|\Gamma|$ is the length of the boundary of the regions Ω_i not in ∂D. Mumford and Shah showed, using geometric measure theory, that for a continuous f a minimizing Γ^* exists which is piecewise C^2. We prove this result constructively, and also extend it to show for general bounded measurable f that a minimizer exists. Furthermore, we prove that every minimizer must be piecewsie $C^{1,1}$. Our approach is to study $\mathbf{E_0}[\Gamma, \alpha]$ on the class of piecewise linear Γ.

Key words: image segmentaion, Mumford–Shah energy, minimizer, Hausdorff metric.

1 Introduction

The segmentation problem in Computer Vision is the problem of subdividing an image into regions in such a way that in each region, the image is relatively uniform. Mumford and Shah (1989) proposed to do this by minimizing energy functionals that encode penalty measures for properties of a good segmentation. Let $f \in L^\infty(D)$, where D is a domain in $\mathbf{R}^2$, represent the light intensity of the image. A decomposition of D is

$$D = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N$$

where each region Ω_i is closed and has a boundary $\partial\Omega_i$ which is piecewise C^1. Let $\Gamma = \cup_i \overline{\partial\Omega_i \setminus \partial D}$ be the boundary of the segmentation. They propose to find such a decomposition and an approximating function $u(x)$ by minimizing the *Mumford–Shah energy*

$$\mathbf{E}[u, \Gamma, \mu, \nu] = \mu^2 \int_D (u(x) - f(x))^2 \, dx + \sum_i \int_{\Omega_i} |\nabla u|^2 \, dx + \nu^2 |\Gamma| \tag{1}$$

where $\mu, \nu > 0$ are constants (weight parameters) and $|\Gamma|$ is the length of Γ. In addition to this energy functional Mumford and Shah introduced a simplified functional obtained by letting μ, $\nu \to 0$ and $\mu^2/\nu^2 \to \alpha > 0$. Then $\nabla u \equiv 0$, and the *simplified Mumford–Shah energy* is

$$\mathbf{E_0}[\Gamma, \alpha] = \sum_{i=1}^{N} \int_{\Omega_i} \left(f(x) - c_{\Omega_i}\right)^2 dx + \alpha|\Gamma|, \tag{2}$$

where c_{Ω_i} is the average of $f(x)$ over Ω_i. It is this simplified Mumford–Shah energy functional that is the subject of this paper.

Mumford and Shah (1989) proved, using methods from geometric measure theory, that if $f(x)$ is continuous on D then there exists a solution Γ^* to

$$\mathbf{E_0}[\Gamma^*, \alpha] = \inf_{\Gamma} \mathbf{E_0}[\Gamma, \alpha],$$

where Γ^* is piecewise C^2. In this paper, we present a constructive approach to find minimizers of $\mathbf{E_0}[\Gamma, \alpha]$, which is based on studying $\mathbf{E_0}[\Gamma, \alpha]$ over piecewise linear boundaries Γ. Using it we rederive Mumford and Shah's existence result, and we show, more generally, that for any bounded measurable $f \in L^\infty(D)$ there exists a minimizer Γ^* which is piecewise C^1. Furthermore we show that for *every* minimizer Γ^* it must satisfy a *weak curvature bound*: the unit tangent vector of γ (parametrized by arc length) of any C^1–segment of Γ^* satisfies a Lipschitz condition where the Lipschitz constant depends only on D, α and $\max_D |f(x)|$. The constructive nature of our approach is in obtaining such Γ^* as a suitable limit of piecewise linear Γ_n's, and constraints on the behavior of the Γ_n's, e.g. angles between segments. It is in principle possible to develop a computer implementation of this approach.

Now reconsider the general Mumford–Shah problem (1). It is much harder. Mumford and Shah conjectured there is a minimizing solution to

$$\mathbf{E}[u^*, \Gamma^*, \mu, \nu] = \inf_{u,\Gamma} \mathbf{E}[u, \Gamma, \mu, \nu],$$

where Γ^* is piecewise C^1 and $u^* \in W^{1,2}(D \setminus \Gamma^*)$, but this conjecture has never been proved. Existence results have been achieved for a weaker problem where Γ is only required to be a relatively closed set and $|\Gamma|$ is replaced by Hausdorff 1–dimensional measure, cf [8], [7]. Such existence result has recently been obtained [1], [5]. Shah (1992) obtained some results on a 1–dimensional simplification of the problem, and Richardson (1992) obtains asymptotic information on solutions as $\mu \to \infty$. All these authors use a geometric measure theory approach relying heavily on existence theorems. The elementary constructive approach of this paper offers a potentially promising approach to some of the questions.

This paper is part of the author's dissertation under Professor David Mumford at Harvard University. The author is extremely indebted to Professor Mumford, without whose supervision as well as his kindness and inspiration this work will never be completed. The author is also greatly indebted to Taka Shiota for his immense help. Finally, the author would like to thank the anonymous referees for their very constructive comments.

2 Basic Results

Let $\mathcal{C} = \{A \subset \mathbf{R}^2 \mid A \text{ is compact }\}$. The *Hausdorff metric* on $\mathcal{C}$ is defined as

$$d_H(A, B) = \sup_{x \in A} \inf_{y \in B} |x - y| + \sup_{x \in B} \inf_{y \in A} |x - y|$$

for A, $B \in \mathcal{C}$. It is easy to show that d_H is indeed a metric on $\mathcal{C}$. The following are well–established facts (see [6]).

Proposition 2.1 *1.* $(\mathcal{C}, d_H)$ *is a complete metric space.*

2. *Let* $\{A_i\} \subset \mathcal{C}$ *and* $A_1 \supseteq A_2 \supseteq A_3 \cdots$. *Then*

$$\lim_{i \to \infty} A_i = \bigcap_{i=1}^{\infty} A_i$$

in the metric space $(\mathcal{C}, d_H)$.

3. *Suppose* $\{A_i\}$ *is a sequence in* $\mathcal{C}$ *and* $\lim_{n \to \infty} A_n = A$ *in* $(\mathcal{C}, d_H)$. *Then*

$$A = \bigcap_{n=1}^{\infty} \overline{\left(\bigcup_{i=n}^{\infty} A_i \right)}.$$

4. *Let* D *be any compact subset of* $\mathbf{R}^2$ *and* $\mathcal{C}_D = \{A \subseteq D \mid A \text{ is closed in } D\}$. *Then* $\mathcal{C}_D$ *is a compact subset of* $\mathcal{C}$ *in* $(\mathcal{C}, d_H)$.

For simplicity we shall write $\mathbf{E_0}[\Gamma]$ in place of $\mathbf{E_0}[\Gamma, \alpha]$ from now on. Given any Γ and $D \setminus \Gamma = \bigcup_i \Omega_i$ where each Ω_i is a connected component of $D \setminus \Gamma$, we can separate the energy

$$\mathbf{E_0}[\Gamma] = \sum_i \int_{\Omega_i} (f - c_{\Omega_i})^2 dx + \alpha |\Gamma|$$

into two parts:

$$\begin{aligned} \text{square energy:} \quad \mathbf{E_S}[\Gamma] &= \sum_i \int_{\Omega_i} (f - c_{\Omega_i})^2 dx; \\ \text{length energy:} \quad \mathbf{E_L}[\Gamma] &= \alpha |\Gamma|. \end{aligned}$$

Recall that c_{Ω_i} denotes the average of $f(x)$ over Ω_i. Notice that the square energy $\mathbf{E_S}[.]$ can be defined for any closed subset $A \subseteq D$.

Lemma 2.2 *Let* $\{A_i\}_{i>0} \subset \mathcal{C}_D$. *If* $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$, $\{A_i\}_{i>0} \subset \mathcal{C}_D$, *and* $A = \cap_{i=1}^{\infty} A_i$, *then*

$$\lim_{i \to \infty} \mathbf{E_S}[A_i] = \mathbf{E_S}[A].$$

Proof: For any compact subset $B \subseteq D$ and $(x,y) \in D \times D$, define

$$\chi_B(x,y) = \begin{cases} 0 & \text{if } x \in B \text{ or } y \in B, \\ 0 & \text{if } x \text{ and } y \text{ do not belong to the same} \\ & \text{connected component of } D \setminus B, \\ 1 & \text{otherwise.} \end{cases}$$

$\chi_B : D \times D \longrightarrow \mathbf{R}$ is measurable and $|\chi_B| \le 1$. It is easy to see that for any $(x,y) \in D \times D$, $\lim_{i\to\infty} \chi_{A_i}(x,y) = \chi_A(x,y)$.

Note that for any $B \subset \mathcal{C}_D$,

$$\mathbf{E_S}[B] = \int_D (f(x) - g_B(x))^2 dx,$$

where

$$g_B(x) = \begin{cases} f(x) & \text{if } x \in B, \\ \dfrac{\displaystyle\int_D \chi_B(x,y) f(y) dy}{\displaystyle\int_D \chi_B(x,y) dy} & \text{otherwise.} \end{cases}$$

Since $\lim_{i\to\infty} \chi_{A_i} = \chi_A$, by the Lebesgue Dominated Theorem,

$$\begin{aligned} \lim_{i\to\infty} \int_D \chi_{A_i}(x,y)\, dy &= \int_D \chi_A(x,y)\, dy, \quad \text{and} \\ \lim_{i\to\infty} \int_D \chi_{A_i}(x,y) f(y)\, dy &= \int_D \chi_A(x,y) f(y)\, dy \end{aligned}$$

for any $x \in D$; hence

$$\lim_{i\to\infty} g_{A_i}(x) = g_A(x).$$

Applying the Lebesgue Dominated Theorem again we obtain

$$\begin{aligned} \lim_{i\to\infty} \mathbf{E_S}[A_i] &= \lim_{i\to\infty} \int_D (f(x) - g_{A_i}(x))^2 dx \\ &= \int_D (f(x) - g_A(x))^2 dx \\ &= \mathbf{E_S}[A]. \end{aligned}$$

∎

Proposition 2.3 (Lower Semicontinuity) *If* $\{A_i\} \subset \mathcal{C}_D$ *and* $\lim_{i\to\infty} A_i = A \in \mathcal{C}_D$ *in* $(\mathcal{C}_D, d_H)$*, then*

$$\liminf_{i\to\infty} \mathbf{E_S}[A_i] \ge \mathbf{E_S}[A].$$

Proof: For any $B_1, B_2 \in \mathcal{C}_D$ such that $B_1 \subset B_2$, $\mathbf{E_S}[B_1] \ge \mathbf{E_S}[B_2]$. Let $B_n = \overline{\bigcup_{i=n}^{\infty} A_i}$. Then $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$. Hence

$$A = \bigcap_{i=1}^{\infty} B_i = \lim_{i\to\infty} B_i.$$

By Lemma 2.2,

$$\lim_{n\to\infty} \mathbf{E_S}[B_n] = \mathbf{E_S}[A].$$

Since $B_n \supseteq A_m$ for $m \geq n$, $\mathbf{E_S}[B_n] \leq \mathbf{E_S}[A_n]$; thus

$$\liminf_{n\to\infty} \mathbf{E_S}[A_n] \geq \liminf_{n\to\infty} \mathbf{E_S}[B_n] = \mathbf{E_S}[A].$$

∎

3 Properties of the Segmentation

In this section, we shall examine the segmentations of the domain D by using *piecewise linear* line segments. For simplicity we shall restrict our discussions to the domain $D = [0, L] \times [0, L]$. When a piecewise linear Γ gives a *locally minimal* $\mathbf{E_0}[\Gamma]$, meaning that $\mathbf{E_0}[\Gamma]$ cannot be reduced through a small perturbation of Γ (in the topology induced by the Hausdorff metric), there are some restrictions as to the number of edges, regions, etc.

We devote essentially the entire section to prove the following two facts: if a piecewise linear Γ is a local minimum of the energy $\mathbf{E_0}[.]$ then

(1) the number of connected components in $D \setminus \Gamma$ must be bounded by some constant depending only on D, α and $\max_D |f|$, and so are the numbers of edges and junctions (see Definition 3.2) in Γ (Proposition 3.8);

(2) the angle between any two adjacent linear segments must be sufficiently close to π (Proposition 3.12).

These two facts are central to our main existence and regularity results, which we prove by obtaining a piecewise C^1 global minimizer from a sequence of locally minimizing piecewise linear Γ_j's.

Definition 3.1 $\Gamma \in \mathcal{C}_D$ *is called* piecewise linear *if*

$$\Gamma = \bigcup_{i=1}^{N} l_i$$

for some finite collection $\{l_i\}_{i=1}^N$ *where each* $l_i \subset D$ *is a linear segment, no* l_i *is part of the boundary* ∂D*, and for any two different* $i, j \leq N$*,* l_i *and* l_j *intersect at most a common end point of* l_i *and* l_j*.* $(l_1, l_2, \ldots, l_N)$ *is called a* piecewise linear representation *of* Γ*.*

Note that the angle between any two adjacent segments is allowed to be π. For each piecewise linear Γ there are infinitely many ways of choosing l_i's, and hence there are infinitely many different piecewise linear representations. Denote

$$\mathbf{SL}(D) = \{\Gamma \in \mathcal{C}_D \mid \Gamma \text{ is piecewise linear}\}.$$

Before any further discussion we define the following terms related to Γ.

Definition 3.2 *Let $\Gamma \in \mathbf{SL}(D)$ and let $(l_i, l_2, \dots l_N)$ be a piecewise linear representation of Γ. Define the following terms related to Γ and its piecewise linear representation:*

region: *A region defined by Γ is a connected component of $D \setminus \Gamma$.*

node: *A node of Γ is an end point of any linear segment l_i.*

junction: *A junction of Γ is a node that satisfies any of the following:*

1. *It is on ∂D; or*
2. *It connects to at least 3 linear segments; or*
3. *It connects to only one linear segment (tip of a crack).*

edge: *An edge of Γ is defined as the closure of any connected component of $\Gamma \setminus \{\text{the junctions of } \Gamma\}$.*

edge-element: *An edge-element of Γ is a linear segment l_i.*

boundary-component: *A boundary-component defined by Γ is the closure of any connected component of $\partial D \setminus \{\text{the junctions of } \Gamma\}$.*

For any representation of Γ, we also define

$$\begin{aligned}
R(\Gamma) &= \text{Number of regions defined by } \Gamma; \\
n(\Gamma) &= \text{Number of nodes in the representation of } \Gamma; \\
J(\Gamma) &= \text{Number of junctions in } \Gamma; \\
E(\Gamma) &= \text{Number of edges in } \Gamma; \\
e(\Gamma) &= \text{Number of edge-elements in the representation of } \Gamma; \\
B(\Gamma) &= \text{Number of boundary-components defined by } \Gamma.
\end{aligned}$$

Note that among all the terms defined, only nodes, edge-elements, $n(\Gamma)$, and $e(\Gamma)$ actually depend on the piecewise linear representation of Γ.

Definition 3.3 *Let $\Gamma \in \mathbf{SL}(D)$. An edge γ in Γ is called a* crack *if the two sides of γ belong to the same connected component of $D \setminus \Gamma$.*

Proposition 3.4 *Let $\Gamma \in \mathbf{SL}(D)$, Γ^* is derived from Γ by removing all cracks in Γ. Then $\mathbf{E_0}[\Gamma^*] \leq \mathbf{E_0}[\Gamma]$. The equality $\mathbf{E_0}[\Gamma^*] = \mathbf{E_0}[\Gamma]$ holds only when $\Gamma^* = \Gamma$,* i.e. *Γ is crack-free. Hence*

$$\inf_{\Gamma \in \mathbf{SL}(D)} \mathbf{E_0}[\Gamma] = \inf_{\substack{\Gamma \in \mathbf{SL}(D) \\ \Gamma \text{ is crack-free}}} \mathbf{E_0}[\Gamma].$$

Proof: Notice that $\mathbf{E_S}(\Gamma^*) = \mathbf{E_S}(\Gamma)$, and obviously $\mathbf{E_L}(\Gamma^*) \le \mathbf{E_L}(\Gamma)$ with the equality being true only when $\Gamma^* = \Gamma$. ∎

Lemma 3.5 (Euler) *For any crack-free $\Gamma \in \mathbf{SL}(D)$, an edge γ of Γ is called a* simple loop *if it contains no junction. Let $l(\Gamma)$ be the number of edges of Γ that are simple loops and $c(\Gamma)$ be the number of connected components of $\Gamma \cup \partial D$. Then*

$$\begin{aligned} E(\Gamma) + B(\Gamma) - R(\Gamma) - J(\Gamma) &= -c(\Gamma) + l(\Gamma) + \delta(\Gamma), \\ e(\Gamma) + B(\Gamma) - R(\Gamma) - n(\Gamma) &= -c(\Gamma) + \delta(\Gamma), \end{aligned}$$

where $\delta(\Gamma) = 1$ if there is no junction on ∂D, and $\delta(\Gamma) = 0$ otherwise.

Proof: The Theorem of Euler states that if a simply connected domain is divided into some simply connected subdomains, then

$$e - r - n = -1,$$

where e is the number of edges, r is the number of regions, and n is the number of nodes.

Notice that in our case, the definition of junctions is slightly different from the definition of nodes in The Theorem of Euler . In our definition, whenever an edge forms a closed loop, we do not consider there is a junction on the edge. In The Theorem of Euler, however, such an edge is considered to contain one node. Thus, if $c(\Gamma) = 1$, then

$$\begin{aligned} e &= E(\Gamma) + B(\Gamma), \\ r &= R(\Gamma), \\ n &= J(\Gamma) + \delta(\Gamma). \end{aligned}$$

Therefore,

$$E(\Gamma) + B(\Gamma) - R(\Gamma) - J(\Gamma) = -1 + \delta(\Gamma).$$

If $c(\Gamma) > 1$, applying The Theorem of Euler to each connected component of $\Gamma \cup \partial D$. Summing the equalities up, we have

$$E(\Gamma) + B(\Gamma) - R(\Gamma) - J(\Gamma) = -c(\Gamma) + l(\Gamma) + \delta(\Gamma).$$

For the second part of our lemma, compensating for $l(\Gamma)$ is not needed because a line segment can't form a closed loop. Hence

$$e(\Gamma) + B(\Gamma) - R(\Gamma) - n(\Gamma) = -c(\Gamma) + \delta(\Gamma).$$

∎

Let $A(\Omega)$ denote the area of Ω for any region Ω.

Lemma 3.6 *Let $\Omega \subset D$ be any connected piecewise C^1 domain. Let $\partial\Omega = E \cup B$ where $B = \partial\Omega \cap \partial D$ and $E = \overline{\partial\Omega \setminus B}$. Suppose $A(\Omega) \leq A(D)/2$. Then*

$$|E| \geq \frac{1}{3}|B|.$$

Proof: Recall that $D = [0, L] \times [0, L]$. We consider two cases. If $|E| \geq L$, then it follows from the isoperimetric inequality that

$$|E| + 4L - |B| = |\partial(D \setminus \Omega)| \geq \sqrt{4\pi A(D \setminus \Omega)} \geq \sqrt{2\pi}L.$$

Hence $3|E| > (4 - \sqrt{2\pi})L + |E| \geq B$.

Suppose $|E| < L$. Then each connected component of E either does not intersect ∂D, or the two intersecting points lie on the same side of ∂D or two adjacent side of ∂D. It is easy to see whatever happens, we always have $|E| \geq |B|/\sqrt{2} \geq |B|/3$. ∎

Lemma 3.7 *Let $\Gamma \in \mathbf{SL}(D)$ and Ω be a region defined by Γ. If $\gamma \subseteq \partial\Omega$ is an edge of Γ such that $|\gamma| > a_0 A(\Omega)$ where $a_0 = 4\max_D |f|^2/\alpha$, then*

$$\mathbf{E_0}[\Gamma] > \mathbf{E_0}[\overline{\Gamma \setminus \gamma}].$$

Proof: If γ is a crack, then obviously

$$\mathbf{E_0}[\Gamma] > \mathbf{E_0}[\overline{\Gamma \setminus \gamma}].$$

So we assume that γ is not a crack. Thus γ separates two different regions Ω and Ω^*. Let c_Ω, c_{Ω^*} and $c_{\Omega\cup\Omega^*}$ be the average of $f(x)$ over Ω, Ω^* and $\Omega \cup \Omega^*$ respectively. Then,

$$\begin{aligned}
&\mathbf{E_0}[\Gamma] - \mathbf{E_0}[\overline{\Gamma \setminus \gamma}] \\
&\quad = \int_\Omega (f - c_\Omega)^2 dx + \int_{\Omega^*} (f - c_{\Omega^*})^2 dx - \int_{\Omega\cup\Omega^*} (f - c_{\Omega\cup\Omega^*})^2 dx + \alpha|\gamma| \\
&\quad \geq \int_\Omega (f - c_\Omega)^2 dx + \int_{\Omega^*} (f - c_{\Omega^*})^2 dx - \int_{\Omega\cup\Omega^*} (f - c_{\Omega^*})^2 dx + \alpha|\gamma| \\
&\quad = \int_\Omega \left\{(f - c_\Omega)^2 - (f - c_{\Omega^*})^2\right\} dx + \alpha|\gamma| \\
&\quad \geq -4\max_\Omega |f|^2 A(\Omega) + \alpha|\gamma| \ > \ 0
\end{aligned}$$

Therefore, $\mathbf{E_0}[\Gamma] > \mathbf{E_0}[\overline{\Gamma \setminus \gamma}]$. ∎

Proposition 3.8 *There exists a constant $K_0 = K_0(D, \alpha, \max|f|) > 0$ such that for any $\Gamma \in \mathbf{SL}(D)$, if $R(\Gamma) + E(\Gamma) + B(\Gamma) + J(\Gamma) > K_0$, then $\mathbf{E_0}[\overline{\Gamma \setminus \gamma}] < \mathbf{E_0}[\Gamma]$ for some edge γ of Γ.*

Proof: Let $\Gamma \in \mathbf{SL}(D)$ and Ω be a region defined by Γ with $A(\Omega) < A(D)/2$. Then $|E| \geq |B|/3$ where $\partial\Omega = E \cup B$ as in Lemma 3.6. Hence

$$|E| \geq \frac{1}{4}(|E| + |B|) = \frac{1}{4}|\partial\Omega| \geq \frac{1}{4}\sqrt{4\pi A(\Omega)} = \frac{1}{2}\sqrt{\pi A(\Omega)}.$$

Choose $0 < b < A(D)/2$ so that for all $0 < \varepsilon \leq b$,

$$\frac{1}{10}\left(\frac{1}{2}\sqrt{\pi\varepsilon}\right) > a_0\,\varepsilon,$$

where $a_0 = 4\max_D |f|^2/\alpha$. It follows from Lemma 3.7 that if $A(\Omega) = \varepsilon \leq b$ and γ is an edge of Γ, then $\mathbf{E_0}[\overline{\Gamma \setminus \gamma}] < \mathbf{E_0}[\Gamma]$ whenever $|\gamma| \geq \sqrt{\pi\varepsilon}/20$.

Let $\mathcal{A} \subset \mathbf{SL}(D)$ denote the set of Γ's such that $\mathbf{E_0}[\overline{\Gamma \setminus \gamma}] \geq \mathbf{E_0}[\Gamma]$ for any edge γ of Γ. For any $\Gamma \in \mathcal{A}$ the above implies that given any region Ω defined by Γ with $A(\Omega) \leq b$ and $\partial\Omega = E \cup B$, the edge part E comprises at least 10 edges of Γ. Let R_+ be the number of regions defined by Γ which have area $> b$. Obviously $R_+ < A(D)/b$. Since each edge corresponds to only two regions while except for those with area $> b$ each region corresponds to at least ten edges, we have

$$E(\Gamma) \geq \frac{10}{2}(R(\Gamma) - R_+) \geq 5R(\Gamma) - \frac{5A(D)}{b}.$$

Applying the same argument to junctions, namely each junction corresponds to at least three edges or boundaries, while each edge or boundary corresponds to at most two junctions, we have

$$E(\Gamma) + B(\Gamma) \geq \frac{3}{2}J(\Gamma).$$

Combining the two inequalities we obtain

$$\begin{aligned}
&E(\Gamma) + B(\Gamma) - R(\Gamma) - J(\Gamma) \\
&\quad = \frac{1}{3}E(\Gamma) + \frac{1}{3}B(\Gamma) - R(\Gamma) + \frac{2}{3}\left(E(\Gamma) + B(\Gamma) - \frac{3}{2}J(\Gamma)\right) \\
&\quad \geq \frac{1}{3}E(\Gamma) - R(\Gamma) \\
&\quad \geq \frac{2}{3}R(\Gamma) - \frac{5A(D)}{3b}.
\end{aligned}$$

On the other hand, Lemma 3.5 gives us

$$E(\Gamma) + B(\Gamma) - R(\Gamma) - J(\Gamma) = -c(\Gamma) + l(\Gamma) + \delta(\Gamma),$$

where $l(\Gamma)$ is the number of edges that are simple loops in Γ. Since any region defined by Γ can not be enclosed by such a simple loop if the area of the region is $\leq b$, it implies $l(\Gamma) \leq A(D)/b$ and therefore

$$E(\Gamma) + B(\Gamma) - R(\Gamma) - J(\Gamma) \leq 1 + A(D)/b.$$

Hence $R(\Gamma) \leq C_R$ where $C_R = 8A(D)/b + 3/2$.

The rest follows easily. For any $\Gamma \in \mathcal{A}$, since $E(\Gamma) + B(\Gamma) \geq \frac{3}{2}J(\Gamma)$ we have

$$\begin{aligned}
\frac{1}{3}\left(E(\Gamma) + B(\Gamma)\right) &\leq E(\Gamma) + B(\Gamma) - J(\Gamma) \\
&= \Big(E(\Gamma) + B(\Gamma) - J(\Gamma) - R(\Gamma)\Big) + R(\Gamma) \\
&\leq -1 + \frac{A(D)}{b} + R(\Gamma).
\end{aligned}$$

Hence both $E(\Gamma)$ and $B(\Gamma)$ are uniformly bounded. So $J(\Gamma) \leq \frac{2}{3}(E(\Gamma) + B(\Gamma))$ must also be uniformly bounded. This proves the proposition. ∎

Corollary 3.9 *Let*

$$\mathbf{SL}_0(D) = \Big\{\Gamma \in \mathbf{SL}(D) \,\Big|\, R(\Gamma) + E(\Gamma) + B(\Gamma) + J(\Gamma) \leq K_0\Big\}.$$

Then

$$\inf_{\Gamma \in \mathbf{SL}(D)} \mathbf{E_0}[\Gamma] = \inf_{\Gamma \in \mathbf{SL}_0(D)} \mathbf{E_0}[\Gamma].$$

We now introduce a new subset of $\mathbf{SL}(D)$. Let $\mathbf{SL}(D, m, \varepsilon) \subset \mathbf{SL}(D)$ denote the set of Γ's which have a piecewise linear representation such that $n(\Gamma) \leq m$ and $|e| \leq \varepsilon$ for any edge-element e of Γ. We have

Lemma 3.10 $\mathbf{SL}(D, m, \varepsilon) \subset \mathcal{C}_D$ *is a compact subset in* $\{\mathcal{C}, d_H\}$ *for any* $m > 0$ *and* $\varepsilon > 0$.

Proof: We establish a bound for $e(\Gamma) + B(\Gamma)$ for $\Gamma \in \mathbf{SL}(D, m, \varepsilon)$. Since except for possibly those that contain at least a corner of D, each region corresponds to at least three edge-elements or boundaries while each edge-element or boundary corresponds to no more that two regions, so

$$e(\Gamma) + B(\Gamma) \geq \frac{3}{2}\Big(R(\Gamma) - 4\Big).$$

Hence

$$\frac{1}{3}\Big(e(\Gamma) + B(\Gamma)\Big) \leq e(\Gamma) + B(\Gamma) - R(\Gamma) + 4 = n(\Gamma) - c(\Gamma) + \delta(\Gamma) + 4 \leq n(\Gamma) + 4.$$

Therefore $e(\Gamma) + B(\Gamma) \leq 3m + 12$. We now conclude the compactness of $\mathbf{SL}(D, m, \varepsilon)$ by showing that the limit of $\{\Gamma_i\}$ where $\Gamma_i \in \mathbf{SL}(D, m, \varepsilon)$ must also be in $\mathbf{SL}(D, m, \varepsilon)$. Let

$$\Gamma_i = \bigcup_{j=1}^{n_i} e_j^i,$$

where e_j^i is an edge-element and $e_j^i \cap e_k^i$ for any $j \neq k$ is either empty or a node in Γ_i. Because n_i are bounded for all i, we may without loss of generality assume that $n_i = n^*$ for all i, or we may replace the sequence Γ_i by a subsequence.

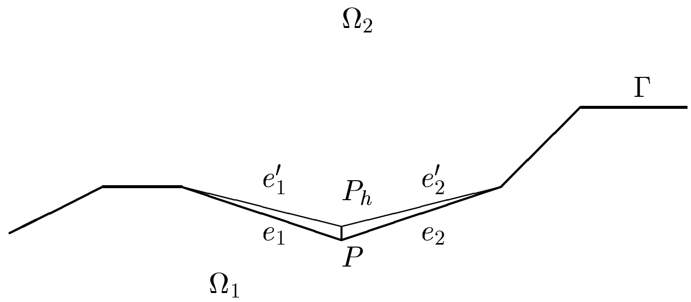

Figure 1: A small perturbation of Γ.

Choose a subsequence $\Gamma_{k_1},\ \Gamma_{k_2},\ \Gamma_{k_3}, \cdots$ of $\{\Gamma_i\}_{i>0}$ such that $\forall j \le n^*$

$$\lim_{i\to\infty} e_j^{k_i} = e_j^*.$$

The for any $j \le n^*$ either e_j^* is a linear segment with $|e_j^*| \le \varepsilon$ or a single point.

Now let Γ^* be the limit of $\{\Gamma_i\}$. Then $\Gamma^* = \bigcup_{j=1}^{n^*} e_j^*$. It is clear that $\Gamma^* \in \mathbf{SL}(D)$. Notice that if $e_{j_0}^*$ is a single point for some j_0 then we still have $\Gamma^* = \bigcup_{j=j_0} e_j^*$. So we may without loss of generality assume that all e_j^* are line segments. If (e_j^*) is a piecewise linear representation of Γ^* then we have $\Gamma^* \in \mathbf{SL}(D,m,\varepsilon)$. Suppose (e_j^*) is not a piecewise linear representation of Γ. Then the following must occur: for some $i \ne j$, a node of e_j^* may lie in the interior of some e_i^*, or $e_j^* \cap e_i^*$ may be a linear segment itself. However, given either of the above cases we can always subdivide $e_i^* \cup e_j^*$ into smaller edge-elements without adding any *new nodes*. So by this procedure we obtain a piecewise linear representation of Γ. Since no new nodes are added all nodes in this representation are limit points of nodes of in Γ_i, so $n(\Gamma) \le m$ and $\Gamma \in \mathbf{SL}(D,m,\varepsilon)$. This implies the compactness of $\mathbf{SL}(D,m,\varepsilon)$. ∎

Proposition 3.11 *For any $m > 0$ and $\varepsilon > 0$, there exists a $\Gamma^* \in \mathbf{SL}(D,m,\varepsilon) \cap \mathbf{SL}_0(D)$ such that*

$$\mathbf{E_0}[\Gamma^*] = \inf_{\Gamma\in\mathbf{SL}(D,m,\varepsilon)} \mathbf{E_0}[\Gamma].$$

Proof: Notice that for any sequence $\{\Gamma_n\} \subset \mathbf{SL}(D,m,\varepsilon)$ such that $\Gamma_n \to \Gamma$, we have $\mathbf{E_L}(\Gamma) \le \liminf_n \mathbf{E_L}(\Gamma_n)$. The existence of $\Gamma^* \in \mathbf{SL}(D,m,\varepsilon)$ follows immediately from the compactness of $\mathbf{SL}(D,m,\varepsilon)$ and the lower semi–continuity of the energy $\mathbf{E_S}(\Gamma)$. $\Gamma^* \in \mathbf{SL}_0(D)$ follows from Proposition 3.8. ∎

Proposition 3.12 *Suppose $\Gamma_0 \in \mathbf{SL}(D,m,\varepsilon)$ and e_2, e_2 are any two adjacent edge-elements of Γ_0 which intersect at a non-junction node. Let $0 \le \theta \le \pi$ be the angle between e_1 and e_2. If*

$$|\pi-\theta| > M_0(|e_1|+|e_2|),$$

where $M_0 = 8 \max_D |f|^2/\alpha$, *then*

$$\mathbf{E_0}[\Gamma_0] > \inf_{\Gamma \in \mathbf{SL}(D,m,\varepsilon)} \mathbf{E_0}[\Gamma].$$

Proof: Suppose that e_1 and e_2 intersect at the node P. Let Ω_1 and Ω_2 be the regions separated by the edge containing e_1 and e_2, as illustrated in Figure 1.

Consider a new $\Gamma^h \in \mathbf{SL}(D, m, \varepsilon)$ which is obtained from Γ_0 by slightly perturbing e_1 and e_2, also shown in Figure 1. As Γ_0 becomes Γ^h, the node P becomes P_h so that the line segment PP_h satisfies $|PP_h| = h$ and it bisects the angle θ. Let the domain formed by the polygon $e_1 e_2 e_2' e_1'$ be Ω_h. Then

$$\begin{aligned}
&\mathbf{E_0}[\Gamma^h] - \mathbf{E_0}[\Gamma_0] \\
&\quad = \int_{\Omega_1 \cup \Omega_h} (f - c_{\Omega_1 \cup \Omega_h})^2 dx + \int_{\Omega_2 \setminus \Omega_h} (f - c_{\Omega_2 \setminus \Omega_h})^2 dx + \alpha\Big(|e_1'| + |e_2'|\Big) \\
&\qquad - \int_{\Omega_1} (f - c_{\Omega_1})^2 dx - \int_{\Omega_2} (f - c_{\Omega_2})^2 dx - \alpha\Big(|e_1| + |e_2|\Big).
\end{aligned}$$

(As usual, for any domain $\Omega \subset D$, c_Ω is the mean value of $f(x)$ over Ω.) Elementary trigonometry gives

$$A(\Omega_h) = \Big(|e_1| + |e_2|\Big) h \sin\frac{\theta}{2}, \quad \text{and} \quad |e_i| = |e_i| - h\cos\frac{\theta}{2} + o(h), \quad i = 1, 2.$$

Because

$$A(\Omega_1 \cup \Omega_h) c_{\Omega_1 \cup \Omega_h} - A(\Omega_1) c_{\Omega_1} = \int_{\Omega_1 \cup \Omega_h} f dx - \int_{\Omega_1} f dx = O(A(\Omega_h)),$$

it is immediate that $c_{\Omega_1 \cup \Omega_h} - c_{\Omega_1} = O(A(\Omega_h))$. Similarly, $c_{\Omega_2 \setminus \Omega_h} - c_{\Omega_2} = O(A(\Omega_h))$. We have therefore

$$\begin{aligned}
&\int_{\Omega_1 \cup \Omega_h} (f - c_{\Omega_1 \cup \Omega_h})^2 dx - \int_{\Omega_1} (f - c_{\Omega_1})^2 dx \\
&\qquad = \int_{\Omega_1 \cup \Omega_h} (f - c_{\Omega_1} + c_{\Omega_1} - c_{\Omega_1 \cup \Omega_h})^2 dx - \int_{\Omega_1} (f - c_{\Omega_1})^2 dx \\
&\qquad = \int_{\Omega_h} (f - c_{\Omega_1 \cup \Omega_h})^2 dx + \Big(c_{\Omega_1} - c_{\Omega_1 \cup \Omega_h}\Big)^2 A(\Omega_1) \\
&\qquad \le 4 \sup_D |f|^2 A(\Omega_h) + o(h),
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega_2 \setminus \Omega_h} (f - c_{\Omega_2 \setminus \Omega_h})^2 dx - \int_{\Omega_2} (f - c_{\Omega_2})^2 dx \\
&\qquad \le 4 \sup_D |f|^2 A(\Omega_h) + o(h).
\end{aligned}$$

Hence,

$$\mathbf{E_0}[\Gamma^h] - \mathbf{E_0}[\Gamma_0] \quad \le \quad 8 \sup_D |f|^2 A(\Omega_h) - 2\alpha h \cos\frac{\theta}{2} + o(h)$$

$$
\begin{aligned}
&= 8\sup_D |f|^2 \Big(|e_1| + |e_2|\Big) h \sin\frac{\theta}{2} - 2\alpha h \cos\frac{\theta}{2} + o(h) \\
&= \alpha M_0 \Big(|e_1| + |e_2|\Big) h \sin\frac{\theta}{2} - 2\alpha h \cos\frac{\theta}{2} + o(h)
\end{aligned}
$$

Let $M_0 = 8\sup_D |f|^2/\alpha$. If $|\pi - \theta| > M_0(|e_1| + |e_2|)$, then

$$
\begin{aligned}
&\limsup_{h\to 0^+} \frac{\mathbf{E_0}[\Gamma^h] - \mathbf{E_0}[\Gamma_0]}{h} \\
&\leq \alpha M_0 \Big(|e_1| + |e_2|\Big) \sin\frac{\theta}{2} - 2\alpha \cos\frac{\theta}{2} \\
&= \alpha \sin\frac{\theta}{2} \left\{ M_0\alpha\Big(|e_1| + |e_2|\Big) - 2\cot\frac{\theta}{2} \right\} \\
&< \alpha \sin\frac{\theta}{2} \left\{ |\pi - \theta| - 2\tan\frac{\pi-\theta}{2} \right\} \\
&\leq \alpha \sin\frac{\theta}{2} \left\{ |\pi - \theta| - 2\left|\frac{\pi-\theta}{2}\right| \right\} \\
&= 0.
\end{aligned}
$$

Therefore $\mathbf{E_0}[\Gamma^h] < \mathbf{E_0}[\Gamma_0]$ for sufficiently small $h > 0$. ∎

4 Approximation

In this section we shall be looking at more general segmentations of D, namely those formed by *piecewise* C^1 Γ's. The key idea in this section is to show that for any locally minimizing piecewise linear Γ, the restriction on the angle between any two adjacent linear segments of Γ stated in Proposition 3.12 implies that each edge of Γ can be approximated well by a $C^{1,1}$ curve. Using this fact we prove our existence result (Proposition 4.6).

We call a curve γ a *simple* C^k curve if there exists a C^k map $c : [0,1] \longrightarrow \mathbf{R}^2$ with $c'(t) \neq 0$ for all $t \in [0,1]$ such that $c(t_1) \neq c(t_2)$ for any $t_1 \in [0,1]$, $t_2 \in (0,1)$ and $t_1 \neq t_2$.

Definition 4.1 *$\Gamma \in \mathcal{C}_D$ is called* piecewise C^1 *if*

$$
\Gamma = \bigcup_{i=1}^{N} \gamma_i
$$

for some finite collection $\{\gamma_i\}_{i=1}^N$ where each $\gamma_i \subset D$ is a simple C^1 curve and for any i and $j \neq i$, both $\gamma_i \cap \partial D$ and $\gamma_j \cap \gamma_j$ are either empty or contain one or two end points of γ_i. $(\gamma_1, \gamma_2, \ldots, \gamma_N)$ is called a piecewise C^1 *representation of Γ.*

Denote

$$
\mathbf{S}^1(D) = \left\{ \Gamma \in \mathcal{C}_D \mid \Gamma \text{ is piecewise } C^1 \right\}.
$$

It is obvious that $\mathbf{SL}(D) \subset \mathbf{S}^1(D)$. We have the following generalization of Definition 3.2.

Definition 4.2 *Let $\Gamma \in \mathbf{S}^1(D)$. We define the following terms related to Γ:*

region: *A region defined by Γ is a connected component of $D \setminus \Gamma$.*

junction: *A junction of Γ is a point in D where some γ_i and ∂D meet, or where at least three different γ_i's meet.*

edge: *An edge of Γ is defined as the closure of any connected component of $\Gamma \setminus \{junctions\ of\ \Gamma\}$.*

boundary: *A boundary defined by Γ is the closure of a connected component of $\partial D \setminus \{junctions\ of\ \Gamma\}$.*

Notice that none of the terms defined above depends on the representation of Γ. The following are also independent of the representation of Γ:

$$\begin{aligned} R(\Gamma) &= \text{Number of regions defined by } \Gamma; \\ J(\Gamma) &= \text{Number of junctions in } \Gamma; \\ E(\Gamma) &= \text{Number of edges in } \Gamma; \\ B(\Gamma) &= \text{Number of boundaries defined by } \Gamma. \end{aligned}$$

Lemma 4.3

$$\inf_{\Gamma \in \mathbf{S}^1(D)} \mathbf{E_0}[\Gamma] = \inf_{\Gamma \in \mathbf{SL}(D)} \mathbf{E_0}[\Gamma]$$

Proof: It is obvious that

$$\inf_{\Gamma \in \mathbf{S}^1(D)} \mathbf{E_0}[\Gamma] \leq \inf_{\Gamma \in \mathbf{SL}(D)} \mathbf{E_0}[\Gamma].$$

But since any $\Gamma \in \mathbf{S}^1(D)$ can be approximated to arbitrary degree of accuracy, it is easy to see that $\inf_{\Gamma \in \mathbf{S}^1(D)} \mathbf{E_0}[\Gamma] \geq \inf_{\Gamma \in \mathbf{SL}(D)} \mathbf{E_0}[\Gamma]$. ∎

Lemma 4.4 *For any $\varepsilon > 0$,*

$$\inf_{\Gamma \in \mathbf{SL}(D)} \mathbf{E_0}[\Gamma] = \lim_{m \to \infty} \inf_{\Gamma \in \mathbf{SL}(D,m,\varepsilon)} \mathbf{E_0}[\Gamma].$$

Proof: Notice that any linear segment can be broken up and viewed as the union of linear segments of length $\leq \varepsilon$. Therefore,

$$\mathbf{SL}(D) = \bigcup_{m=1}^{\infty} \mathbf{SL}(D,m,\varepsilon) = \lim_{m \to \infty} \mathbf{SL}(D,m,\varepsilon).$$

The lemma follows immediately. ∎

Lemma 4.3 and Lemma 4.4 indicate that in order to minimize $\mathbf{E_0}[\Gamma]$ for $\Gamma \in \mathbf{S}^1(D)$, we can first minimize $\mathbf{E_0}[\Gamma]$ over $\mathbf{SL}(D,m,\varepsilon) \cap \mathbf{SL}_0(D)$ and consider the limit of $\mathbf{E_0}[\Gamma]$ as $m \to \infty$.

Lemma 4.5 *Let $f(t) : [a, b) \longrightarrow \mathbf{R}^2$ be a piecewise constant map such that $f(t) = X_i$ for $t \in [t_i, t_{i+1})$, where $a = t_0 < t_1 < \cdots < t_n = b$. Suppose $|t_{i+1} - t_i| \leq \varepsilon$ for any $0 \leq i < n$ and there exists a constant M such that $|X_{i+1} - X_i| \leq M(|t_{i+2} - t_i|)$ for any $0 \leq i < n-1$. Then there is a $F(t) : [a, b) \longrightarrow \mathbf{R}^2$ which has the following properties:*

1. *$F(t)$ is C^1.*
2. *$|F'(t)| \leq 6M$.*
3. *$|F(t) - f(t)| \leq 8M\varepsilon$.*
4. *$\int_a^b F(t)dt = \int_a^b f(t)dt$.*

Proof: Consider

$$a = \bar{t}_0 < \bar{t}_1 < \cdots < \bar{t}_n = b,$$

where for any $0 < i < n$, $\bar{t}_i = (t_i + t_{i+1})/2$. Let $F_1(t)$ be a cubic spline approximation of $f(t)$ on $[a, b]$ defined as follows: for $t \in [\bar{t}_i, \bar{t}_{i+1})$ where $0 \leq i < n-1$,

$$F_1(t) = \frac{X_i - X_{i+1}}{(\bar{t}_{i+1} - \bar{t}_i)^3}\Big(2(t - \bar{t}_i)^3 - 3(\bar{t}_{i+1} - \bar{t}_i)(t - \bar{t}_i)^2\Big) + X_i,$$

and for $t \in [\bar{t}_{n-1}, \bar{t}_n]$

$$F_1(t) = X_{n-1}.$$

Clearly, $F_1(\bar{t}_i) = X_i$ and $F_1'(\bar{t}_i) = 0$ for any $i < n$; hence $F_1(t)$ is C^1.

For any $t \in [a, b]$, if $t \in [t_i, t_{i+1})$, then either $f(t) = X_i$ or $f(t) = X_{i+1}$. Notice that if $t \in [t_i, t_{i+1})$ then

$$0 \leq 3(\bar{t}_{i+1} - \bar{t}_i)(t - \bar{t}_i)^2 - 2(t - \bar{t}_i)^3 \leq (\bar{t}_{i+1} - \bar{t}_i)^3.$$

Hence we have

$$\begin{aligned}
|F_1(t) - f(t)| &\leq |F_1(t) - X_i| + |X_i - f(t)| \\
&\leq \frac{|X_i - X_{i+1}|}{(\bar{t}_{i+1} - \bar{t}_i)^3} \cdot \Big|2(t - \bar{t}_i)^3 - 3(\bar{t}_{i+1} - \bar{t}_i)(t - \bar{t}_i)^2\Big| \\
&\quad + |X_i - X_{i+1}| \\
&\leq |X_i - X_{i+1}| + |X_i - X_{i+1}| \\
&\leq 2|X_i - X_{i+1}| \\
&\leq 2M|t_i - t_{i+2}| \\
&\leq 4M\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
|F_1'(t)| &= \frac{|X_i - X_{i+1}|}{(\bar{t}_{i+1}-\bar{t}_i)^3} \cdot \left|6(t-\bar{t}_i)^2 - 6(\bar{t}_{i+1}-\bar{t}_i)(t-\bar{t}_i)\right| \\
&\leq \frac{|X_i - X_{i+1}|}{(\bar{t}_{i+1}-\bar{t}_i)^3} \cdot |(t-\bar{t}_i)(\bar{t}_{i+1}-t)| \\
&\leq \frac{2 \cdot 3M|t_i - t_{i+2}|}{|t_{i+2}+t_{i+1}-t_{i+1}+t_i|} \\
&= 6M.
\end{aligned}$$

Let $F(t) = F_1(t) + \delta$ where

$$\delta = \frac{\int_a^b (f(t) - F_1(t))dt}{b-a};$$

then $F'(t) = F_1'(t)$ and

$$\begin{aligned}
|F(t) - f(t)| &\leq |F(t) - F_1(t)| + |F_1(t) - f(t)| \\
&\leq |\delta| + 4M\varepsilon \\
&\leq \frac{\int_a^b |f(t) - F_1(t)|dt}{b-a} + 4M\varepsilon \\
&\leq 8M\varepsilon.
\end{aligned}$$

Hence $F(t)$ satisfies the listed properties. ∎

Proposition 4.6 *Let $\{\Gamma_i\}_{i>0}$ be a sequence in $\mathcal{C}_D$ such that $\Gamma_i \in \mathbf{SL}(D, m_i, \varepsilon_i)$, where $\lim_{i\to\infty} m_i = \infty$ and $\lim_{i\to\infty} \varepsilon_i = 0$. Suppose*

$$\mathbf{E_0}[\Gamma_i] = \inf_{\Gamma \in \mathbf{SL}(D, m_i, \varepsilon_i)} \mathbf{E_0}[\Gamma].$$

Then there exists a $\Gamma^* \in \mathcal{C}_D$ which is limit point of $\{\Gamma_i\}_{i>0}$ such that Γ^* satisfies the following properties:

1. $$\Gamma^* = \bigcup_{j=1}^{N^*} \gamma_j,$$

   with $N^* \leq 2K_0$ where $K_0 = K_0(D, \alpha, \max|f|)$ is defined in Proposition 3.8 and each γ_j is a simple C^1 curve.

2. *For any $j \leq N^*$ let $T_j(s)$ denote the unit tangent vector of γ_j parametrized by the arc length s of γ_j. Then,*

   $$|T_j(s_1) - T_j(s_2)| \leq C_0|s_1 - s_2|,$$

   where $C_0 = C_0(D, \alpha, \max|f|)$ is a constant.

3. *For any i and $j \neq i$, both $\gamma_i \cap \partial D$ and $\gamma_i \cap \gamma_j^*$ are either empty or contain some endpoints of γ_i. Hence $\Gamma^* \in \mathbf{S}^1(D)$.*

4.

$$\mathbf{E_0}[\Gamma^*] = \inf_{\Gamma \in \mathbf{S}^1(D)} \mathbf{E_0}[\Gamma].$$

Proof: Let $\Gamma_i = \bigcup_{j=1}^{m_i} \gamma_j^i$ where γ_j^i are the edges of Γ_i. According to Corollary 3.9, since $m_i \leq \mathbf{K_0}$, we may without loss of generality assume that $m_i = N_0$ because we can always find an $N_0 \leq K_0$ such that there are infinitely many i's for which $m_i = N_0$.

Now, fix a j and consider the family $\{\gamma_j^i\}_{i>0}$. Define f_i by

$$\gamma_j^i(s) = \gamma_j^i(0) + \int_0^s f_i(t)dt,$$

where s is the arc length parameter of γ_j^i and $0 \leq s \leq l_i^j = |\gamma_j^i|$. Since γ_j^i is piecewise linear, f_i is piecewise constant. Because Γ_i minimizes $\mathbf{E_0}[\Gamma]$ in $\mathbf{SL}(D, m_i, \varepsilon_i)$, Proposition 3.12 implies that $f_i(t)$ satisfies the conditions stated in Lemma 4.5 for some constant $M = M(D, \alpha, \max|f|)$. Hence there is a C^1 function $F_i(t)$ defined on $[0, l_i^j]$ such that

$$|F_i(t) - f_i(t)| \leq 8M\varepsilon_i, \qquad |F_i'(t)| \leq 6M, \quad \text{and}$$

$$\int_0^{l_i^j} f_i(t)dt = \int_0^{l_i^j} F_i(t)dt.$$

For a given i let $\hat{\gamma}_j^i$ be the parameterized curve

$$\hat{\gamma}_j^i(s) = \gamma_j^i(0) + \int_0^s F_i(t)dt$$

for $0 \leq s \leq l_i^j$. Since $|F_i'(t)|$ is a uniformly bounded sequence of functions, there exists a subsequence $\{\hat{\gamma}_j^{n_i}\}_{i>0}$ of $\{\hat{\gamma}_j^i\}_{i>0}$ which converges uniformly to some $\gamma_j^*(s)$ which is either C^1 or in the degenerate case, a single point. Let $\Gamma^* = \cup_{j=1}^{N_0} \gamma_j^*$. If $\gamma_{j_0}^*$ is a single point for some j_0, then we still have $\Gamma^* = \cup_{j \neq j_0} \gamma_j^*$. So without loss of generality we may assume that γ_j^* is not a single point for all j, and that $\lim_{i\to\infty} \gamma_j^i = \gamma_j^*$.

We now prove that $\Gamma^* \in \mathbf{S}^1(D)$ by showing that it has a piecewise C^1 representation. If (γ_j^*) is a piecewise C^1 representation of Γ^* then we are done. Suppose it is not a piecewise C^1 representation of Γ^*. Then there must be some $n \neq m$ such that the set $\gamma_n^* \cap \gamma_m^*$ contains a point x which is not an endpoint of both γ_n^* and γ_m^*, or there is an $x \in \gamma_n^* \cap \partial D$ such that x is not an end point of γ_n^* for some n. We show that in the former case x must be an endpoint of either γ_n^* or γ_m^*. If not, since $\gamma_n^i \cap \gamma_m^i$ for any i contains only endpoints of both γ_n^i and γ_m^i, γ_n^* and γ_m^* must be tangent to each other at x (there cannot cross each other at x, otherwise γ_n^i and γ_m^i will intersect for sufficiently large i). For any $a > 0$, let $\delta = \delta(a) = d_H(\gamma_n^* \cap B_a(x), \gamma_m^* \cap B_a(x))$. Since γ_n^* and γ_m^* are C^1 and tangent to each other at x, we can make δ/a arbitrarily small by choosing a sufficiently small $a > 0$.

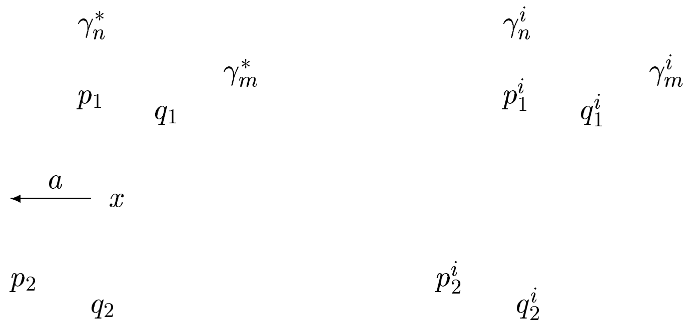

Figure 2: A case in which γ_n^* is tangent to γ_m^*.

Let $p_1,\ p_2$ be the endpoints of $B_a(x) \cap \gamma_n^*$ and $q_1,\ q_2$ of $B_a(x) \cap \gamma_m^*$. Since p_1, p_2, q_1, and q_2 are all on $\partial B_a(x)$, we assume that on $\partial B_a(x)$, q_1 is in between p_1 and q_2 while q_2 is in between p_2 and q_1. This is shown in Figure 2. Consider $p_1^i,\ p_2^i \in \gamma_n^i$ and $q_1^i,\ q_2^i \in \gamma_m^i$ which satisfy

$$\lim_{i\to\infty} p_1^i = p_1, \qquad \lim_{i\to\infty} p_2^i = p_2$$

and

$$\lim_{i\to\infty} q_1^i = q_1, \qquad \lim_{i\to\infty} q_2^i = q_2.$$

Let

$$\hat{\Gamma}_i = \Gamma_i \cup \{\text{line segments } p_1^i q_1^i \text{ and } p_2^i q_2^i\};$$

$\hat{\Gamma}_i \in \mathbf{SL}(D, m_i + n_i, \varepsilon_i)$ for some $n_i > 0$. Denote the portion of γ_n^i between p_1^i and p_2^i by ψ_n^i and the portion of γ_m^i between q_1^i and q_2^i by ψ_m^i. Let Ω be the region enclosed by ψ_n^i, $p_2^i q_2^i$, ψ_m^i, and $p_1^i q_1^i$. Since for sufficiently large i,

$$d_H(\psi_n^i, \psi_m^i) \le 2\delta\,, \quad |\psi_n^i| \le 3a, \quad \text{and} \quad |\psi_m^i| \le 3a,$$

we have

$$A(\Omega_i) \le 2\delta \max\{\psi_n^i, \psi_m^i\} \le 6\delta a.$$

Therefore

$$|\mathbf{E}_{\mathbf{S}}[\hat{\Gamma}_i \setminus \psi_n^i] - \mathbf{E}_{\mathbf{S}}[\Gamma_i]| \le C_1 A(\Omega_i) \le 6C_1\delta a,$$

and

$$\mathbf{E}_{\mathbf{0}}[\hat{\Gamma}_i \setminus \psi_n^i] - \mathbf{E}_{\mathbf{0}}[\Gamma_i] \le 6C_1\delta a + 2\delta - |\psi_n^i| \le 6C_1\delta a + 2\delta - a.$$

Since δ/a can be made arbitrarily small by choosing a sufficiently small $a > 0$, we can choose an $a > 0$ such that

$$6C_1\delta a + 2\delta - a \le -\frac{a}{2}.$$

Thus for i sufficiently large,

$$\mathbf{E_0}[\hat{\Gamma}_i \setminus \psi_n^i] \leq \mathbf{E_0}[\Gamma_i] - \frac{a}{2}.$$

But

$$\lim_{i\to\infty} \mathbf{E_0}[\Gamma_i] = \inf_{\Gamma\in\mathbf{S}^1(D)} \mathbf{E_0}[\Gamma] \leq \inf_{\Gamma\in\mathbf{SL}(D,m,\varepsilon)} \mathbf{E_0}[\Gamma]$$

for any $N_0 > 0$ and $\varepsilon > 0$. This is a contradiction. Hence any $x \in \gamma_n^* \cap \gamma_m^*$ must be an endpoint of either γ_n^* or γ_m^*. The same argument shows that any $x \in \gamma_n^* \cap \partial D$ must be an endpoint of γ_n^*, and that if γ_n^* has a self-intersection at x then x must be an endpoint of γ_n^*.

So we may now refine each γ_n^* into $\gamma_n^* = \cup\gamma_{n,k}^*$ such that each endpoint of $\gamma_{n,k}^*$ is and endpoint of some γ_m^*, and that every $x \in \gamma_{n,k}^* \cap \gamma_{m,l}^*$ must be an endpoint of both $\gamma_{n,k}^*$ and $\gamma_{m,l}^*$. Each $\gamma_{n,k}^*$ is simple. Furthermore, since the total number of endpoints of all γ_n^* is bounded by K_0, the number of $\gamma_{n,k}^*$ is bounded by $2K_0$. So $(\gamma_{n,k}^*)$ is a piecewise C^1 representation of Γ^*.

It is clear that $\Gamma^* = \cup\gamma_{n,k}^*$ satisfies properties 1, 3 of the proposition. Because each $\gamma_{n,k}^*$ is simple, property 2 follows immediately from Lemma 4.5. We now prove property 4. Since $\lim_{i\to\infty}\Gamma_i = \Gamma^*$, we have

$$\liminf_{i\to\infty} \mathbf{E_S}[\Gamma_i] \geq \mathbf{E_S}[\Gamma^*].$$

The proof of (1) also shows that

$$\lim_{i\to\infty} \mathbf{E_L}[\Gamma_i] \geq \mathbf{E_L}[\Gamma^*].$$

Therefore,

$$\inf_{\Gamma\in\mathbf{S}^1(D)} \mathbf{E_0}[\Gamma] \leq \mathbf{E_0}[\Gamma^*] \leq \liminf_{i\to\infty} \mathbf{E_0}[\Gamma_i] = \inf_{\Gamma\in\mathbf{S}^1(D)} \mathbf{E_0}[\Gamma].$$

∎

5 Conclusion

Theorem 5.1 *Let $f(x) \in L^\infty(D)$ where $D = [0,L]\times[0,L]$. Then there exists a $\Gamma^* \in \mathbf{S}^1(D)$ such that $\mathbf{E_0}[\Gamma^*] = \inf_{\Gamma\in\mathbf{S}^1(D)} \mathbf{E_0}[\Gamma]$. Moreover, every such Γ^* satisfies the following properties:*

1. *Γ^* is crack-free and there exists a constant $K = K(D, \alpha, \max_D |f|)$ such that*

$$R(\Gamma^*) + E(\Gamma^*) + J(\Gamma^*) + B(\Gamma^*) \leq K.$$

2. *Every edge in Γ^* is C^1.*

3. *Suppose $\gamma = \gamma(s)$ is an edge of Γ^* parametrized by the arc length. Let $T(s) = \gamma'(s)$ be the unit tangent vector of γ. Then*

$$|T(s_1) - T(s_2)| \le C_0 |s_1 - s_2|.$$

where $C_0 = 18 \max_D |f|^2/\alpha$.

4. *Every junction in D° connects to exactly three edges such that the angle between any two edges is $2\pi/3$. Every junction on ∂D connects one edge to ∂D such that the edge meets ∂D perpendicularly.*

Before proving Theorem 5.1, we first examine the effect a small perturbation of Γ will have on $\mathbf{E_0}[\Gamma]$. Let $\Gamma^* \in \mathbf{S}^1(D)$ and $\gamma \subset \Gamma^*$ be a piece of C^1 curve parametrized by its arc length s,

$$\gamma(s) : [0, l] \longrightarrow \mathbf{R}^2.$$

Consider a perturbation of $\gamma(s)$ with a sufficiently small $h > 0$:

$$\gamma_h(s) = \gamma(s) - ha(s)S_0,$$

where $a(s) \in C_0^\infty[0, l]$ and S_0 is a unit vector pointing to a fixed side of γ on the support of $a(s)$. This can be achieved if the support of $a(s)$ is sufficiently small. We have

$$|\gamma_h'(x)|^2 = |T(s) - ha'(s)S_0|^2 = 1 - 2ha'(s)g_1(s) + o(h),$$

where $g_1(s) = \langle S_0, T(s) \rangle$ with $\langle ., . \rangle$ being the inner product in $\mathbf{R}^2$. Hence

$$|\gamma_h'(x)| = 1 - ha'(s)g_1(s) + o(h).$$

Denote the region on the left side of γ (with respect to the orientation of $\gamma(s)$) by Ω_L and the region on the right side of γ by Ω_R. Let Ω_h be the domain sandwiched by γ and γ_h.

$$\Omega_h = \left\{ \gamma(s) - ta(s)S_0 \;\middle|\; 0 \le s \le l,\ 0 \le t \le h \right\}.$$

Then

$$A(\Omega_h) = \int_0^l \int_0^h |J(\gamma_t(s)| dt ds = O(h),$$

where $J(\gamma_t(s))$ is the Jacobian of $\gamma_t(s)$:

$$|J(\gamma_t(s))| = \left| \det \begin{pmatrix} \gamma'(s) - ta'(s)S_0 \\ -a(s)S_0 \end{pmatrix} \right| = \left| \det \begin{pmatrix} T(s) \\ -a(s)S_0 \end{pmatrix} \right| = a(s)g_2(s).$$

Lemma 5.2 *Let* $\Gamma^h = (\Gamma^* \setminus \gamma) \cup \gamma_h$. *Then*

$$\mathbf{E_0}[\Gamma^h] - \mathbf{E_0}[\Gamma^*] = \int_0^l \int_0^h F(\gamma_t(s)) g_2(s) a(s)\, dt\, ds - \alpha h \int_0^l g_1(s) a'(s)\, ds + o(h),$$

where $F(x) = (c_{\Omega_R} - c_{\Omega_L})(2f(x) - c_{\Omega_L} - c_{\Omega_R})$.

Proof: Without loss of generality, we assume that $\Omega_h \subset \Omega_R$. Calculations in the proof of Proposition 3.12 have shown that

$$c_{\Omega_L \cup \Omega_h} = c_{\Omega_L} + \varepsilon_L, \quad c_{\Omega_R \setminus \Omega_h} = c_{\Omega_R} + \varepsilon_R,$$

where $\varepsilon_L,\ \varepsilon_R = O(A(\Omega_h)) = O(h)$.

$$\begin{aligned}
&\int_{\Omega_L \cup \Omega_h} (f - c_{\Omega_L \cup \Omega_h})^2 dx - \int_{\Omega_L} (f - c_{\Omega_L})^2 dx \\
&\quad = \int_{\Omega_L} (f - c_{\Omega_L} - \varepsilon_L)^2 dx + \int_{\Omega_h} (f - c_{\Omega_L} - \varepsilon_L)^2 dx - \int_{\Omega_L} (f - c_{\Omega_L})^2 dx \\
&\quad = \int_{\Omega_L} -2\varepsilon_L (f - c_{\Omega_L}) dx + \int_{\Omega_L} \varepsilon_L^2 dx + \int_{\Omega_h} (f - c_{\Omega_L} - \varepsilon_L)^2 dx \\
&\quad = \int_{\Omega_h} (f - c_{\Omega_L})^2 dx + o(h).
\end{aligned}$$

Similarly,

$$\int_{\Omega_R \setminus \Omega_h} (f - c_{\Omega_R \setminus \Omega_h})^2 dx - \int_{\Omega_R} (f - c_{\Omega_R})^2 dx = - \int_{\Omega_h} (f - c_{\Omega_R})^2 dx + o(h).$$

Therefore

$$\begin{aligned}
&\mathbf{E_S}[\Gamma^h] - \mathbf{E_S}[\Gamma^*] \\
&\quad = \int_{\Omega_h} \left\{ (f - c_{\Omega_L})^2 - (f - c_{\Omega_R})^2 \right\} dx + o(h) \\
&\quad = \int_0^l \int_0^h \left\{ \Big(f(\gamma_t(s)) - c_{\Omega_L} \Big)^2 - \Big(f(\gamma_t(s)) - c_{\Omega_R} \Big)^2 \right\} |J(\gamma_t(s))| dt ds + o(h) \\
&\quad = \int_0^l \int_0^h \left\{ \Big(f(\gamma_t(s)) - c_{\Omega_L} \Big)^2 - \Big(f(\gamma_t(s)) - c_{\Omega_R} \Big)^2 \right\} a(s) g_2(s) dt ds + o(h) \\
&\quad = h \int_0^l F(\gamma_t(s)) g_2(s) a(s) ds + o(h).
\end{aligned}$$

$$\mathbf{E_L}[\Gamma^h] - \mathbf{E_L}[\Gamma^*] = \alpha \int_0^l \left(|\gamma_h'(s)| - |\gamma'(s)| \right) ds = -\alpha h \int_0^l g_1(s) a'(s) ds + o(h).$$

This proves the lemma. ∎

Proof of Theorem 5.1: It is clear from Proposition 4.6 that there exists a $\Gamma^* \in \mathbf{S}^1(D)$ such that $E(\Gamma^*) = \inf_{\Gamma \in \mathbf{S}^1(D)} E(\Gamma)$, and that for each such Γ^* it must satisfy properties 1. We show that Γ^* satisfy property 2–4.

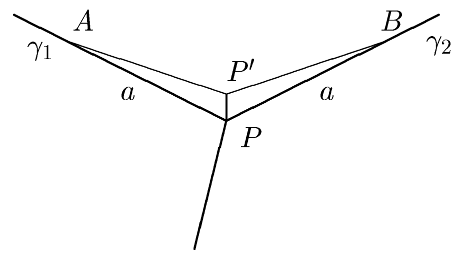

Figure 3: Close-up of a junction.

First we prove that the unit tangent vector of every C^1 curve in Γ^* must satisfy Lipschitz condition. Property 3 will follow easily from property 2, which we prove later. Let $\gamma(s)\colon [0,l]\longrightarrow \mathbf{R}^2$ be an edge of Γ^* parametrized by its arc length. Using the same notations as in Lemma 5.2 we have

$$\mathbf{E_0}[\Gamma^h]-\mathbf{E_0}[\Gamma^*]=\int_0^l\int_0^h F(\gamma_t(s))g_2(s)a(s)\,dt\,ds-\alpha h\int_0^l g_1(s)a'(s)\,ds+o(h).$$

Let $N(s)$ be the unit normal vector of $\gamma(s)$ pointing to the region Ω_L. Suppose $1>|T(s_1)-T(s_0)|>C|s_1-s_0|$ where $s_0,s_1\in[0,l]$ are sufficiently close. Choose $S_0=N(s_0)$. Then $g_1(s_0)=\langle S_0,T(s_0)\rangle=0$ and

$$|g_1(s_1)|=|\langle S_0,T(s_1)\rangle|>\frac{C}{2}|s_1-s_0|.$$

Let $\operatorname{supp}(a(s))\subseteq[s_0,s_1]$. So

$$\begin{aligned}\mathbf{E_0}[\Gamma^h]-\mathbf{E_0}[\Gamma^*] &= \int_0^l\int_0^h F(\gamma_t(s))g_2(s)a(s)\,dt\,ds-\alpha h\int_0^l g_1(s)a'(s)\,ds+o(h)\\ &= h\int_{s_0}^{s_1}\Big(\int_{s_0}^{s} G(t)g_2(t)\,dt-\alpha g_1(s)\Big)a'(s)\,ds+o(h),\end{aligned}$$

where $G(s)=\frac{1}{h}\int_0^l\int_0^h F(\gamma_t(s))\,dt$. Note that

$$\Big|\int_{s_0}^{s}G(t)g_2(t)\,dt\Big|\le\int_{s_0}^{s}|G(t)g_2(t)|\,dt<8\max_D|f|^2|s-s_0|.$$

So if $C>18\max_D|f|^2/\alpha$ then

$$\begin{aligned}&\max_{s\in[s_0,s_1]}\Big(\int_{s_0}^{s}G(t)g_2(t)\,dt-\alpha g_1(s)\Big)-\min_{s\in[s_0,s_1]}\Big(\int_{s_0}^{s}G(t)g_2(t)\,dt\\ \ge\;&\Big|\Big(\int_{s_0}^{s_1}G(t)g_2(t)\,dt-\alpha g_1(s_1)\Big)-\Big(\int_{s_0}^{s_0}G(t)g_2(t)\,dt-\alpha g_1(s_0)\Big)\Big|\\ \ge\;&\frac{18\max_D|f|^2}{2\alpha}\alpha|s_1-s_0|-8\max_D|f|^2|s_1-s_0|\\ =\;&\max_D|f|^2|s_1-s_0|\;>\;0.\end{aligned}$$

Therefore we can find an $a(s) \in C_0^\infty([0,l])$ such that $\mathbf{E_0}[\Gamma^h] < \mathbf{E_0}[\Gamma^*]$ by choosing a sufficiently small h. This is impossible. Hence $|T(s_1) - T(s_0)| < (18 \max_D |f|^2/\alpha)\, |s_1 - s_0|$.

Next we prove that property 4 must be satisfied by Γ^*. Let P be any junction in Γ^* such that $P \notin \partial D$; assume that two edges γ_1 and γ_2 meet at P at angle $0 < \theta < \pi$. Consider $P' \in D$ such that $|PP'| = h$ and PP' bisect angle θ, as illustrated in Figure 3. Let $A \in \gamma_1$ and $B \in \gamma_2$ be sufficiently close to P and $|AP| = |BP| = a$. We first assume that both γ_1 and γ_2 are locally linear around P. Denote the domain enclosed by the polygon $APBP'$ by Ω. Elementary trigonometry shows that

$$|AP'| = |BP'| = a - h\cos\frac{\theta}{2} + o(h),$$

$$A(\Omega) = ah\sin\frac{\theta}{2}.$$

Let

$$\Gamma^h = \Big(\Gamma^* \setminus \{\text{line segment } AP,\ BP\}\Big) \cup \{\text{line segments } PP',\ AP',\ BP'\}.$$

Calculations in the proof of Proposition 3.12 have shown that

$$\begin{aligned}
\mathbf{E_S}[\Gamma^h] - \mathbf{E_S}[\Gamma^*] &\le C_1 A(\Omega) = C_1 ah\sin\frac{\theta}{2}, \\
\mathbf{E_L}[\Gamma^h] - \mathbf{E_L}[\Gamma^*] &= \alpha h\left(1 - 2\cos\frac{\theta}{2}\right) + o(h).
\end{aligned}$$

Thus,

$$\mathbf{E_0}[\Gamma^h] - \mathbf{E_0}[\Gamma^*] \le \left(C_1 a\sin\frac{\theta}{2} + \alpha\left(1 - 2\cos\frac{\theta}{2}\right)\right) h + o(h).$$

If $0 < \theta < 2\pi/3$ then we can choose sufficiently small a and h so that $\mathbf{E_0}[\Gamma^h] - \mathbf{E_0}[\Gamma^*] < 0$. This contradicts the fact that Γ^* minimizes $\mathbf{E_0}[\Gamma]$. Thus, the angle at which γ_1 and γ_2 meet must be $2\pi/3$ or more.

In general γ_1 and γ_2 are not locally linear. Let the length of γ_1 between A and P be s_1 and let the length of γ_2 between B and P be s_2. Because both γ_1 and γ_2 are C^1 and the unit tangent vectors parametrized by their respected arc length satisfy the Lipschitz condition, for $i = 1, 2$ we have

$$\begin{aligned}
a &= \left|\int_0^{s_i} \gamma_i'(s)\,ds\right| \\
&= \left|\int_0^{s_i} \gamma_i'(0)\,ds + \int_0^{s_i} (\gamma_i'(s) - \gamma_i'(0))\,ds\right| \\
&\ge s - \int_0^{s_i} C_0 s\,ds \\
&= s_i - \frac{C_0 s_i^2}{2};
\end{aligned}$$

hence $s_i - a = O(a^2)$. Similarly, we can show that the area enclosed by γ_1 and AP and that by γ_2 and BP are both $O(a^3)$. Thus all arguments used in the locally linear case will not be affected. So $\theta \geq 2\pi/3$.

Therefore at any junction $P \in D$, the angle at which every two edges meet should be no less than $2\pi/3$. Consequently, the junction must connect exactly 3 edges and the angle at which every two edges meet must be $2\pi/3$.

Suppose $P \in \partial D$ is a junction. Let $0 < \theta \leq \pi/2$ be the angle at which an edge γ meets ∂D. Same as in the $P \notin \partial D$ case, we may assume that γ is locally linear at P. Consider $A \in \gamma$ and $B \in \partial D$ such that the line segment AB is perpendicular to the boundary. Assume that $|PA| = h$; thus $|AB| = h \sin\theta$. Let

$$\Gamma^h = \Big(\Gamma^* \setminus \{\text{line segment } PA\}\Big) \cup \{\text{line segments } AB\}.$$

Then

$$\begin{aligned}\mathbf{E_0}[\Gamma^h] - \mathbf{E_0}[\Gamma^*] &\leq \frac{C_1}{2} h^2 \sin\theta\cos\theta - (1 - \sin\theta)h + o(h) \\ &\leq -(1 - \sin\theta)h + o(h).\end{aligned}$$

Hence $\theta = \pi/2$, *i.e.* γ must meet ∂D perpendicularly.

Property 2 can be proved by using essentially the same idea. If two C^1 curves meet at a nonjunction point such that they form a corner at that point, then we can decrease $\mathbf{E_0}$ by cutting the corner. We omit the detail of the proof here. ∎

Theorem 5.3 *Let $f(x)$ be continuous on $D = [0, L] \times [0, L]$. Suppose $\Gamma^* \in \mathbf{S}^1(D)$ and*

$$\mathbf{E_0}[\Gamma^*] = \inf_{\Gamma \in \mathbf{S}^1(D)} \mathbf{E_0}[\Gamma].$$

Then $\Gamma^* \in \mathbf{S}^2(D)$. Moreover, let γ be any edge of Γ^* and $x \in \gamma$ be a nonjuction point. Then

$$\alpha\kappa(x) = (c_{\Omega_R} - c_{\Omega_L})(c_{\Omega_R} + c_{\Omega_L} - 2f(x)),$$

where γ is oriented with Ω_L and Ω_R being the region on its left and right respectively, and $\kappa(x)$ is the curvature of γ at x.

Proof: Again, we use the same notations as in Lemma 5.2. Let $\gamma(s)\colon [0, l] \longrightarrow \mathbf{R}^2$ be any edge of Γ^* parametrized by its arc length s. Then according to Lemma 5.2,

$$\mathbf{E_0}[\Gamma^h] - \mathbf{E_0}[\Gamma^*] = \int_0^l \int_0^h F(\gamma_t(s)) g_2(s) a(s)\, dt\, ds - \alpha h \int_0^l g_1(s) a'(s)\, ds + o(h).$$

Denote $F_0(s) = (c_{\Omega_R} - c_{\Omega_L})(2f(\gamma(s)) - c_{\Omega_L} - c_{\Omega_R})$. Then because $f(x)$ is continuous, $F(\gamma_t(s)) - F_0(s) \to 0$ as $h \to 0$. Thus

$$\mathbf{E_0}[\Gamma^h] - \mathbf{E_0}[\Gamma^*] = h \int_0^l \Big(F_0(s) g_2(s) a(s) - \alpha g_1(s) a'(s)\Big)\, ds + o(h).$$

Hence

$$\int_0^l \Big(F(s)g_2(s)a(s) - \alpha g_1(s)a'(s)\Big)ds = 0,$$

$$\int_0^l \left\{-\Big(\int_0^s F(t)g_2(t)dt\Big)a'(s) - \alpha g_1(s)a'(s)\right\} ds = 0.$$

Because $a(s)$ can be any function in $C_0^\infty([0,l])$ as long as S_0 points to a fixed side of γ on the support of $a(s)$, it implies

$$\int_0^s F(t)g_2(t)dt + \alpha g_1(s) = \text{constant}$$

on any interval $[b,c] \subseteq (0,l)$ in which S_0 points to a fixed side of γ. Thus on $[b,c]$, $g_1(s) = \langle S_0, T(s)\rangle$ is C^1. Since S_0 is arbitrary, $T(s)$ is C^1 and hence $\gamma(s)$ is C^2. Let $x = \gamma(s_0)$ where $s_0 \in [0,l]$ and choose $S_0 = N(s_0)$ where $N(s)$ is the unit normal vector of $\gamma(s)$ pointing to the region Ω_L. Then

$$\alpha\langle S_0, -\kappa(s)N(s)\rangle = \alpha\langle S_0, T'(s)\rangle = \alpha g_1'(s) = -F(s)g_2(s).$$

It is clear that $S_0 = N(s_0)$ implies $g_2(s_0) = 1$; hence

$$\begin{aligned}\alpha\kappa(x) &= -\Big(f(\gamma(s)) - c_{\Omega_L}\Big)^2 + \Big(f(\gamma(s)) - c_{\Omega_R}\Big)^2 \\ &= (c_{\Omega_R} - c_{\Omega_L})\Big(c_{\Omega_R} + c_{\Omega_L} - 2f(x)\Big).\end{aligned}$$

■

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