On the Use of High Order Ambiguity Function for Multicomponent Polynomial Phase Signals*

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Abstract

Nonstationary signals appear often in real-life applications and many of them can be modeled as polynomial phase signals (PPS). High-order ambiguity function (HAF) was first introduced to estimate the parameters of a single component PPS, but has not been widely used for multi-component PPS because of its nonlinearity. Multi-component PPS arise for example, in Doppler radar applications when multiple objects are tracked simultaneously. In this paper, we show that HAF is virtually additive for multi-component PPS and suggest an algorithm to estimate their parameters. Numerical examples are presented to illustrate the theories.

Keywords: Polynomial phase signal, chirp signal, high-order ambiguity function, radar, Weyl sum.

*Original manuscript was submitted to *Signal Processing* 6/96; revised 12/96, 12/97.
†Research supported in part by the National Science Foundation, grant DMS–9307601.

1 Introduction

Signals encountered in engineering applications such as communications, radar, and sonar often involve amplitude (AM) and/or frequency modulation (FM). AM-FM is inherent in certain natural signals as well. For example, experimental evidence has shown that voiced speech segments exhibit amplitude and frequency modulation, cf. Teager and Teager [14]. In Chen *et al* [2], spatial-temporal motion was viewed as an FM problem in the frequency domain, and motion estimation was carried out by tracking the frequency variations in the Fourier transforms.

An AM-FM signal can be written as $x(t) = \rho(t)e^{j\phi(t)}$, where $\rho(t)$ represents the time-varying amplitude, $\phi(t)$ stands for the phase, and instantaneous frequency is defined as the derivative of the phase, $f(t) = d\phi(t)/dt$. Although non-parametric techniques are available to track amplitude and frequency variations, we focus on parametric models here because they offer parsimony and inherently unlimited resolution.

The phase function of a large class of AM-FM processes can be modeled by a polynomial function of t. It is known that in active systems, the radar echo from a maneuvering target has nonlinear phase characteristics, which depend on the target trajectory. In Kelly [5], the radar echo is expressed as $x(t) = \rho(t)e^{jr(t)}$, and the trajectory is approximated by

$$r(t) \doteq r_0 + v_y t + \frac{1}{2}\left(a_y + \frac{v_x^2}{r_0}\right)t^2 + \frac{1}{2r_0}\, a_x v_x t^3 + \frac{1}{8r_0}\, a_x^2 t^4 + \cdots . \tag{1.1}$$

From (1.1) we see that radial velocity v_y introduces a linear phase term in $x(t)$; radial acceleration a_y and cross-range velocity v_x induce a quadratic phase term; a_x and v_x induce a cubic phase term, and so forth. The coefficients of the power series of $r(t)$ are thus related to the kinetic parameters of the moving target, cf. also Rihaczek [10]. According to the Stone-Weierstrass theorem, any continuous function (such as $r(t)$) over a closed interval can be approximated uniformly by a polynomial function. Therefore the class of polynomial phase signals is rather broad.

Single-component polynomial phase signals (PPS) have been investigated extensively in recent years using the high-order ambiguity function (HAF), introduced by Peleg and Porat [6] (see for example, Porat [9, Ch. 12]). HAF has proven to be effective in parameter estimation of single-component PPS, and results on constant, random, or time-varying amplitudes have appeared in Peleg and Porat [6], Shamsunder, Giannakis and Friedlander [12], Swami [13], and Zhou, Giannakis and Swami [17].

Signals arising from real life applications often have multiple components, and their estimation poses a great challenge. When HAF is applied to multi-component PPS, a large number of cross-terms, which are themselves PPS, are introduced. The cross-terms are nuisance, and in estimation involving multi-component PPS, many authors have imposed severe constraints on the parameters (e.g., [3], [8]), or, avoided discussing them altogether (e.g., [7]). In Barbarossa, Scaglione and Giannakis [1] however, PPS parameter identifiability was resolved using the product multi-lag HAF, when the components have equal amplitudes. The focus of this paper is on the application of single-lag HAF to multi-component PPS. In this respect, except for Zhou and Swami [18], there has been no quantitative study on the magnitudes of the cross-terms. Our main contribution here is to show that these cross-terms are almost always negligible, and that HAF is virtually additive. This knowledge is then incorporated into parameter estimation schemes for multi-component PPS.

The organization of the paper is as follows: in §2, we give a brief overview of the HAF and the related single-component PPS parameter estimation issues. In §3, we focus on multi-component chirp signals and examine in detail, the cross-terms in the ambiguity function domain. The results of §3 are then generalized in §4 to the

Mth-order multi-component PPS. An algorithm for parameter estimation of multi-component PPS is derived based on these results. Throughout the paper, numerical examples are provided along with theoretical expositions. Finally, conclusions are drawn in §5, followed by an Appendix which includes the rather technical proofs of the three major theorems of this paper.

2 HAF and Single-Component PPS

HAF was originally devised for estimation of single-component, constant amplitude polynomial phase signals (PPS) of the form

$$y(t) = \rho\, e^{j\phi(t)} = \rho\, e^{j\sum_{m=0}^{M} a_m t^m}. \tag{2.1}$$

Throughout this paper, we consider $y(t)$ given by (2.1) in discrete time $t = 0, \ldots, T-1$, and refer to it as a PPS of order M. Note that $y(t)$ corresponds to a harmonic when $M = 1$ and a chirp when $M = 2$.

For an integer $\tau \neq 0$, define $\mathcal{P}_2[y(t);\tau] = y(t)y^*(t-\tau)$, which can be viewed as a second-order instantaneous moment of $y(t)$. Since multiplying $y(t)$ by its conjugated lagged copy $y^*(t-\tau)$ is equivalent to differencing in the phase of $y(t)$, it follows easily that $\mathcal{P}_2[y(t);\tau]$ is a new PPS of order $M-1$. The above operation can be iterated to eventually reduce a PPS of any order to a complex constant. These iterates are called the *high-order instantaneous moments* (HIM) of $y(t)$.

General properties of the HIM operations are discussed in Porat [9, Ch. 12]. For self-containment, we include below, some of the basic results of [9, Ch. 12].

Let $y(t)$ be a complex valued signal, and define for any integer q

$$y^{(*q)}(t) \triangleq \begin{cases} y(t), & \text{if } q \text{ is even,} \\ y^*(t), & \text{if } q \text{ is odd.} \end{cases} \tag{2.2}$$

For integers $M, \tau > 0$, the corresponding HIM operator is defined as

$$\mathcal{P}_M[y(t);\tau] \triangleq \prod_{q=0}^{M-1} [y^{(*q)}(t-q\tau)]^{\binom{M-1}{q}}, \tag{2.3}$$

where $\binom{M-1}{q}$ are the binomial coefficients, and M is the *order* of $\mathcal{P}_M$. For $y(t)$ defined in (2.1), it holds that

$$\mathcal{P}_M[y(t);\tau] = \rho^{2^{M-1}} e^{j\tilde{\omega}t+\tilde{\phi}}, \tag{2.4}$$

where

$$\tilde{\omega} \triangleq M!\, \tau^{M-1}\, a_M, \tag{2.5}$$

$$\tilde{\phi} \triangleq (M-1)!\, \tau^{M-1}\, a_{M-1} - 0.5M!(M-1)\tau^M a_M. \tag{2.6}$$

We notice that $\mathcal{P}_M[y(t);\tau]$ of an Mth-order PPS is reduced to a constant amplitude harmonic with amplitude $\rho^{2^{M-1}}$, frequency $\tilde{\omega}$ and phase $\tilde{\phi}$. HIM of order $> M$ will reduce an Mth-order PPS to a simple complex constant.

Since $\mathcal{P}_M[y(t);\tau]$ is (almost) periodic,[1] we consider its Fourier series (FS) coefficient function (of $\alpha \in [-\pi, \pi)$) defined as,

$$P_M[y;\alpha,\tau] \triangleq \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathcal{P}_M[y(t);\tau]\, e^{-j\alpha t}. \tag{2.7}$$

We call P_M a *high-order ambiguity function* (HAF) of $y(t)$, following the terminology of Porat [9]. Note that up to a scalar constant, HAF of order $M = 2$ is the same as the classical ambiguity function.

Substituting (2.4) into (2.7), we obtain

$$P_M[y;\alpha,\tau] = \rho^{2^{M-1}} e^{j\tilde{\phi}}\, \delta(\alpha - \tilde{\omega}), \tag{2.8}$$

[1] Although $e^{j\omega t}$ is strictly periodic in continuous time $\forall\omega$, it is strictly periodic in discrete time only when ω is a rational multiple of π. This is why we refer to $e^{j\omega t}$ as almost periodic $\forall\omega$ when t is discrete.

where $\delta(\cdot)$ denotes the Kronecker delta function

$$\delta(\alpha) \triangleq \begin{cases} 1, & \text{if } \alpha = 0 \pmod{2\pi}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

The HAF in (2.8) peaks when α is at $\tilde{\omega} = M!\,\tau^{M-1} a_M$. Hence we may obtain the highest order polynomial phase coefficient a_M from the peak location of (2.8)

$$a_M = \frac{1}{M!\,\tau^{M-1}} \ \arg\max_{\alpha} \left| P_M[y;\alpha,\tau] \right|. \tag{2.10}$$

Now, by multiplying $\exp\{-j a_M t^M\}$ with $y(t)$, we obtain a PPS of order $M-1$. The above procedure is then repeated to obtain an estimate of a_{M-1}. Subsequent iterations yield estimates for $a_{M-2}, \ldots, a_1$, cf. [9, Ch. 12]. Finally, $\rho \exp\{j a_0\}$ can be estimated via linear least squares (LS) method. Note that the estimation of a_M via (2.10) requires $\tau \neq 0$ and $|a_M| < \pi/(M!\tau^{M-1})$. Unless otherwise stated, we shall choose $\tau = 1$ throughout the paper to ensure the maximum range of a_M.

In practice, additive noise $v(t)$ may be present and we actually observe

$$x(t) = y(t) + v(t) = \rho\ e^{j\sum_{m=0}^{M} a_m t^m} + v(t). \tag{2.11}$$

Sample estimate of $P_M[y;\alpha,\tau]$ is computed from $\{x(t)\}_{t=0}^{T-1}$ as follows:

$$\hat{\mathcal{P}}_M[y(t);\tau] \triangleq \mathcal{P}_M[x(t);\tau] = \prod_{q=0}^{M-1} [x^{(*q)}(t - q\tau)]^{\binom{M-1}{q}}, \tag{2.12}$$

$$\hat{P}_M[y;\alpha,\tau] \triangleq \frac{1}{T} \sum_{t=0}^{T-1} \hat{\mathcal{P}}_M[y(t);\tau]\ e^{-j\alpha t}. \tag{2.13}$$

Asymptotic unbiasedness and consistency of (2.13) were established in Zhou, Giannakis and Swami [17] (see also [9, Ch. 12]) where $v(t)$ is zero-mean and white complex Gaussian. Once $\hat{P}_M[y;\alpha,\tau]$ is computed, a_M can be estimated by substituting $\hat{P}_M[y;\alpha,\tau]$ for $P_M[y;\alpha,\tau]$ into (2.10).

3 Multi-Component Chirp Signals

In Doppler applications and when dealing with multiple moving targets, the returned echo can be modeled as a multi-component PPS,

$$y(t) = \sum_{l=1}^{L} y_l(t) = \sum_{l=1}^{L} \rho_l e^{j \sum_{m=0}^{M_l} a_{l,m} t^m} \tag{3.1}$$

where each $y_l(t)$ is a constant amplitude PPS of order M_l. When the nonlinear HIM operator $\mathcal{P}_M$ is applied to $y(t)$, many cross-terms will emerge,

$$\mathcal{P}_M[y(t);\tau] = \sum_{l=1}^{L} \mathcal{P}_M[y_l(t);\tau] + \text{cross-terms.} \tag{3.2}$$

For an L-component signal with all $M_l = M$, the number of cross-terms is $L^{2^{M-1}} - L$, which gives two cross-terms for $L = 2$, $M = 2$ and 14 for $L = 2$, $M = 3$. These cross-terms are themselves PPS. We shall study the $M = 2$ case in detail here and generalize to the $M \geq 2$ case in the next section.

If the targets are moving along radial directions and have constant accelerations, then it is appropriate to model the echo as a multi-component chirp. We shall focus on constant amplitude chirps in this paper; generalizations to the time-varying amplitude chirps are rather straightforward, by applying the results of [17].

A discrete time L-component chirp signal with constant amplitudes is give by

$$y(t) = \sum_{l=1}^{L} y_l(t) = \sum_{l=1}^{L} \rho_l e^{j(a_{l0} + a_{l1} t + a_{l2} t^2)}, \qquad t = 0, 1, \ldots, T-1. \tag{3.3}$$

For example, a two-component ($L = 2$) chirp signal with constant amplitude is given by

$$y(t) = \rho_1 e^{j(a_{10} + a_{11} t + a_{12} t^2)} + \rho_2 e^{j(a_{20} + a_{21} t + a_{22} t^2)}. \tag{3.4}$$

The two components will be considered distinct if their respective instantaneous frequencies, $d\phi_l(t)/dt = a_{l1} + 2a_{l2} t$, are different.

It is not difficult to show that the 2nd-order instantaneous moment of $y(t)$ in (3.4) is (assuming $\tau = 1$)

$$\begin{aligned}
\mathcal{P}_2[y(t);1] &\stackrel{\triangle}{=} y(t)y^*(t-1) \\
&= \rho_1^2\, e^{2ja_{12}t}\, e^{j(a_{11}-a_{12})} + \rho_2^2\, e^{2ja_{22}t}\, e^{j(a_{21}-a_{22})} \\
&\quad + \underbrace{2\rho_1\rho_2\, e^{j\{(a_{12}-a_{22})t^2+(a_{11}-a_{21}+2a_{22})t+(a_{21}-a_{22}+a_{10}-a_{20})\}}}_{\mathcal{T}_1(t)} \\
&\quad + \underbrace{2\rho_1\rho_2\, e^{j\{(a_{22}-a_{12})t^2+(a_{21}-a_{11}+2a_{12})t+(a_{11}-a_{12}+a_{20}-a_{10})\}}}_{\mathcal{T}_2(t)}. \qquad (3.5)
\end{aligned}$$

The FS coefficient function of $\mathcal{P}_2[y(t);1]$ is given by

$$\begin{aligned}
P_2[y;\alpha,1] &= \rho_1^2\, e^{j(a_{11}-a_{12})}\, \delta(\alpha-2a_{12}) + \rho_2^2\, e^{j(a_{21}-a_{22})}\, \delta(\alpha-2a_{22}) \\
&\quad + \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathcal{T}_1(t)\, e^{-j\alpha t} + \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathcal{T}_2(t)\, e^{-j\alpha t}. \qquad (3.6)
\end{aligned}$$

We refer to the first two terms on the r.h.s. of (3.6) as *auto-peaks* because their locations yield the highest order polynomial phase coefficients a_{12} and a_{22}. To obtain good estimates of a_{12} and a_{22} based on the locations of the auto peaks, it is highly desirable that the last two (cross-) terms on the r.h.s. of (3.6); i.e. the FS coefficient functions of $\mathcal{T}_1(t)$ and $\mathcal{T}_2(t)$, be negligible as compared to the auto-peaks.

3.1 FS Coefficient Function of a Single Chirp

For simplicity, let us rewrite $\mathcal{T}_1(t)$ defined in (3.5) as

$$\mathcal{T}_1(t) = 2\rho_1\rho_2\, e^{j\gamma_2 t^2}\, e^{j\gamma_1 t}\, e^{j\gamma_0}, \qquad (3.7)$$

where

$$\gamma_2 \stackrel{\triangle}{=} a_{12}-a_{22}, \quad \gamma_1 \stackrel{\triangle}{=} a_{11}-a_{21}+2a_{22}, \quad \gamma_0 \stackrel{\triangle}{=} a_{21}-a_{22}+a_{10}-a_{20}.$$

Let us denote the FS coefficient function of $e^{j\gamma_2 t^2}$ by $h(\alpha)$,

$$h(\alpha) \stackrel{\triangle}{=} \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} e^{j\gamma_2 t^2}\, e^{-j\alpha t}, \qquad (3.8)$$

and express the third term on the r.h.s. of (3.6) as

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathcal{T}_1(t)\ e^{-j\alpha t} = 2\rho_1\rho_2\ e^{j\gamma_0}\ h(\alpha - \gamma_1). \tag{3.9}$$

We would like to compare its magnitude with ρ_l^2, the magnitude of the auto-peaks.

Interestingly, although $\exp(j\gamma_2 t^2)$ is aperiodic in continuous time, it is periodic in discrete time only when γ_2 is a rational multiple of π. To see this, write $\gamma_2 = 2\pi N/D$ where N and $D > 0$ are co-prime integers. For all integers t, we have

$$e^{j\gamma_2 (t+D)^2 N/D} = e^{j2\pi(t^2+2tD+D^2)N/D} = e^{j2\pi t^2 N/D}, \tag{3.10}$$

which is periodic in t with period D (D may not be the minimum period though). Now since $\exp(j2\pi t^2 N/D)$ is periodic, $h(\alpha)$ contains spectral lines. Theorem 1 below establishes a bound on $|h(\alpha)|$ and is crucial for assessing the contribution of the cross-terms in (3.6).

Theorem 1 *Suppose that $\gamma_2 = 2\pi N/D$ where $D > 0$ and N are co-prime integers. Then the FS coefficient function $h(\alpha)$ of $e^{j\gamma_2 t^2}$ satisfies*

$$\max_\alpha |h(\alpha)| = \begin{cases} \sqrt{2/D}, & \text{if } D \text{ is even,} \\ \sqrt{1/D}, & \text{if } D \text{ is odd.} \end{cases} \tag{3.11}$$

Note that the r.h.s of the above expression does not depend on N.

Proof. See the appendix. ∎

Our Theorem 3 (in §4) asserts that the limit (3.8) defining $h(\alpha)$ tends to zero uniformly in α when γ_2 is an irrational multiple of π. Although any irrational number can be approximated to arbitrary precision by rational numbers, the denominators of these rationals tend to infinity as the precision increases. Theorem 1 then predicts that the corresponding $h(\alpha)$ is negligible in general. The following example illustrates the difference between the two scenarios.

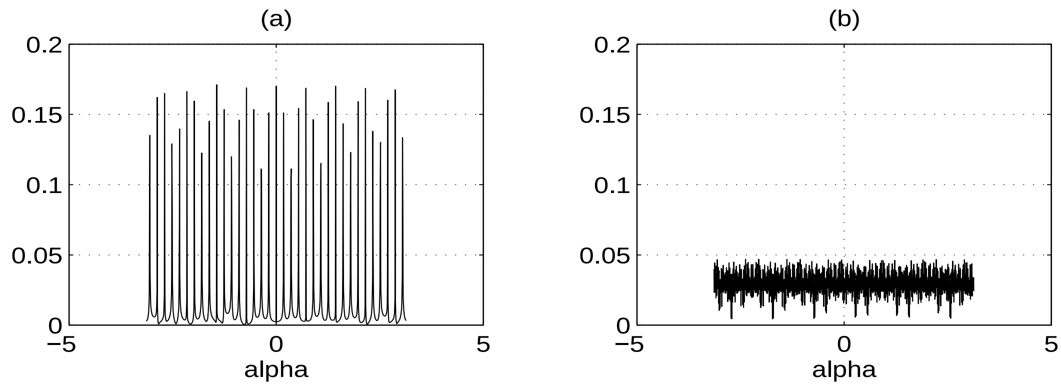

Figure 1: FS coefficient function of $e^{j\gamma_2 t^2}$ for $\gamma_2 = 2\pi \times 24/35$ and $\gamma_2 = 0.5$.

Example 1. Figure 1(a) shows $|h(\alpha)|$ (calculated with $T = 1,024$) as a function of $\alpha \in [-\pi, \pi)$ for $\gamma_2 = 2\pi \times 24/35$. We observe 35 spectral lines[2], and their magnitudes do not exceed $1/\sqrt{D} = 0.169$. In Figure 1(b), we have $\gamma_2 = 0.5$, which cannot be expressed as $2\pi N/D$ for integers D, N. There are no discernible peaks in Figure 1(b) and $|h(\alpha)|$ here is much smaller than that in Figure 1(a).

Because line spectra are produced only when γ_2 is a rational multiple of π, and almost all real numbers are irrational, we conclude that spectral lines appear in $h(\alpha)$ with probability zero and $h(\alpha)$ is very small for large T. Even if a given signal does have coefficients that are all rational multiples of π, unless the common denominator D is very small such as $D = 3$, 4, the algorithm should still work (see the analysis on the worst case scenarios in the next subsection). In fact, in real applications such small D's can only be due to *very low and insufficient* sampling rates.

Considering the above conclusion and together with (3.9), we infer that the two cross-terms in (3.6) are negligible and hence HAF is virtually additive:

$$P_2[y; \alpha, 1] \approx \rho_1^2 \, e^{j(a_{11} - a_{12})} \, \delta(\alpha - 2a_{12}) + \rho_2^2 \, e^{j(a_{21} - a_{22})} \, \delta(\alpha - 2a_{22}). \tag{3.12}$$

[2] It can be shown that $h(\alpha)$ contains $D/2$ spectral lines when D is even, and D spectral lines when D is odd.

3.2 Worst Case Scenarios

In [8], Polad and Friedlander proposed a procedure for tracking multi-component PPS parameters. Those of the strongest component are first identified. The component is then removed and the estimation process is continued with the other $L-1$ components. Relation $\rho_1/\rho_2 > 2$ was assumed in [8] in order for $\rho_1^2 > 2\rho_1\rho_2$ and to ensure that the cross-terms are never more than the strongest auto peak. With the help of Theorem 1 (and Theorem 3 in §4), however, we can show that such an assumption is unnecessarily strong.

We observe that the contribution from the cross-term in (3.9) is no more than $2\rho_1\rho_2 \max_\alpha |h(\alpha)|$; i.e.

$$\lim_{T\to\infty} \left| \frac{1}{T} \sum_{t=0}^{T-1} \mathcal{T}_1(t)\, e^{-j\alpha t} \right| \le 2\rho_1\rho_2 \max_\alpha |h(\alpha)|. \tag{3.13}$$

The r.h.s. tends to zero (Theorem 3) when γ_2 is an irrational multiple of π, and is nonzero otherwise. The worst cases are when $\gamma_2 = 2\pi N/D$ with D small, and we shall examine them below.

First, we recognize that with $\tau = 1$, the leading chirp coefficients must satisfy $|a_{12}| < \pi/2$ and $|a_{22}| < \pi/2$ in order to satisfy the HAF-based identifiability condition stated following (2.10). This implies that $|\gamma_2| = |a_{12} - a_{22}| < \pi$, and hence $N/D < 1/2$. Without loss of generality, we assume that $\rho_1 \ge \rho_2$. Worst case scenarios are identified as follows:

(c1) $D = 4$, $N = 1$ and $|a_{12} - a_{22}| = \pi/2$. The r.h.s. of (3.13) is then $\sqrt{2}\,\rho_1\rho_2$. In order for the cross-terms not to exceed the strongest auto peak ρ_1^2, we must have $\rho_1/\rho_2 > \sqrt{2}$.

(c2) $D = 3$, $N = 1$ and $|a_{12} - a_{22}| = 2\pi/3$. The r.h.s. of (3.13) is then $2\rho_1\rho_2/\sqrt{3}$. In order for the cross-terms not to exceed the strongest auto peak ρ_1^2, we must have

$\rho_1/\rho_2 > 2/\sqrt{3}$.

For all other D's, Theorem 1 ensures that the the cross-term in (3.13) is never more than the strongest auto-peak. Hence, we conclude that if $|a_{12} - a_{22}| \neq \pi/2$ or $2\pi/3$, then the successive estimation algorithm described in [8] can be implemented for *any* $\rho_1/\rho_2 > 1$. Otherwise, one needs to ensure $\rho_1/\rho_2 > \sqrt{2}$ or $2/\sqrt{3}$. This is a much weaker condition than the one stated in [8].

We further infer from Theorem 1 that if $D \geq 8$ is even and $1 \leq \rho_1/\rho_2 < \sqrt{D/8}$, or, if $D \geq 5$ is odd and $1 \leq \rho_1/\rho_2 < \sqrt{D}/2$, then the two strongest peaks in $P_2[y;\alpha,1]$ will always be due to the auto-terms, because the r.h.s. of (3.13) will always be smaller than ρ_2^2 (and hence ρ_1^2). For a generic $\gamma_2 = a_{12} - a_{22}$ to be well approximated by $2\pi N/D$, D would have to be fairly large, and the above condition is then easily met. This implies that in general, $P_2[y;\alpha,1]$ can be regarded as virtually additive, and it is safe to use the locations of the L largest peaks to estimate a_{l2} for $l = 1, 2, \ldots, L$.

Example 2. We generated $T = 1024$ samples of a two-component PPS $y(t)$ given by (3.4), where each a_{lm} is an i.i.d. uniform random variable in $[0, 1)$. Figures 2(a) and 2(b) show particular realizations of $\hat{P}_2[y;\alpha,1]$ with amplitudes $\rho_1 = \rho_2 = 1$ and $\rho_1 = 2.5$, $\rho_2 = 1$, respectively. We observe two distinct peaks in $\hat{P}_2[y;\alpha,1]$, the locations of which correspond to $2a_{12}$ and $2a_{22}$, illustrating the virtual additivity of $\hat{P}_2[y;\alpha,1]$. This experiment was repeated 100 times, and two peaks were observed for all 100 realizations. Note that although the dynamic range in Figure 2(b) is large due to $\rho_1 \neq \rho_2$, the two strongest peaks nevertheless yield the correct $2a_{12}$ and $2a_{22}$.

Proceeding arguments assume that the leading chirp coefficients are different. The picture changes when $a_{12} = a_{22}$, because then the two auto-peaks merge to one,

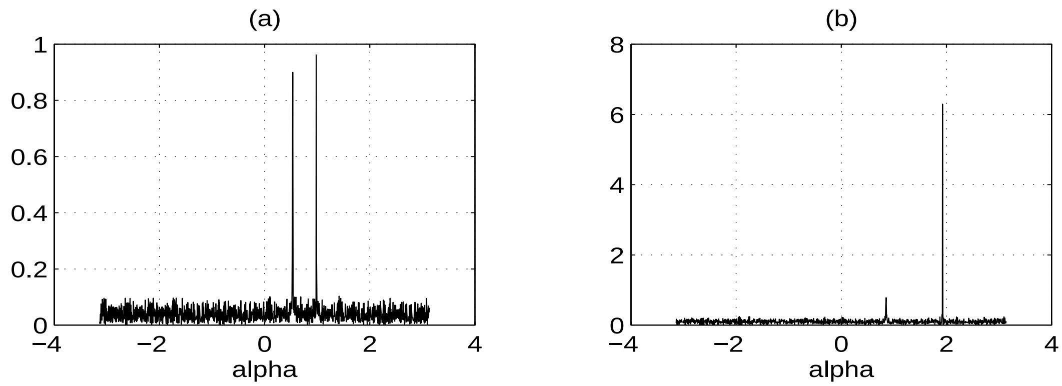

Figure 2: $\hat{P}_2[y;\alpha,1]$ of a two component chirp.

and $\gamma_2 = 0$ makes both cross-terms $\mathcal{T}_1(t)$ and $\mathcal{T}_2(t)$ behave like harmonics. Under the assumption that the instantaneous frequencies of the different components must be different, we infer that $a_{11} \neq a_{21}$ when $a_{12} = a_{22}$ and $\mathcal{T}_1(t)$ and $\mathcal{T}_2(t)$ generate peaks at $2a_{12} + (a_{11} - a_{21})$ and $2a_{12} - (a_{11} - a_{21})$ in $P_2[y;\alpha,1]$, which are equidistant from the auto-peak at $2a_{12}$.

When the leading chirp coefficients are different but are close to each other, we observe two closely spaced peaks in $P_2[y;\alpha,1]$. High resolution algorithms such as MUSIC, matrix pencil, and the Tufts-Kumaresan method can then be employed (see [17]) to estimate the coefficients of the polynomial phase.

4 Multi-Component PPS of Order M

Recall that a general L-component, constant amplitude PPS is defined as

$$y(t) = \sum_{l=1}^{L} y_l(t) = \sum_{l=1}^{L} \rho_l \, e^{j\sum_{m=0}^{M_l} a_{l,m} t^m}.$$

We may assume without loss of generality that the polynomial phase orders satisfy $M_1 \geq M_2 \geq \ldots \geq M_L$. The HAF of order M_1, $P_{M_1}[y;\alpha,\tau]$, exhibits peaks at $M_1!\tau^{M_1-1}a_{l,M_1}$ for all l such that $a_{l,M_1} \neq 0$, but a large number of cross-terms are also present. As in §3.1, we shall examine the magnitude of the FS coefficient function of

$$c(t) = e^{j \sum_{m=2}^{M} a_m t^m}, \tag{4.1}$$

in order to make inference about the contribution of these cross-terms.

4.1 FS Coefficient Function of the Cross-Terms

As with the case of a chirp, $c(t)$ given in (4.1) is periodic in discrete time t if and only if all a_m are rational multiples of π, in which case the FS coefficient function of $c(t)$ contains spectral lines. Unlike the chirp, there is no general formula for the largest magnitude of these spectral lines when $M > 2$. Instead, a bound can be established as stated in the following theorem:

Theorem 2 *Consider the polynomial phase signal* $c(t) = e^{j \sum_{m=2}^{M} a_m t^m}$ *and suppose that* $a_m = 2\pi b_m / D$*, where* $D, b_1, \ldots, b_m$ *are relatively prime integers,* $D > 0$*. Then*

$$\lim_{T \to \infty} \max_{\alpha} \left| \frac{1}{T} \sum_{t=0}^{T-1} c(t)\ e^{-j\alpha t} \right| \leq \min\left(1,\ c_D / D^{1/M}\right) \tag{4.2}$$

where $c_D = d_D^{\log_2 M}$ *and* d_D *denotes the number of divisors of* D*. Hence the FS coefficient function of* $c(t)$ *is uniformly bounded by* $\min\left(1,\ c_D / D^{1/M}\right)$*.*

Proof. See the appendix. ∎

Remark 1. If D has the prime factorization $D = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, then it follows that $d_D = (\beta_1 + 1) \cdots (\beta_k + 1)$.

Remark 2. It can be shown that the rate of growth of d_D as D increases is approximately logarithmic or less, depending on how many factors D has, see Rosen [11]. Therefore, $\lim_{D \to \infty} d_D / D^{\varepsilon} = 0$ for any $\varepsilon > 0$, and the r.h.s. of (4.2) tends to

TABLE I. EXAMPLES OF (4.2)

b_2	b_3	D	d_D	l.h.s.	r.h.s.
1024708	7286213	11142379	8	0.0813	0.1038
1135718	950919	1247601	8	0.0864	0.2155
1545555	279513	5888885	16	0.0880	0.4085

zero as $D \to \infty$ as a result.

We note here that Theorem 2 provides bounds on the FS coefficient functions of all Mth-order PPS. Since they include worst case scenarios, these bounds may not be always optimal. However, the established bounds do point out the qualitative dependence of the magnitude of FS coefficient function on $D^{-1/M}$, which tends to zero as $D \to \infty$. The works by Hua [4], Vinogradov [16] and others also indicate that the exponent $-1/M$ of D on the r.h.s. of (4.2) is optimal and cannot be improved. From (4.2), we infer that the larger the D and the smaller the d_D, the tighter the bound. For generic $a_m = 2\pi b_m / D$, such will be the case. Table 4.1 gives numerical examples on the use of (4.2).

In practice, it is unlikely for an arbitrarily chosen a_m to be a rational multiple of π, and it is even less likely for *all* $\{a_m\}$, $m = 2, \ldots, M$, to be rational multiples of π. The following theorem show the magnitude of the FS coefficient function tends to zero uniformly when the conditions of Theorem 2 are not met.

Theorem 3 *Consider the polynomial phase signal* $c(t) = e^{j \sum_{m=2}^{M} a_m t^m}$ *and suppose that at least one* a_m *is an irrational multiple of* π*. Then*

$$\lim_{T \to \infty} \max_{\alpha} \left| \frac{1}{T} \sum_{t=0}^{T-1} c(t)\ e^{-j\alpha t} \right| = 0. \tag{4.3}$$

Proof. See the appendix. ∎

The importance of Theorem 2 and Theorem 3 is to guarantee that except in the pathological case where all a_m are rational multiples of π with a *very small* common denominator, the FS coefficient function of the cross-term $c(t)$ is negligible. Hence in a multi-component PPS setting, the HAF is virtually additive.

4.2 Parametric Estimation of Multi-Component PPS

The fact that the HAF of a multi-component PPS is virtually additive allows us to develop an algorithm that estimates the PPS parameters in a straightforward manner. We illustrate the algorithm by way of two examples.

Example 3. Two-component cubic FM signals.

In this example, we have available $T = 2048$ samples of

$$x(t) = y(t) + v(t) = \sum_{l=1}^{2} e^{j(a_{l1}t + a_{l2}t^2 + a_{l3}t^3)} + v(t),$$

where $a_{11} = 1$, $a_{12} = -1$, $a_{13} = -0.25$, $a_{21} = 2$, $a_{22} = 0.5$, $a_{23} = 0.5$, and $v(t)$ is a zero-mean, white complex Gaussian process with variance $\sigma_v^2 = 0.01$. Note that we assume without loss of generality that the amplitudes are 1 and the initial phases are zero, because otherwise these parameters can be estimated from the standard LS after all other parameters have been estimated. Our procedure consists of the following steps:

Step 1. We compute $\hat{\mathcal{P}}_3[y(t);1] \stackrel{\triangle}{=} \mathcal{P}_3[x(t);1] = x(t)[x^*(t-1)]^2 x(t-2)$, the FS coefficient function of which should produce spectral lines at $3!a_{13}$ and $3!a_{23}$. Indeed, we observe peaks at $6a_{13} = -1.5$ and $6a_{23} = 3.0$ from Figure 3(a). These two peaks give estimates $\hat{a}_{13} = -0.2500$, and $\hat{a}_{23} = 0.5000$.

Step 2. In this step, we form two new PPS

$$x_{11}(t) = x(t)e^{-j\hat{a}_{13}t^3} \quad \text{and} \quad x_{12}(t) = x(t)e^{-j\hat{a}_{23}t^3}.$$

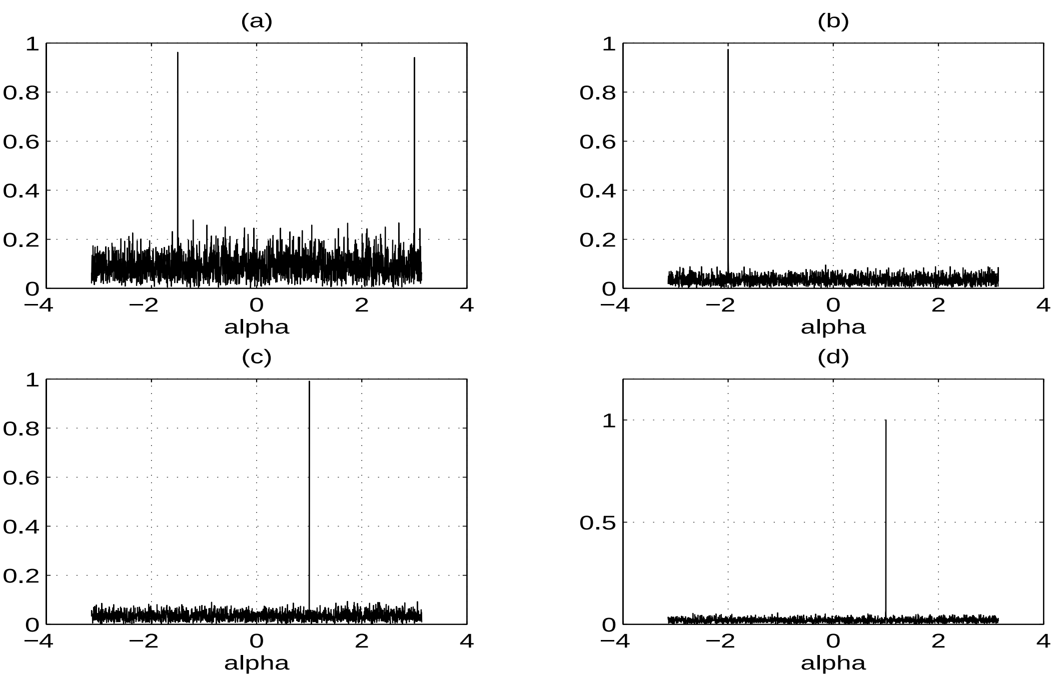

Figure 3: Parameter estimation of a two-component cubic FM signal

We compute $\hat{\mathcal{P}}_2[x_{1l}(t);1] \stackrel{\triangle}{=} x_{1l}(t)x_{1l}^*(t-1)$ for $l=1,\ 2$. Note that each $\hat{\mathcal{P}}_2[x_{1l}(t);1]$ is the sum of one harmonic and some other higher order PPS. Therefore by our theorems their FS coefficient functions should each produce a single peak. This is precisely the case as shown in Figure 3(b) and Figure 3(c). These two peaks yield estimates $\hat{a}_{12} = -1.000$ and $\hat{a}_{22} = 0.5001$ respectively.

Step 3. In this final step, we form two still new PPS

$$x_{21}(t) = x_{11}(t)e^{-j\hat{a}_{12}t^2} \quad \text{and} \quad x_{22}(t) = x_{12}(t)e^{-j\hat{a}_{22}t^t}.$$

Now each x_{2l} is the sum of a harmonic and some other higher order PPS, so again by our theorems the FS coefficient function of each should have a single peak. The single peak of the FS coefficient function of $x_{21}(t)$ is shown in Figure 3(d), its location yields the estimate $\hat{a}_{11} = 1.0002$. Similarly, the single peak of the FS coefficient function of $x_{22}(t)$ yields the estimate $\hat{a}_{21} = 1.9999$.

The FFT length used in this example is $N = 2^{14}$. ■

Example 4. Often, the components of a multi-component PPS are not all of the same order. The algorithm outlined in Example 3 can be easily extended to these settings. Here we explain how the parameters can be estimated when the given multi-component PPS consists of a cubic phase signal and a quadratic phase signal (a chirp) corrupted by a zero-mean white complex Gaussian noise $v(t)$,

$$x(t) = y(t) + v(t) = e^{j(a_{11}t + a_{12}t^2 + a_{13}t^3)} + e^{j(a_{21}t + a_{22}t^2)} + v(t).$$

As in Example 3, we first compute $\mathcal{P}_3[x(t);1] = x(t)[x^*(t-1)]^2 x(t-2)$, the FS coefficient function of which should produce one peak at $3!a_{13}$. This allows us to obtain estimate $\hat{a}_{13}$ of a_{13}. In our next step, we form two PPS $x_{11}(t)$ and $x_{12}(t)$ by

$$x_{11}(t) = x(t)e^{-j\hat{a}_{13}t^3}, \quad \text{and} \quad x_{12}(t) = x(t).$$

Now, we can just repeat Step 2 and Step 3 in Example 3 to find the remaining parameters. ∎

5 Conclusions

Multi-component AM-FM models describe a large class of nonstationary processes, among which multi-component polynomial phase signals (PPS) form a particularly important subclass. The so-called high-order ambiguity function (HAF) was originally introduced by Peleg and Porat to estimate the parameters of single-component PPS, but has not been widely used for multi-component problems due to the appearance of many cross-terms. In this paper, we have carefully examined the magnitudes of the cross-terms and shown that they are almost always negligible in comparison with the peaks due to the original signal components. Thus HAF can be regarded as virtually additive and be applied to multi-component PPS.

Our simulations show that cross-terms rarely cause false peaks in the HAF domain. Problems may arise when the components share the same (highest order) polynomial phase coefficients or when the dynamic range of the component amplitudes is large. We have used examples to illustrate the use of HAF to estimate multi-component PPS parameters.

Acknowledgment. We thank Prof. L.D. Pitt of the University of Virginia (UVA) for introducing the second author to relevant results on Gauss sums. The work presented in this paper is an extension of the second author's joint work with Prof. G.B. Giannakis of UVA and Dr. A. Swami of the US Army Research Laboratories.

Appendix. Proofs of the Theorems

We prove our main theorems by using several well-known estimates of the Weyl sum $\sum_{t=0}^{T-1} e^{jf(t)}$ where $f(t)$ is a polynomial. Weyl sums have been used in analytic number theory to study, among many other problems, Goldbach's Conjecture and Waring's Problem. Several deep results, which we shall use in this paper, were obtained by Weyl, Vinogradov, Hua, Vaughan, and others, see [4], [15], [16].

To simplify the exposition, we shall use $e(t)$ to denote $e^{j2\pi t}$ throughout the appendix.

Lemma 1 *Let $f(x) = \sum_{m=1}^{M} b_m x^m$ with all $b_m \in \mathbf{Z}$. Let $D > 0$ and $T = Dn$. Then*

$$\sum_{t=0}^{T-1} e(\tfrac{1}{D} f(t) - \alpha t) = \sum_{t=0}^{D-1} e(\tfrac{1}{D} f(t) - \alpha t) \cdot \sum_{k=0}^{n-1} e(-\alpha D k).$$

Proof. Note that $\frac{1}{D} f(t+D) \equiv \frac{1}{D} f(t) \pmod 1$ for all $t \in \mathbf{Z}$. Therefore,

$$\begin{aligned}\sum_{t=0}^{T-1} e(\tfrac{1}{D} f(t) - \alpha t) &= \sum_{k=0}^{n-1} \sum_{t=0}^{D-1} e(\tfrac{1}{D} f(Dk+t) - \alpha(Dk+t)) \\ &= \sum_{k=0}^{n-1} \sum_{t=0}^{D-1} e(\tfrac{1}{D} f(t) - \alpha t) e(-\alpha D k) \\ &= \sum_{t=0}^{D-1} e(\tfrac{1}{D} f(t) - \alpha t) \cdot \sum_{k=0}^{n-1} e(-\alpha D k),\end{aligned}$$

proving the lemma. ∎

Lemma 2 *Let $f(x)$ be as in Lemma 1 and $D > 0$. Then*

$$\lim_{T\to\infty} \max_{\alpha} \left| \frac{1}{T} \sum_{t=0}^{T-1} e(\tfrac{1}{D} f(t) - \alpha t) \right| = \max_{c \in \mathbf{Z}} \left| \frac{1}{D} \sum_{t=0}^{D-1} e(\tfrac{1}{D} f(t) - \tfrac{c}{D} t) \right|. \tag{A.1}$$

Proof. For any $T > 0$, write $T = Dn + r_0$ where $0 \le r_0 < D$. Then by Lemma 1,

$$\sum_{t=0}^{T-1} e(\tfrac{1}{D} f(t) - \alpha t) = \sum_{t=0}^{D-1} e(\tfrac{1}{D} f(t) - \alpha t) \cdot \sum_{k=0}^{n-1} e(-\alpha D k) + \sum_{t=Dn}^{T-1} e(\tfrac{1}{D} f(t) - \alpha t).$$

Now the last term is bounded by $r_0 < D$. If $\alpha D \notin \mathbf{Z}$ then $|\sum_{k=0}^{n-1} e(-\alpha D k)| \leq 2/|1 - e(-\alpha D)|$, which is independent of T. Therefore

$$\lim_{T\to\infty} \frac{1}{T} \left| \sum_{t=0}^{T-1} e(\tfrac{1}{D} f(t) - \alpha t) \right| = 0.$$

Therefore the maximum of $\left| \frac{1}{T} \sum_{t=0}^{T-1} e(\frac{1}{D} f(t) - \alpha t) \right|$ can only be attained when $\alpha D \in \mathbf{Z}$. But if $\alpha = c/D$ for some $c \in \mathbf{Z}$ then $\sum_{k=0}^{n-1} e(-\alpha D k) = n$. Hence

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} e(\tfrac{1}{D} f(t) - \alpha t) = \frac{1}{D} \sum_{t=0}^{D-1} e(\tfrac{1}{D} f(t) - \tfrac{c}{D} t),$$

and the lemma follows. ∎

A.1 Proof of Theorem 1

We first restate Theorem 1 in the following slightly different (but stronger) form.

Theorem 1 *Let $D > 0$ and N be co-prime integers. Then*

$$\lim_{T\to\infty} \max_{\alpha} \left| \frac{1}{T} \sum_{t=0}^{T-1} e(\tfrac{N}{D} t^2 - \alpha t) \right| = \begin{cases} \sqrt{2/D}, & \textit{if } D \textit{ is even}, \\ \sqrt{1/D}, & \textit{if } D \textit{ is odd}. \end{cases} \tag{A.2}$$

Theorem 1 is proved by applying the following result on the so-called *Gauss sum* $\sum_{t=0}^{D-1} e(\frac{N}{D} t^2)$.

Lemma 3 *Let N, D be co-prime integers, $D > 0$. Then*

$$\left| \sum_{t=0}^{D-1} e(\tfrac{N}{D} t^2) \right| = \begin{cases} \sqrt{D}, & \textit{if } D \textit{ is odd}, \\ \sqrt{2D}, & \textit{if } D \textit{ is a multiple of } 4, \\ 0, & \textit{otherwise}. \end{cases} \tag{A.3}$$

Proof. Let $D = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of D, where $p_1 < p_2 < \cdots < p_k$ are primes. Denote $S(D, N) = \sum_{t=0}^{D-1} e(\frac{N}{D} t^2)$. Then there exist integers

N_j, $1 \le j \le k$, such that

$$S(D, N) = \prod_{j=1}^{k} S(p_j^{\alpha_j}, N_j)$$

with $\gcd(N_j, p_j) = 1$, cf. Vaughan [15], Lemma 2.10. Now, for any prime p and any integer b such that $\gcd(b, p) = 1$ we have $|S(p^\alpha, b)| = p^{\alpha/2}$, except that $S(2, b) = 0$ (see Vinogradov [16], Chapter II, Lemma 4 and Lemma 5). This proves the lemma. ∎

Proof of Theorem 1. By Lemma 3 we only need to evaluate the maximum of $|\sum_{t=0}^{D-1} e(\frac{N}{D}t^2 - \frac{c}{D}t)|$ for all $c \in \mathbf{Z}$. Note that because N and D are co-prime, there exists some $b \in \mathbf{Z}$ such that $c \equiv Nb \pmod{D}$. Therefore

$$\frac{1}{D}\left|\sum_{t=0}^{D-1} e(\tfrac{N}{D}t^2 - \tfrac{c}{D}t)\right| = \frac{1}{D}\left|\sum_{t=0}^{D-1} e(\tfrac{N}{D}(t^2 - bt))\right|.$$

Case 1. D is odd.

Without loss of generality, we assume that b is even, because we can replace b by $b + D$ if otherwise. Let $b = 2\tilde{b}$. Then completing square, we obtain

$$\left|\sum_{t=0}^{D-1} e(\tfrac{N}{D}(t^2 - bt))\right| = \left|\sum_{t=0}^{D-1} e(\tfrac{N}{D}(t - \tilde{b})^2)\right|. \tag{A.4}$$

As t runs through a complete residue system (mod D), in this case from $t = 0$ to $t = D - 1$, so does $t - \tilde{b}$. Hence the r.h.s. of (A.4) has

$$\left|\sum_{t=0}^{D-1} e(\tfrac{N}{D}(t - \tilde{b})^2)\right| = \left|\sum_{t=0}^{D-1} e(\tfrac{N}{D}t^2)\right| = \sqrt{D}$$

following Lemma 3, and proving the theorem in this case.

Case 2. D is even but not a multiple of 4.

If $b = 2\tilde{b}$, then as in (A.4), by completing square and applying Lemma 3, we have

$$\sum_{t=0}^{D-1} e(\tfrac{N}{D}(t^2 - bt)) = 0.$$

Suppose now b is odd. Let $D = 2D_1$. We break the sum $\sum_{t=0}^{D-1} e(\frac{N}{D}(t^2 - bt))$ into two sums, one for all even t and the other for all odd t. First, for even t's,

$$\sum_{t=2s} e(\tfrac{N}{D}(t^2 - bt)) = \sum_{s=0}^{D_1-1} e(\tfrac{N}{D_1}(2s^2 - bs)) = \sqrt{D_1}$$

following Case 1 because D_1 is odd. For the sum over odd t's, note that each odd t can be written as $\tilde{t} + D_1$ for some even $\tilde{t}$. Furthermore, one can check that $e(\frac{N}{D}(t^2 - bt)) = e(\frac{N}{D}(\tilde{t}^2 - b\tilde{t}))$. As t runs through all odd residue classes (mod D), $\tilde{t}$ runs through all even residue classes (mod D). Thus the sum of $e(\frac{N}{D}(t^2 - bt))$ over odd residue classes t (mod D) is identical to that over all even residue classes t (mod D); i.e.

$$\sum_{t=2s+1} e(\tfrac{N}{D}(t^2 - bt)) = \sum_{t=2s} e(\tfrac{N}{D}(t^2 - bt)) = \sqrt{D_1}.$$

Therefore,

$$\sum_{t=0}^{D-1} e(\tfrac{N}{D}(t^2 - bt)) = 2\sqrt{D_1} = \sqrt{2D},$$

proving the theorem in this case.

Case 3. D is a multiple of 4.

Again, if $b = 2\tilde{b}$ then as in (A.4), by completing square, we immediately have

$$\sum_{t=0}^{D-1} e(\tfrac{N}{D}(t^2 - bt)) = \sqrt{2D}.$$

Suppose that b is odd. We show that

$$\left| \sum_{t=0}^{D-1} e(\tfrac{N}{D}(t^2 - bt)) \right| = 0. \tag{A.5}$$

To see (A.5), let $D = 2D_1$. For each $0 \leq t < D_1$, denote $s = t + D_1$. One can check that

$$e(\tfrac{N}{D}(t^2 - bt)) = -e(\tfrac{N}{D}(s^2 - bs)).$$

Therefore, the l.h.s sum in (A.5) can be grouped into pairs that cancel each other. This immediately yields (A.5) and hence the theorem in this case.

∎

A.2 Proof of Theorem 2

We first restate Theorem 2:

Theorem 2 *Let* $c(t) = e^{j\sum_{m=2}^{M} a_m t^m}$ *and suppose that* $a_m = 2\pi b_m/D$, *where* $D, b_1, \ldots, b_m$ *are relatively prime integers,* $D > 0$. *Then*

$$\lim_{T\to\infty} \max_{\alpha} \left| \frac{1}{T} \sum_{t=0}^{T-1} c(t)\ e^{-j\alpha t} \right| \leq \min\left(1,\ c_D/D^{1/M}\right) \tag{A.6}$$

where $c_D = d_D^{\log_2 M}$ *and* d_D *denotes the number of divisors of* D.

Proof. Let $f(t) = \sum_{m=2}^{M} b_m t^m$. Then the l.h.s. of (A.6) is precisely

$$\lim_{T\to\infty} \max_{\alpha} \left| \frac{1}{T} \sum_{t=0}^{T-1} e(\tfrac{1}{D} f(t) - \alpha t) \right| = \max_{c\in\mathbf{Z}} \left| \frac{1}{D} \sum_{t=0}^{D-1} e(\tfrac{1}{D} f(t) - \tfrac{c}{D} t) \right|,$$

following Lemma 2. Now, write $g(t) = f(t) - ct$. A deep result of Hua [4] states that

$$\frac{1}{D} \left| \sum_{t=0}^{D-1} e(\tfrac{1}{D} g(t)) \right| \leq d_D^{\log_2 M} \cdot D^{-1/M} \tag{A.7}$$

where d_D is the number of divisors of D (cf. [4], Theorem 1 and its proof). This proves the theorem. ∎

A.3 Proof of Theorem 3

Theorem 3 *Let* $c(t) = e^{j\sum_{m=2}^{M} a_m t^m}$ *and suppose that at least one* a_m *is an irrational multiple of* π. *Then*

$$\lim_{T\to\infty} \max_{\alpha} \left| \frac{1}{T} \sum_{t=0}^{T-1} c(t)\ e^{-j\alpha t} \right| = 0.$$

The proof of Theorem 3 is the most technical of all three, and relies heavily on the difficult estimates of the Weyl sum $\sum_{t=0}^{T-1} e(f(t))$ where some of the coefficients of $f(t)$ are irrational. We shall apply estimates by Vinogradov [16] and Vaughan [15]. The general idea of the proof is to approximate irrationals with rationals by continued fractions. Once this is done, we may apply the theorem of Hua (see the proof of Theorem 2), and the fact that the denominators of these rationals tend to infinity.

First, let us recall that for any irrational $\alpha \in \mathbf{R}$, its continued fraction gives no worse than quadratic approximations of α by rational numbers. More precisely, let p/q and P/Q be two consecutive convergents of the continued fraction of α, $q < Q$. Then

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{qQ}. \tag{A.8}$$

As an example, one can check (A.8) for $\alpha = \sqrt{2}$; the first 6 convergents of its continued fraction are $1/1, 3/2, 7/5, 17/12, 41/29, 99/77$. For a reference on continued fractions, see Rosen [11].

Lemma 4 (Vinogradov) *Let $f(t) = \sum_{t=0}^{M} b_m t^m$ with all b_m real. Assume that b_r is irrational for some $2 \le r \le M$. Let $p/q,\ P/Q$ be two consecutive convergents of the continued fraction of b_r. Then there exist positive constants c, ρ depending only on M such that*

$$\left| \sum_{t=0}^{T-1} e(f(t)) \right| \le c \cdot T^{1-\rho}. \tag{A.9}$$

for all T satisfying $q \le T \le q^{2M}$ or $T \le Q \le T^{r-1/4}$.

Proof. See of Vinogradov [16], Chapter IV, Theorem I. ∎

Lemma 5 *Let $f(t) = \sum_{t=1}^{M} a_m t^m$ with all a_m real. Assume that $a_m = b_m/q + \beta_m$ for $2 \le m \le M$ where $q > 0$ and $\gcd(q, b_2, \ldots, b_M) = 1$. Let $g(t) = a_1 t + \frac{1}{q}\sum_{m=2}^{M} b_m t^m$.*

Then
$$\sum_{t=0}^{T-1} e(f(t)) = \frac{1}{q}\Big(\sum_{t=0}^{q-1} e(g(t))\Big) \cdot A + B, \tag{A.10}$$
where $|A| \le T$ *and* $|B| \le q(1 + |\beta_2|T^2 + \cdots + |\beta_M|T^M)$.

Proof. See Vaughan [15], Theorem 7.2 and Theorem 7.3. We remark that our lemma is stated slightly differently from Vaughan's Theorem 7.2 in that we allow the first coefficient of $g(t)$ to be irrational while in Vaughan's theorem a_1 is also approximated by a rational b_1/q. However, from the proof of Vaughan's theorem it is quite obvious that his result still holds in our case. ∎

Proof of Theorem 3. Let $f(t) = \sum_{m=2}^{M} b_m t^m \triangleq \frac{1}{2\pi}\sum_{m=2}^{M} a_m t^m$. Then
$$\max_{\alpha}\left|\frac{1}{T}\sum_{t=0}^{T-1} c(t)\; e^{-j\alpha t}\right| = \max_{\alpha}\left|\frac{1}{T}\sum_{t=0}^{T-1} e(f(t)-\alpha t)\right|.$$
b_m may be rational for some $2 \le m \le M$, and we let L_0 denote the least common denominator of these rational b_m's.

Let $T \ge L_0^{2M}$. For any irrational b_r, let p_r/q_r and P_r/Q_r be the consecutive convergents of the continued fraction of b_r such that $q_r \le T < Q_r$. By Lemma 4, if $q_r \le T \le q_r^{2M}$ or $T \le Q_r \le T^{r-1/4}$ for some r then
$$\left|\sum_{t=0}^{T-1} e(f(t)-\alpha t)\right| \le c \cdot T^{1-\rho}, \tag{A.11}$$
where $c > 0,\ \rho > 0$ depend only on M.

Now, suppose $q_r^{2M} < T$ and $Q_r > T^{r-1/4}$ for all r where b_r is irrational. Let $q > 0$ be the least common multiple of L_0 and the q_r's. Then for each irrational b_r,
$$b_r = \frac{p_r}{q_r} + \beta_r = \frac{k_r}{q} + \beta_r$$

with $|\beta_r| < 1/q_r Q_r$, while for each rational b_m, $b_m = k_m/q + \beta_m$ with $\beta_m = 0$. Let $g(t) = \frac{1}{q}\sum_{m=2}^{M} k_m t^m$. By Lemma 5,

$$\frac{1}{T}\left|\sum_{t=0}^{T-1} e(f(t) - \alpha t)\right| \leq \frac{1}{q}\left|\sum_{t=0}^{q-1} e(g(t) - \alpha t)\right| + \frac{|B|}{T}, \tag{A.12}$$

where $|B| \leq q(1 + \beta_2 T^2 + \cdots + \beta_M T^M)$. We show that $|B|/T$ is small. For each $\beta_r \neq 0$,

$$|\beta_r| T^r < \frac{T^r}{q_r Q_r} = \frac{T^{r-1/4} \cdot T^{1/4}}{q_r Q_r} \leq \frac{T^{1/4}}{q_r}.$$

Hence

$$\frac{|B|}{T} \leq \frac{q}{T}\Big(1 + |\beta_2| T^2 + \cdots + |\beta_M| T^M\Big) \leq \frac{Mq}{T^{3/4}}. \tag{A.13}$$

Notice that

$$q \leq L_0 \prod q_r < \left(T^{1/2M}\right)^M = T^{1/2}.$$

Therefore by (A.13),

$$\frac{|B|}{T} < \frac{M \cdot T^{1/2}}{T^{3/4}} = M \cdot T^{-1/4}. \tag{A.14}$$

Combining (A.12), (A.14) and Theorem 2, we obtain

$$\frac{1}{T}\left|\sum_{t=0}^{T-1} e(f(t) - \alpha t)\right| \leq (1 + d_q^{\log_2 M}) \cdot q^{-1/M} + M \cdot T^{-1/4}, \tag{A.15}$$

whenever T is sufficiently large. Since $q \to \infty$ as $T \to \infty$, (A.15) combines with (A.11) to give

$$\lim_{T\to\infty} \max_{\alpha} \left| \frac{1}{T} \sum_{t=0}^{T-1} e(f(t) - \alpha t) \right| = 0.$$

∎

References

[1] S. Barbarossa, A. Scaglione, and G. B. Giannakis, "Product multilag high-order ambiguity function for multicomponent polynomial phase signal modeling," *IEEE Trans. on Signal Processing,* 1998 (to appear).

[2] W. Chen, G. B. Giannakis, and N. Nandhakumar, "Spatio-temporal approach for time-varying image motion estimation," *IEEE Trans. on Image Processing*, vol. 10, October 1996, pp. 1448-1461.

[3] J. M. Francos and B. Friedlander, "Parametric estimation of multi-component signals with random amplitude and deterministic phase," *Proc. of 7th IEEE-SP Workshop on Statistical Signal and Array Processing*, Quebec City, Canada, June 1994, pp. 11-14.

[4] L.-K. Hua, "On exponential sums over an algebraic number field," *Canadian J. Math.* vol. 3, 1951, pp. 44-51.

[5] E. J. Kelly, "The radar measurement of range, velocity, and acceleration," *IRE Trans. on Military Electronics*, MIL-5, April 1961, pp. 51-57.

[6] S. Peleg and B. Porat, "The Cramer-Rao lower bound for signals with constant amplitude and polynomial phase," *IEEE Trans. on Signal Processing*, vol. 39, March 1991, pp. 749-752.

[7] S. Peleg and B. Friedlander, "A technique for estimating the parameters of multiple polynomial phase signals," *Proc. IEEE-SP Intl. Symp. on Time-Frequency and Time-Scale Signal Analysis,* Victoria, British Columbia, Canada, October 1992, pp. 119-122.

[8] M. Polad and B. Friedlander, *Proc. of Sixth IEEE Signal Processing Workshop on DSP*, Yosemite, October 1994, pp. 3-6.

[9] B. Porat, *Digital Processing of Random Signals, Theory & Methods*, Prentice Hall, Englewood Cliffs, NJ, 1994.

[10] A. W. Rihaczek, *Principles of High-Resolution Radar*, Peninsula Pub., CA, 1985.

[11] K. H. Rosen, *Elementary Number Theory and Its Applications*, Third Edition, Addison-Wesley, 1993.

[12] S. Shamsunder, G. B. Giannakis, and B. Friedlander, "Estimating random amplitude polynomial phase signals: a cyclostationary approach," *IEEE Trans. on Signal Processing*, vol. 43, February 1995, pp. 492-505.

[13] A. Swami, "Polyphase signals in additive and multiplicative noise: CRLB and HOS," *Proc. of Sixth IEEE Signal Processing Workshop on DSP*, Yosemite, October 1994, pp. 109-112.

[14] H. M. Teager and S. M. Teager, "Evidence of nonlinear sound production mechanisms in the vocal tract," in *Speech production and speech modeling*, NATO ASI, Ser. D, Bonas, France, July 1989; Kluwer Acad. 1990, vol. 55, pp. 241-261.

[15] R. C. Vaughan, *The Hardy-Littlewood Method*, Cambridge Univ. Press, 1981.

[16] I. M. Vinogradov, *The Method of Trigonometrical Sums in the Theory of Numbers,* Interscience Publishers (New York and London), 1954.

[17] G. Zhou, G. B. Giannakis, and A. Swami, "On polynomial phase signals with time-varying amplitudes," *IEEE Trans. on Signal Processing*, vol. 44, April 1996, pp. 848-861.

[18] G. Zhou and A. Swami, "Performance analysis for a class of amplitude modulated polynomial phase signals," *Proc. Intl. Conference on Acoustics, Speech, and Signal Processing*, Detroit, MI, May 1995, pp. 1593-1596.