LETTER TO THE EDITOR

SPARSE COMPLETE GABOR SYSTEMS ON A LATTICE

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Abstract. It is well known that if a Gabor system $\mathbf{G}(\Lambda, g)$ is complete and Λ is a lattice then $D(\Lambda) \geq 1$, where $D(\cdot)$ denotes the Beurling density. But what if Λ is a subset of a lattice but is not itself a lattice? We investigate this question here. We show that the upper Beurling density of Λ can be arbitrarily small, provided that the lattice containing Λ has density greater than 1. We conjecture that this cannot be done if the lattice has density exactly equal to 1.

1. Introduction

Let Λ be a discrete subset in $\mathbb{R}^d \times \mathbb{R}^d$, and let $g(x) \in L^2(\mathbb{R}^d)$. The *Gabor system* (also known as the *Weyl-Heisenberg system*) with respect to Λ and g is the following family of functions in $L^2(\mathbb{R}^d)$:

$$\mathbf{G}(\Lambda, g) := \left\{ g_{\lambda,p} := e^{2\pi i \lambda x} g(x-p) \,\Big|\, (\lambda, p) \in \Lambda \right\}. \tag{1}$$

Such a family was first introduced by Gabor [6] in 1946 for signal processing, and is still widely used today.

A well-known question concerning a Gabor system $\mathbf{G}(\Lambda, g)$ is how sparse the set Λ can be if the system is complete, see [13]. For any $\mathcal{J} \in \mathbb{R}^m$ define the *upper and lower Beurling density* $D^+(\mathcal{J})$ and $D^-(\mathcal{J})$ respectively by

$$\begin{aligned} D^+(\mathcal{J}) &= \limsup_{r\to\infty} \sup_{x\in\mathbb{R}^m} \frac{\#\mathcal{J} \cap (x+[0,r]^m)}{r^m}, \\ D^-(\mathcal{J}) &= \liminf_{r\to\infty} \inf_{x\in\mathbb{R}^m} \frac{\#\mathcal{J} \cap (x+[0,r]^m)}{r^m}. \end{aligned}$$

If $D^+(\mathcal{J}) = D^-(\mathcal{J})$ then $D(\mathcal{J}) = D^+(\mathcal{J}) = D^-(\mathcal{J})$ is the *Beurling density* of $\mathcal{J}$. In [13] Ramanathan and Steger made the following conjecture, which can actually be traced back much earlier:

Conjecture (Ramanathan and Steger [13]): *Suppose that a Gabor system* $\mathbf{G}(\Lambda, g)$ *is complete in* $L^2(\mathbb{R}^d)$ *then* $D^+(\Lambda) \geq 1$.

The above conjecture is shown to be true under the additional assumption that Λ is a lattice in $\mathbb{R}^d \times \mathbb{R}^d$, see Rieffel [14] and Ramanathan and Steger [13]. In the non-lattice

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setting the conjecture is true for Gabor frames, see Janssen [9]. However, the conjecture is false in general. Benedetto, Heil and Walnut [2] constructed a family of complete Gabor systems $\mathbf{G}(\Lambda, g)$ in which $D^+(\Lambda)$ can be made arbitrarily small. In their counter-examples Λ is not contained in a lattice.

The work of Benedetto, Heil and Walnut [2] raises the following question: Are there complete Gabor systems $\mathbf{G}(\Lambda, g)$ in which Λ is contained in a lattice $\mathcal{L}$ such that $D^+(\Lambda)$ can be made arbitrarily small? If so, how sparse can the lattice $\mathcal{L}$ be? Obviously $D^+(\mathcal{L}) \geq 1$ because the completeness of $\mathbf{G}(\Lambda, g)$ is preserved when Λ is replaced by $\mathcal{L}$. But can $D^+(\mathcal{L})$ be 1, or arbitrarily close to 1?

A related question concerns the density of a complete set of exponential functions. Landau [10] showed that there exists a small perturbation Γ of $\mathbb{Z}$ such that $\{e^{2\pi i\lambda x} : \lambda \in \Gamma\}$ is complete in $C(\Omega)$ where $\mu(\Omega)$ can be arbitrarily large. This is not possible if Γ is a lattice, in which case we must have $D^+(\Gamma) \geq \mu(\Omega)$. So again we may ask: For any $\varepsilon > 0$ does there exist a $\Gamma \subset \mathbb{Z}$ and a $\Omega \subset [0,1]$ such that $D^+(\Gamma) < \varepsilon$ and $\mu(\Omega) > 1 - \varepsilon$?

The objective of this note is to answer these questions. Part of the answers can be derived from the earlier work of Landau [10].

Theorem 1. *Let $\mathcal{L}$ be a lattice in $\mathbb{R} \times \mathbb{R}$ such that $D(\mathcal{L}) > 1$. For any $\varepsilon > 0$ there exists a $g \in L^2(\mathbb{R})$ and a subset $\Lambda \subset \mathcal{L}$ such that $D^+(\Lambda) < \varepsilon$, $D^-(\Lambda) = 0$ and $\mathbf{G}(\Lambda, g)$ is complete in $L^2(\mathbb{R})$.*

We remark that if the lattice $\mathcal{L}$ is *separable*, i.e. $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ where each $\mathcal{L}_i$ is a lattice in $\mathbb{R}$, then we may choose a compactly supported g.

It is not clear whether the result holds in dimension $d > 1$. But it does lead to:

Theorem 2. *For any $a > 1$ there exists a separable lattice $\mathcal{L}$ in $\mathbb{R}^d \times \mathbb{R}^d$ with $D(\mathcal{L}) = a$ with the following property: For any $\varepsilon > 0$ there exists a compactly supported $g \in L^2(\mathbb{R}^d)$ and a subset $\Lambda \subset \mathcal{L}$ such that $D^+(\Lambda) < \varepsilon$, $D^-(\Lambda) = 0$ and $\mathbf{G}(\Lambda, g)$ is complete in $L^2(\mathbb{R}^d)$.*

We also construct examples showing that for any $\varepsilon > 0$ there exist a $\Gamma \subset \mathbb{Z}$ and an $\Omega \subset [0,1]$ such that $D(\Gamma) < \varepsilon$, $\mu(\Omega) > 1 - \varepsilon$ and $\{e^{2\pi i\lambda x} : \lambda \in \Gamma\}$ is complete in $C(\Omega)$.

It should be pointed out that complete Gabor systems are less studied and not as well understood, comparing to Gabor bases or frames. We pose the following conjecture:

Conjecture:

(i) *Let $\mathcal{L}$ be a lattice in $\mathbb{R}^d \times \mathbb{R}^d$ with $D(\mathcal{L}) > 1$. Let $\Lambda \subseteq \mathcal{L}$ such that $\mathbf{G}(\Lambda, g)$ is complete in $L^2(\mathbb{R}^d)$ for some compactly supported $g \in L^2(\mathbb{R}^d)$. Then $D^+(\Lambda) > 0$.*

(ii) *Let $\mathcal{L}$ be a lattice in $\mathbb{R}^d \times \mathbb{R}^d$ with $D(\mathcal{L}) = 1$. Let $\Lambda \subseteq \mathcal{L}$ such that $\mathbf{G}(\Lambda, g)$ is complete in $L^2(\mathbb{R}^d)$ for some compactly supported $g \in L^2(\mathbb{R}^d)$. Then $D^+(\Lambda) = 1$.*

The conjecture is false when we replace $D^+(\Lambda)$ with $D^-(\Lambda)$. We give an example in the next section of a complete Gabor system $\mathbf{G}(\Lambda, g)$ in which $D^(\mathcal{L}) = 1$ and $D^-(\Lambda) = 0$.

Nevertheless we establish the following special case of the conjecture for symplectic lattices (details on symplectic lattices can be found in Gröchenig [7]).

Theorem 3. *Let $\mathcal{L}$ be a symplectic lattice in $\mathbb{R}^d \times \mathbb{R}^d$ with $D(\mathcal{L}) = 1$. Let $\Lambda \subseteq \mathcal{L}$ such that $\mathbf{G}(\Lambda, g)$ is complete for some $g \in L^2(\mathbb{R}^d)$. Then $\mathcal{L} \setminus \Lambda$ does not contain any (possibly translated) sublattice of $\mathcal{L}$.*

We thank Russ Lyons for bringing to our attention the question concerning the completeness and density of exponentials for subsets of $[0, 1]$. We are especially indebted to Alex Powell for the example in the end of the paper, and to the anonymous referee for very carefully reading the manuscript and pointing out a mistake in the earlier version of the paper. We also thank John Benedetto, Chris Heil and Gitta Kutyniok for very helpful discussions.

2. Proof of Results

We first construct, for any given $\varepsilon > 0$, a set $\Gamma \subset \mathbb{Z}$ and an $\Omega \subset [0, 1]$ such that $D(\Gamma) < \varepsilon$, $\mu(\Omega) > 1 - \varepsilon$ and $\{e^{2\pi i\lambda x} : \lambda \in \Gamma\}$ is complete in $C(\Omega)$. This construction is only slightly modified from the examples in [10].

Lemma 4 (Landau [10])**.** *We may partition $\mathbb{N}$ into infinitely may disjoint sequences $S_r = \{k_n^{(r)}\}$ (in increasing order), $r = 0, 1, 2, \ldots$, such that $\limsup_{n\to\infty} n/k_n^{(r)} = 1$ for each r.*

It follows from a well-known result that $\{e^{2\pi i\lambda x} : \lambda \in S_r\}$ is complete in $L^2([a, b])$ whenever $b - a < 1$, see [15]. Now for any $N, p \geq 1$ and $0 < \delta \leq \frac{1}{N}$ denote

$$\Gamma_N = \bigcup_{r=0}^{N-1} (S_r + \tfrac{r}{N}), \qquad \Omega_{N,p,\delta} = \bigcup_{k=0}^{N-1} ([\delta, 1 - \delta] + kp). \tag{2}$$

Lemma 5. *Suppose that $p, N \in \mathbb{Z}$ are relatively prime. Then the set of exponentials $\{e^{2\pi i\lambda x} : \lambda \in \Gamma_N\}$ is complete in $L^2(\Omega_{N,p,\delta})$.*

Proof. Again the basic idea is in [10]. We include a proof for self-containment. Assume that the lemma is false. Then there exists a nonzero $f \in L^2(\Omega_{N,p,\delta})$ such that f is orthogonal to all $e^{2\pi i\lambda x}$, $\lambda \in \Gamma_N$. Denote $I_0 = [\delta, 1-\delta]$. It is easy to check that for each $\lambda = \lambda' + \frac{r}{N} \in S_r + \frac{r}{N}$ we have

$$\begin{aligned} \int_{\Omega_{N,p,\delta}} f(x) e^{-2\pi i\lambda x}\, dx &= \sum_{k=0}^{N-1} \int_{I_0+kp} f(x) e^{-2\pi i(\lambda' + \frac{r}{N})x}\, dx \\ &= \sum_{k=0}^{N-1} e^{-2\pi i \frac{rkp}{N}} \int_{I_0} f(x + kp) e^{-2\pi i\lambda x}\, dx. \end{aligned}$$

Choose $\lambda_r \in S_r + \frac{r}{N}$ arbitrarily for $0 \leq r < N$. The orthogonality assumption now yields $\int_{\Omega_{N,p,\delta}} f(x) e^{-2\pi i\lambda_r x}\, dx = 0$. It follows that

$$\sum_{k=0}^{N-1} e^{-2\pi i \frac{rkp}{N}} \int_{I_0} f(x + kp) e^{-2\pi i\lambda_r x}\, dx = 0, \qquad r = 0, 1, \ldots, N - 1. \tag{3}$$

Observe that the matrix $[c_{rk}]$ with entries $c_{rk} = e^{-2\pi i \frac{rk}{N}}$ is a Vandermonde matrix. It is nonsingular because its rows are distinct. Therefore $\int_{I_0} f(x+kp)e^{2\pi i \lambda_r x}\,dx = 0$ for all r and k. Since $\lambda_r \in S_r + \frac{r}{N}$ is arbitrarily chosen, and $\{e^{2\pi i\lambda x} : \lambda \in S_r + \frac{r}{N}\}$ is complete in $L^2(I_0)$, we must have $f(x+kp) = 0$ for $x \in I_0$ and all $0 \le k < N$. Therefore $f \equiv 0$ on $\Omega_{N,p,\delta}$. This is a contradiction. ∎

To construct our example, for any $\varepsilon > 0$ let $N, p \ge 1$ and $\delta > 0$ such that N, p are relatively prime, $\frac{1}{N} < \varepsilon$ and $\delta < \frac{1}{2N}$. Set

$$\Omega = \frac{1}{N}\Omega_{N,p,\delta} \quad \text{and} \quad \Gamma = N\Gamma_N. \tag{4}$$

Then $\{e^{2\pi i\lambda x} : \lambda \in \Gamma\}$ is complete in $L^2(\Omega)$ by Lemma 5. Observe that Γ_N is a small perturbation of a subset of $\mathbb{N}$, so $D^+(\Gamma_N) \le 1$ and $D^-(\Gamma_N) = 0$. Hence $D^+(\Gamma) \le \frac{1}{N} < \varepsilon$ and $D^-(\Gamma) = 0$. Note that $\Gamma \subset \mathbb{N}$ and $\mu(\Omega) = 1 - 2\delta > 1 - \varepsilon$. By taking $p = 1$ we also have $\Omega \subset [0,1]$. This yields an example of $\Gamma \subset \mathbb{Z}$ and $\Omega \subset [0,1]$ such that $\{e^{2\pi i\lambda x} : \lambda \in \Gamma\}$ is complete in $L^2(\Omega)$, with $D^+(\Gamma) < \varepsilon$ and $D^-(\Gamma) = 0$.

We use a the construction in 4 to prove Theorem 1.

Lemma 6. *Let $\mathcal{T}$ be a uniformly discrete subset of $\mathbb{R}^d$ and $\Omega \subset \mathbb{R}^d$ be measurable such that $\bigcup_{p\in\mathcal{T}}(\Omega + p) = \mathbb{R}^d$. Let Γ be a discrete subset of $\mathbb{R}^d$ such that $\{e^{2\pi i\lambda x} : \lambda \in \Gamma\}$ is complete in $L^2(\Omega)$. Then for $g(x) = \chi_\Omega(x)$ and $\Lambda = \Gamma \times \mathcal{T}$ the Gabor system $\mathbf{G}(\Lambda, g)$ is complete in $L^2(\mathbb{R}^d)$.*

Proof. Let $\varphi(x) = \sum_{p\in\mathcal{T}} \chi_\Omega(x-p)$. Since $\bigcup_{p\in\mathcal{T}}(\Omega+p) = \mathbb{R}^d$ and $\mathcal{T}$ is uniformly discrete, we have $1 \le \varphi(x) \le K$ for some $K \ge 1$ for all x. For each $p \in \mathcal{T}$ set $h_p(x) = \chi_\Omega(x-p)/\varphi(x)$. Then h_p is supported on $\Omega + p$ and $h_p \in L^2(\Omega + p)$, with $\sum_{p\in\mathcal{T}} h_p(x) = 1$ for all x.

Now for any $f(x) \in L^2(\mathbb{R}^d)$ let $f_p(x) := f(x)h_p(x)$. Then f_p is supported on $\Omega + p$. Hence f_p is in the closure of the span of $\{e^{2\pi i\lambda x}g(x-p) : \lambda \in \Gamma\}$. However, $\sum_{p\in\mathcal{T}} f_p(x) = f(x)$. Therefore $f(x)$ is in the closure of the span of $\mathbf{G}(\Lambda, g)$. This proves the lemma. ∎

Proof of Theorem 1. We first establish the theorem for the lattice $\mathcal{L} = \mathbb{Z} \times a^{-1}\mathbb{Z}$ with $D(\mathcal{L}) = a > 1$. For any $\varepsilon > 0$ let Γ, Ω be as in (4) with N sufficiently large so that $\frac{1}{N} < \varepsilon/a$. We show that there exist relatively prime N, p and $\delta > 0$ sufficiently small such that $\bigcup_{q\in\mathcal{T}}(\Omega + q) = \mathbb{R}$ for $\mathcal{T} = a^{-1}\mathbb{Z}$. If this is the case then it follows from Lemma 6 that $\mathbf{G}(\Lambda, g)$ is complete in $L^2(\mathbb{R})$, where $g = \chi_\Omega$ and $\Lambda = \Gamma \times a^{-1}\mathbb{Z}$. Furthermore $\Lambda \subset \mathcal{L}$ has $D^+(\Lambda) = D^+(\Gamma)D^+(a^{-1}\mathbb{Z}) < \varepsilon$ and $D^-(\Lambda) = 0$. g is compactly supported. This would prove the theorem for $\mathcal{L} = \mathbb{Z} \times a^{-1}\mathbb{Z}$.

To see the existence of N, p and δ, We first choose $\delta > 0$ so that $b := a(1-2\delta) > 1$. Note that b is the length of the interval $[-a\delta, a - a\delta]$. Observe that

$$\bigcup_{q\in\mathcal{T}}(\Omega + q) = \frac{1}{N}[-\delta, 1-\delta] + \Big\{0, \frac{p}{N}, \dots, \frac{(N-1)p}{N}\Big\} + a^{-1}\mathbb{Z}.$$

Hence

$$aN\bigcup_{q\in\mathcal{T}}(\Omega+q)=[-a\delta,a-a\delta]+\big\{0,ap,\dots,(N-1)ap\big\}+N\mathbb{Z}.$$

By continued fraction approximation we may find integers A_n, B_n such that $a=\frac{A_n}{B_n}+\varepsilon_n$ with $|\varepsilon_n|<\frac{1}{B_n^2}$, and $B_n\to\infty$ as $n\to\infty$ (if $a\in\mathbb{Q}$ then can obviously be achieved with $\varepsilon_n=0$ for sufficiently large n. In this case do cannot make A_n and B_n coprime, but such is not required in the proof). Now set $p=B_n$ and choose N to be coprime to both B_n and A_n, with n sufficiently large so that $\frac{N}{B_n}<\frac{1}{2}(b-1)$. Then

$$aN\bigcup_{q\in\mathcal{T}}(\Omega+q)=[-a\delta,a-a\delta]+\Big\{kA_n+kB_n\varepsilon_n:\ 0\le k<N\Big\}+N\mathbb{Z}.$$

Note that $\big\{kA_n:\ 0\le k<N\big\}+N\mathbb{Z}=\mathbb{Z}$ as $\big\{kA_n:\ 0\le k<N\big\}$ is a complete residue system modulo N. Hence $\big\{kA_n+kB_n\varepsilon_n:\ 0\le k<N\big\}+N\mathbb{Z}$ is a small perturbation of $\mathbb{Z}$ with distance between any two adjacent points no more that $1+2(N-1)B_n\varepsilon_n<b$. Since the interval $[-a\delta,a-a\delta]$ have length b, it follows that $aN\bigcup_{q\in\mathcal{T}}(\Omega+q)=\mathbb{R}$. Hence $\bigcup_{q\in\mathcal{T}}(\Omega+q)=\mathbb{R}$.

The proof can be extended to all separable lattices $\mathcal{L}=b_1\mathbb{Z}\times b_2\mathbb{Z}$ with $D(\mathcal{L})=|b_1b_2|<1$. This is easily obtained with a rescaling of the Gabor system in the previous case for $a=|b_1b_2|^{-1}$, by letting $g(x)=\chi_{b_1\Omega}(x)$ and $\Lambda=(b_1\Gamma)\times(b_2\mathbb{Z})$. Then $\mathbf{G}(\Lambda,g)$ is complete in $L^2(\mathbb{R})$, proving the theorem for $\mathcal{L}=b_1\mathbb{Z}\times b_2\mathbb{Z}$.

Finally, let $\mathcal{L}$ be any lattice in $\mathbb{R}\times\mathbb{R}$ with $D(\mathcal{L})=a>1$. It is known that $\mathcal{L}$ is symplectic, and there is a unitary transformation taking the elements of $\mathbf{G}(b(\mathbb{Z}\times\mathbb{Z}),g)$ to the elements of a Gabor system $\mathbf{G}(\mathcal{L},\tilde{g})$, see Gröchenig [7], pp. 199-200. In particular the unitary transformation takes the complete Gabor system $\mathbf{G}(\Lambda,g)$ for the previous case to a new complete Gabor system $\mathbf{G}(\tilde{\Lambda},\tilde{g})$, with $\tilde{\Lambda}\subset\mathcal{L}$ and $D(\tilde{\Lambda})<\varepsilon$. This completes the proof of the theorem. ∎

Proof of Theorem 2. The proof follows easily from Theorem 1. For any $a>1$ and $\varepsilon>0$ let $\mathcal{L}_1=\mathbb{Z}^d$ and $\mathcal{T}=(a^{-1}\mathbb{Z})\times\mathbb{Z}^{d-1}$. Let Γ and Ω be as in (4). Set $\mathcal{L}=\mathcal{L}_1\times\mathcal{T}$, $\Lambda=(\Gamma\times\mathbb{Z}^{d-1})\times\mathcal{T}$ and $g(x)=\chi_{\Omega\times[0,1]^{d-1}}(x)$. Then $\mathbf{G}(\Lambda,g)$ is complete in $L^2(\mathbb{R}^d)$. Furthermore $\Lambda\subset\mathcal{L}$ and $D(\Lambda)<\varepsilon$. ∎

It is not clear whether the result of Theorem 1 holds in the higher dimension, since not all lattices are symplectic in higher dimensions.

To prove Theorem 3 we apply the Zak transform. For any $f\in L^2(\mathbb{R}^d)$ the Zak transform $Z[f]$ of f is

$$Z[f](x,y):=\sum_{\alpha\in\mathbb{Z}^d}f(x+\alpha)e(\alpha\cdot y),\qquad(x,y)\in Q_d\times Q_d \tag{5}$$

where $Q_d=[0,1]^d$, where $e(t):=e^{2\pi it}$ here and throughout the rest of the paper. It is well known that the Zak transform Z is a unitary operator from $L^2(\mathbb{R}^d)$ to $L^2(Q_d\times Q_d)$. Furthermore, for $g_{\lambda,p}(x):=e(\lambda\cdot x)g(x-p)$ we have $Z[g_{\lambda,p}](x,y)=e(\lambda\cdot x)e(p\cdot y)Z[g](x,y)$.

In this paper, we identify $L^2(Q_d \times Q_d)$ with $L^2(\mathbb{T}^d \times \mathbb{T}^d)$ in the obvious fashion. Hence Z is a unitary operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{T}^d \times \mathbb{T}^d)$.

Lemma 7. *Let $\Lambda \subseteq \mathbb{Z}^d \times \mathbb{Z}^d$ and $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\Lambda, g)$ is complete in $L^2(\mathbb{R}^d)$. Then for any nonzero $f(x,y) \in L^2(\mathbb{T}^d \times \mathbb{T}^d)$ we have*

$$\overline{Z[g](x,y)}f(x,y) \neq \sum_{(\lambda,p)\in\Lambda^c} c_{\lambda,p} e(\lambda\cdot x)e(p\cdot y) \tag{6}$$

for any $c_{\lambda,p}$ with $\sum_{(\lambda,p)\in\Lambda^c} |c_{\lambda,p}|^2 < \infty$, where $\Lambda^c := \mathbb{Z}^d \times \mathbb{Z}^d \setminus \Lambda$. The converse is also true if Λ^c is finite.

Proof. We prove that

$$\overline{Z[g](x,y)}f(x,y) \neq \sum_{(\lambda,p)\in\Lambda^c} c_{\lambda,p} e(\lambda\cdot x)e(p\cdot y)$$

for any nonzero $f \in L^2(\mathbb{T}^d \times \mathbb{T}^d)$ if $\mathbf{G}(\Lambda, g)$ is complete in $L^2(\mathbb{R}^d)$. Assume it were false. Then

$$\overline{Z[g](x,y)}f(x,y) = \sum_{(\lambda,p)\in\Lambda^c} c_{\lambda,p} e(\lambda\cdot x)e(p\cdot y)$$

for some nonzero $f \in L^2(\mathbb{T}^d \times \mathbb{T}^d)$. It follows that

$$\int_{\mathbb{T}^d\times\mathbb{T}^d} e(\lambda\cdot x)e(p\cdot y)Z[g](x,y)\overline{f(x,y)}\,dxdy = 0$$

for all $(\lambda,p) \in \Lambda$. Hence f is orthogonal to $Z[g_{\lambda,p}]$ for all $(\lambda,p) \in \Lambda$. This means $\{Z[g_{\lambda,p}] : (\lambda,p) \in \Lambda\}$ is incomplete in $L^2(\mathbb{T}^d \times \mathbb{T}^d)$. But the Zak transform is a unitary operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{T}^d \times \mathbb{T}^d)$. This implies $\mathbf{G}(\Lambda, g)$ is incomplete in $L^2(\mathbb{R}^d)$, a contradiction.

Conversely, let Λ^c be finite. Assume that $\mathbf{G}(\Lambda, g)$ is incomplete in $L^2(\mathbb{R}^d)$. Then there exists an $f(x,y) \in L^2(\mathbb{T}^d \times \mathbb{T}^d)$ such that $\int_{\mathbb{T}^d\times\mathbb{T}^d} Z[g_{\lambda,p}]\overline{f} = 0$ for all $(\lambda,p) \in \Lambda$. This means

$$\overline{Z[g](x,y)}f(x,y) = \sum_{(\lambda,p)\in\Lambda^c} c_{\lambda,p} e(\lambda\cdot x)e(p\cdot y)$$

for some $c_{\lambda,p}$, a contradiction. We remark that the assumption that Λ^c is finite is needed because $Z[g_{\lambda,p}]\overline{f}$ may not be in $L^2(\mathbb{T}^d \times \mathbb{T}^d)$. ∎

Proof of Theorem 3. We shall prove the theorem by contradiction. Assume that $\mathcal{L} \setminus \Lambda$ does contain a (possibly translated) lattice. Since $\mathcal{L}$ is symplectic there exists a unitary operator on $L^2(\mathbb{R}^d)$ that maps $\mathbf{G}(\mathcal{L}, g)$ to $\mathbf{G}(\mathbb{Z}^d \times \mathbb{Z}^d, \tilde{g})$ for some $\tilde{g} \in L^2(\mathbb{R}^d)$. Furthermore $\mathbf{G}(\Lambda, g)$ is mapped to $\mathbf{G}(\tilde{\Lambda}, \tilde{g})$ with the property that $\mathbb{Z}^d \times \mathbb{Z}^d \setminus \tilde{\Lambda}$ contains a (possibly translated) lattice Γ and $\mathbf{G}(\tilde{\Lambda}, \tilde{g})$ is complete in $L^2(\mathbb{R}^d)$.

We show that there exists an $f(z) \in L^2(\mathbb{T}^d \times \mathbb{T}^d)$ such that

$$\overline{Z[g](z)}f(z) = \sum_{(\lambda,p)\in\Gamma} c_{\lambda,p} e((\lambda,p)\cdot z).$$

This would be a contradiction, following from Lemma 7.

Let us consider the case Γ is a lattice. Later we shall show that our proof is not affected by a translation of Γ. Denote by Ω a fundamental domain in $\mathbb{R}^{2d}$ of the dual lattice Γ^* of Γ, $\Gamma^* \supseteq \mathbb{Z}^d \times \mathbb{Z}^d$. Let $\gamma_1, \gamma_2, \cdots, \gamma_m$ be a complete set of coset representatives of the group $\Gamma^*/\mathbb{Z}^d \times \mathbb{Z}^d$ and set $\bar{\Omega}_j := \Omega + \gamma_j \pmod 1$. Then $\{\bar{\Omega}_j : 1 \leq j \leq m\}$ is a partition of $\mathbb{T}^d \times \mathbb{T}^d$. In addition, $\{e((\lambda, p) \cdot z) : (\lambda, p) \in \Gamma\}$ is an orthogonal basis of $L^2(\bar{\Omega}_j)$ for each $1 \leq j \leq m$.

It follows from the completeness of $\mathbf{G}(\mathbb{Z}^d \times \mathbb{Z}^d, \tilde{g})$ in $L^2(\mathbb{R}^d)$ that $Z[\tilde{g}](z) \neq 0$ for almost all $z \in \mathbb{T}^d \times \mathbb{T}^d$, see [3]. Hence there exists an $\varepsilon_0 > 0$ such that the set $E = \{z \in \mathbb{T}^d \times \mathbb{T}^d : |Z[\tilde{g}](z)| \leq \varepsilon_0\}$ has measure $\mu(E) \leq \frac{1}{2m}$. Define $F \subseteq \Omega$ by

$$F = \{z \in \Omega : z + \gamma_j \pmod 1 \in E \text{ for some } 1 \leq j \leq m\}.$$

Clearly $\mu(F) \leq \frac{1}{2m}$, so $\Omega \setminus F$ has Lebesgue measure at least $\frac{1}{2m}$. Let $\bar{F}_j = F + \gamma_j \pmod 1$. Then each $\bar{F}_j \subseteq \bar{\Omega}_j$ and $\mu(\bar{\Omega}_j \setminus \bar{F}_j) \geq \frac{1}{2m}$, and $|Z[\tilde{g}](z)| > \varepsilon_0$ for all $z \in \bar{\Omega}_j \setminus \bar{F}_j$.

Now set $f(z) \in L^2(\mathbb{T}^d \times \mathbb{T}^d)$ as $f(z) = 0$ for $z \in \bigcup_{j=1}^m \bar{F}_j$, and $f(z) = (\overline{Z[\tilde{g}](z)})^{-1}$ otherwise. Then $\overline{Z[g](z)} f(z) = \chi_{\bar{\Omega}_j \setminus \bar{F}_j}(z)$ for $z \in \bar{\Omega}_j$. But notice that for any $(\lambda, p) \in \Gamma$ we have $e((\lambda, p) \cdot (z + \gamma_i)) = e((\lambda, p) \cdot (z + \gamma_j))$ for all $1 \leq i, j \leq m$ and $z \in \mathbb{T}^d \times \mathbb{T}^d$. Therefore

$$\chi_{\bar{\Omega}_j \setminus \bar{F}_j}(z) = \sum_{(\lambda,p) \in \Gamma} c_{\lambda,p} e((\lambda, p) \cdot z)$$

for some $(c_{\lambda,p}) \in l^2(\Gamma)$ that is uniform for all $1 \leq j \leq m$. This means

$$\overline{Z[g](z)} f(z) = \sum_{(\lambda,p) \in \Gamma} c_{\lambda,p} e((\lambda, p) \cdot z)$$

for $z \in \mathbb{T}^d \times \mathbb{T}^d$, which contradicts Lemma 7. Hence $\mathbb{Z}^d \times \mathbb{Z}^d \setminus \tilde{\Lambda}$ does not contain a lattice.

If the lattice Γ is translated, say $\Gamma = \Gamma_0 + \beta_0$, then the above proof goes through with Γ_0 in place of Γ. In the end, we simply observe that

$$\sum_{(\lambda,p) \in \Gamma_0} c_{\lambda,p} e((\lambda, p) \cdot z) = e(-\beta_0 \cdot z) \sum_{(\lambda,p) \in \Gamma} \tilde{c}_{\lambda,p} e((\lambda, p) \cdot z).$$

Hence

$$\overline{Z[g](z)} f(z) e(\beta_0 \cdot z) = \sum_{(\lambda,p) \in \Gamma} \tilde{c}_{\lambda,p} e((\lambda, p) \cdot z),$$

resulting in a contradiction. ∎

Example. Let $g(x) = e^{-\frac{1}{x^2}}$ for $x \in (0, 1)$ and $g(x) = 0$ elsewhere. We claim:

(i) For any finite subset of $\Lambda \subseteq \mathbb{Z}^2$ the Gabor system $\mathbf{G}(\mathbb{Z}^2 \setminus \Lambda, g)$ is complete in $L^2(\mathbb{R})$.

(ii) The Gabor system $\mathbf{G}(\mathbb{Z}^+ \times \mathbb{Z}, g)$ is complete in $L^2(\mathbb{R})$. Observe that $D^-(\mathbb{Z}^+ \times \mathbb{Z}) = 0$.

Proof. Note that $Z[g](x, y) = g(x)$ for $(x, y) \in [0, 1]^2$. For any finite $\Lambda \subseteq \mathbb{Z}^2$ and $f \in L^2([0, 1]^2)$ we have $\overline{Z[g](x, y)} f(x, y) = e^{-\frac{1}{x^2}} f(x, y) \neq \sum_{(\lambda,p) \in \Lambda} c_{\lambda,p} e((\lambda, p) \cdot (x, y))$ since at $(x, y) = (0, 0)$ the function has a zero of infinite order. Hence by Lemma 7 the Gabor system $\mathbf{G}(\mathbb{Z}^2 \setminus \Lambda, g)$ is complete in $L^2(\mathbb{R})$.

For the second claim, we note that $\log|g(x)|$ is not in $L^2([0,1])$. By a theorem of Kolmogorov the system $\{g(x)e(nx) : n \in \mathbb{Z}^+\}$ is complete in $L^2([0,1])$, see [1], page 213. This implies the Gabor system $\mathbf{G}(\mathbb{Z}^+ \times \mathbb{Z}, g)$ is complete in $L^2(\mathbb{R})$. ∎

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