UNIVERSAL SPECTRA, UNIVERSAL TILING SETS AND THE SPECTRAL SET CONJECTURE

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Abstract. A subset Ω of $\mathbb{R}^d$ with finite positive Lebesgue measure is called a *spectral set* if there exists a subset $\Lambda \subset \mathbb{R}$ such that $\mathcal{E}_\Lambda := \left\{ e^{i2\pi\langle\lambda,x\rangle} : \lambda \in \Lambda \right\}$ form an orthogonal basis of $L^2(\Omega)$. The set Λ is called a *spectrum* of the set Ω. The Spectral Set Conjecture states that Ω is a spectral set if and only if Ω tiles $\mathbb{R}^d$ by translation. In this paper we prove the Spectral Set Conjecture for a class of sets $\Omega \subset \mathbb{R}$. Specifically we show that a spectral set possessing a spectrum that is a strongly periodic set must tile $\mathbb{R}$ by translates of a strongly periodic set depending only on the spectrum, and vice versa.

1. Introduction

Let Ω be a (Lebesgue) measurable subset of $\mathbb{R}$ with finite positive measure. For $t \in \mathbb{R}$ let $\Omega + t := \{x + t : x \in \Omega\}$ denote the translate of Ω by t. We say that Ω *tiles* $\mathbb{R}$ by translation if there exists a subset $\mathcal{T} \subset \mathbb{R}$ so that $\mathbb{R} = \bigcup_{t\in\mathcal{T}} (\Omega + t) = \mathbb{R}$ and $(\Omega + t) \cap (\Omega + t')$ is a set of measure zero whenever $t, t' \in \mathcal{T}$ are distinct. In the affirmative case $\mathcal{T}$ is called a *tiling set* for Ω, and $(\Omega, \mathcal{T})$ is called a *tiling pair*. Similarly, we say that Ω tiles the non-negative half line $\mathbb{R}^+ = [0, \infty)$ if there exists a subset $\mathcal{T} \subset \mathbb{R}$ such that $\bigcup_{t\in\mathcal{T}} (\Omega + t) = \mathbb{R}^+$ and $(\Omega + t) \cap (\Omega + t')$ is a set of measure zero whenever $t, t' \in \mathcal{T}$ are distinct. Sets that tile the real line by translation have been studied recently, e.g., [Odl78], [LW97], [LW96].

For $\lambda \in \mathbb{R}$ we introduce the functions

$$e_\lambda(x) := e^{i2\pi\lambda x}, \quad x \in \mathbb{R}.$$

We say that Ω is a *spectral set* if there exists a subset $\Lambda \subset \mathbb{R}$ so that the functions $\mathcal{E}_\Lambda := \{e_\lambda : \lambda \in \Lambda\}$ form an orthogonal basis for $L^2(\Omega)$, the Hilbert space of complex valued square integrable functions on Ω with the inner product

$$\langle f, g\rangle := \int_\Omega \overline{f(x)} g(x)\, dx.$$

If the functions in $\mathcal{E}_\Lambda$ form an orthogonal basis for $L^2(\Omega)$, then we call (Ω, Λ) a *spectral pair* and Λ a *spectrum* for Ω. Spectral sets have recently been studied in various contexts, e.g., [Fug74], [JP92], [JP94], [Ped96], [LW97], [JP98].

One of the main open questions concerning spectral sets is the following Spectral Set Conjecture, first proposed by Fuglede [Fug74]:

Spectral Set Conjecture. *Let Ω be a measurable subset of $\mathbb{R}^d$ with finite positive Lebesgue measure. Then Ω is a spectral set if and only if Ω tiles $\mathbb{R}^d$ by translation.*

In this paper we study the one dimensional case of the Spectral Set Conjecture. A special class of sets we study consists of tiles that tile the non-negative half line $\mathbb{R}^+$ by translation. We prove:

Theorem 1.1. *Let Ω be a subset of $\mathbb{R}$ with finite positive Lebesgue measure. Suppose that Ω tiles $\mathbb{R}^+$ by translation. Then Ω tiles $\mathbb{R}$ by translation and is a spectral set.*

Let $\mathbb{N} := \{1, 2, 3, \dots\}$ be the set of natural numbers and $\mathbb{Z}^+ := \{0, 1, 2, \dots\}$ be the set of non-negative integers. For any $n \in \mathbb{N}$ let $\mathbb{Z}_n^+ := \{0, 1, \dots, n-1\}$. For any A, $B \subseteq \mathbb{Z}$ we write

$$A + B := \{a + b : a \in A, b \in B\}$$

for the Minkowski sum of A and B. We will write $A \oplus B$ if each element in $A + B$ has a *unique* decomposition of the form $a + b$ with $a \in A$ and $b \in B$.

Definition 1.2. *We call $A \subset \mathbb{Z}^+$ a* direct summand of $\mathbb{Z}_n^+$ *if there exists a $B \subset \mathbb{Z}^+$ such that $A \oplus B = \mathbb{Z}_n^+$. We call a subset $\mathcal{T}$ of $\mathbb{R}$ a* strongly periodic set *if there exist an $n \in \mathbb{N}$ and a direct summand $A \subset \mathbb{Z}^+$ of $\mathbb{Z}_n^+$ such that $\mathcal{T} = \alpha(A \oplus n\mathbb{Z})$ for some non-zero $\alpha \in \mathbb{R}$.*

In [LW97] it was shown that certain tiles that tile $\mathbb{R}$ by translation are spectral sets that possess the so-called *universal spectra*, in the sense that the spectra depend only on the tiling sets, not the tiles. Our main theorem below strengthens this notion by providing a large new class of tiles that possess universal spectra. It shows that a tile that tiles $\mathbb{R}$ by the translates of a strongly periodic set must have a universal spectrum that is also a strongly periodic set. More importantly, the theorem also gives rise to the notion of *universal tiling set*, which can be viewed as the dual of universal spectrum. We show that a spectral set that possesses a spectrum that is a strongly periodic set must have a univeral tiling set depending only on the spectrum.

Theorem 1.3. *Let Ω be a subset of $\mathbb{R}$ with finite positive measure. Suppose that there exists a strongly periodic set $\Lambda \subset \mathbb{R}$ such that (Ω, Λ) is a spectral pair. Then there exists a strongly periodic set $\mathcal{T} \subset \mathbb{R}$ depending only on Λ such that Ω tiles $\mathbb{R}$ by translates of $\mathcal{T}$. Conversely, suppose that there exists a strongly periodic set $\mathcal{T} \subset \mathbb{R}$ such that Ω tiles $\mathbb{R}$ by translates of $\mathcal{T}$. Then there exists a strongly periodic set $\Lambda \subset \mathbb{R}$ depending only on $\mathcal{T}$ such that (Ω, Λ) is a spectral pair.*

The strongly periodic sets Λ and $\mathcal{T}$ in Theorem 1.3 are *duals* of each other, and for each given one the other is constructed explicitly in §4. In fact we prove a stronger version of Theorem 1.3 there. For the rest of the paper, in §2 we state a result on the structure of strongly periodic sets, first shown in [deB56]. In §3 we classify tiles that tile $\mathbb{R}^+$ by translation. The classification is used to prove Theorem 1.1.

2. Structure of Strongly Periodic Sets

In this section we classify subsets A, B of $\mathbb{Z}^+$ satisfying $A \oplus B = \mathbb{Z}_n^+$ for some $n \in \mathbb{N}$. The classification is based on a theorem of de Bruijn [deB56] establishing the structure of subsets of $\mathbb{Z}^+$ that tile $\mathbb{Z}^+$ by translation. To formulate the result we first introduce some notation regarding divisibility. For $r, s \in \mathbb{Z}$ we use $r \mid s$ to mean that r divides s; for $r \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}$ we use $r \mid A$ to mean that r divides every $a \in A$.

Proposition 2.1 (de Bruijn). *Let $A, B \subseteq \mathbb{Z}^+$ such that $A \oplus B = \mathbb{Z}^+$ and $A \neq \mathbb{Z}^+$, $B \neq \mathbb{Z}^+$. Then there exists an integer $r > 1$ such that $r \mid A$ or $r \mid B$. Furthermore, if $r \mid B$ and $B = r\widetilde{B}$ then there exists an $\widetilde{A} \subseteq \mathbb{Z}^+$ such that*

$$A = \mathbb{Z}_r^+ \oplus r\widetilde{A}, \quad \text{and} \quad \widetilde{A} \oplus \widetilde{B} = \mathbb{Z}^+.$$

Proof. A proof can be found in de Bruijn [deB56]. For the sake of self-containment we give a short proof here.

Without loss of generality we assume $1 \in A$. Let r be the smallest non-zero member of B. For each $m \in \mathbb{N}$ let $A_m \subseteq A$ and $B_m \subseteq B$ be the minimal subsets so that

$$\mathbb{Z}_{mr}^+ \subseteq A_m + B_m.$$

It follows immediately from the minimality and the uniqueness in $A \oplus B$ that

$$A_m = A \cap \mathbb{Z}_{mr}^+, \qquad B_m = B \cap \mathbb{Z}_{mr}^+.$$

Observe that $\mathbb{Z}_{(m+1)r}^+ \setminus \mathbb{Z}_{mr}^+ = \mathbb{Z}_r^+ + mr$. So

$$A_{m+1} \setminus A_m \subseteq \mathbb{Z}_r^+ + mr, \qquad B_{m+1} \setminus B_m \subseteq \mathbb{Z}_r^+ + mr.$$

We show by induction on m that there are subsets C_m and D_m of $\mathbb{Z}^+$ such that

$$A_m = \mathbb{Z}_r^+ + rC_m, \qquad B_m = rD_m.$$

Let $C_1 := \{0\}$ and $D_1 := \{0\}$. Then $A_1 = \mathbb{Z}_r^+ + rC_1$ and $B_1 = rD_1$ as required. Suppose that C_m, $D_m \subseteq \mathbb{Z}^+$ have been constructed so that $A_m = \mathbb{Z}_r^+ + rC_m$ and $B_m = rD_m$. If $\mathbb{Z}_{(m+1)r}^+ \subseteq A_m + B_m$, then $A_{m+1} = A_m$ and $B_{m+1} = B_m$, and so it suffices to set $C_{m+1} := C_m$ and $D_{m+1} := D_m$ to complete the proof.

Now suppose that $\mathbb{Z}^+_{(m+1)r} \not\subseteq A_m + B_m$. Let $j \in \mathbb{Z}^+_r$. If $j + mr \in A_m + B_m = \mathbb{Z}^+_r + r(C_m + D_m)$ then $m \in C_m + D_m$ and therefore $\mathbb{Z}^+_r + mr \subseteq A_m + B_m$, contradicting $\mathbb{Z}^+_{(m+1)r} \not\subseteq A_m + B_m$. Hence,

$$(\mathbb{Z}^+_r + mr) \cap (A_m + B_m) = \emptyset.$$

It follows that $mr \in A_{m+1}$ or $mr \in B_{m+1}$.

If $mr \in B_{m+1}$, then $A_{m+1} = A_m$ and $B_{m+1} = B_m \cup \{rm\}$. Hence we may set $C_{m+1} := C_m$ and $D_{m+1} := D_m \cup \{m\}$.

Assume that $mr \in A_{m+1}$. Let $j \in \mathbb{Z}^+_r$. We have shown above that $j + mr \notin A_m + B_m$, so $j + mr = a + b$ for $a \in A_{m+1} \setminus A_m$, $b \in B_{m+1}$ or $a \in A_m$, $b \in B_{m+1} \setminus B_m$. If $b \in B_{m+1} \setminus B_m$ then $(m+1)r - b \in \mathbb{Z}^+_r$. Thus $mr + r = ((m+1)r - b) + b$ constitute two different decompositions of the same element in $A \oplus B$, a contradiction. This yields $a \in A_{m+1} \setminus A_m$. If $b \neq 0$ then $B_m = rD_m$ and $B_{m+1} \setminus B_m \subseteq \mathbb{Z}^+_r + mr$ implies that $b \geq r$. So $j + mr = a + b \geq mr + r > j + mr$, again a contradiction. So $b = 0$ and therefore $j + mr = a \in A_{m+1}$. It follows that

$$A_{m+1} = A_m \cup (\mathbb{Z}^+_r + mr).$$

The inductions steps are now complete by setting $C_{m+1} := C_m \cup \{m\}$ and $D_{m+1} := D_m$.

Finally, the proposition follows by letting $\widetilde{A} := \bigcup_{m=1}^\infty C_m$ and $\widetilde{B} = \bigcup_{m=1}^\infty D_m$. $\square$

Proposition 2.1 immediately leads to the following classification of strongly periodic sets.

Corollary 2.2. *Let A, $B \subseteq \mathbb{Z}^+$ such that $A \oplus B = \mathbb{Z}^+_n$ and $A \neq \mathbb{Z}^+_n$, $B \neq \mathbb{Z}^+_n$. Then there exists an $r > 1$ such that $r \mid n$ and either $r \mid A$ or $r \mid B$. Furthermore, if $r \mid B$ and $B = r\widetilde{B}$ then there exists an $\widetilde{A} \subset \mathbb{Z}^+$ so that*

$$A = \mathbb{Z}^+_r \oplus r\widetilde{A}, \quad \textit{and} \quad \widetilde{A} \oplus \widetilde{B} = \mathbb{Z}^+_{\frac{n}{r}}.$$

Proof. Suppose that $1 \in A$. Applying Proposition 2.1 to $A \oplus (B \oplus n\mathbb{Z}^+) = \mathbb{Z}^+$ yields an $r > 1$ and a set $\widetilde{A}$ so that $A = \mathbb{Z}^+_r \oplus r\widetilde{A}$ and $r \mid (B \oplus n\mathbb{Z}^+)$. Since $0 \in B$ and $0 \in \mathbb{Z}^+$ it follows that $r \mid n$ and $r \mid B$. Finally, $\mathbb{Z}^+_r \oplus r(\widetilde{A} + \widetilde{B}) = A \oplus B = \mathbb{Z}^+_n$ implies $\widetilde{A} \oplus \widetilde{B} = \mathbb{Z}^+_{\frac{n}{r}}$. $\square$

Corollary 2.3. *Let A, $B \subseteq \mathbb{Z}^+$ such that $A \oplus B = \mathbb{Z}^+_n$. Assume that $1 \in A$. Then there exists a unique finite sequence $d_0 = 1, d_1, \dots, d_{k-1}, d_k = n$ in $\mathbb{N}$ with $r_j := d_j/d_{j-1} \in \mathbb{N}$ and $r_j > 1$ for $1 \leq j \leq k$ such that*

$$A = d_0\mathbb{Z}^+_{r_1} \oplus d_2\mathbb{Z}^+_{r_3} \oplus \cdots, \tag{2.1}$$

$$B = d_1\mathbb{Z}^+_{r_2} \oplus d_3\mathbb{Z}^+_{r_4} \oplus \cdots. \tag{2.2}$$

Proof. Since $1 \in A$, the proof of Proposition 2.1 yields $A = \mathbb{Z}^+_{r_1} \oplus r_1\widetilde{A}$ and $B = r_1\widetilde{B}$ where $r_1 = \min\{b : b \in B, b \neq 0\}$, and $\widetilde{A} \oplus \widetilde{B} = \mathbb{Z}^+_{\frac{n}{r_1}}$. The proof is completed by applying Corollary 2.2 iteratively to $\widetilde{A} \oplus \widetilde{B} = \mathbb{Z}^+_{\frac{n}{r_1}}$. Note that the uniqueness follows from the fact that $r_1 = d_1/d_0 = \min\{b : b \in B, b \neq 0\}$, $r_2 = d_2/d_1 = \{a : a \in \widetilde{A}, a \neq 0\}$, etc. □

Corollary 2.4. *Suppose that $A, B \subseteq \mathbb{Z}^+$ such that $A \oplus B = \mathbb{Z}^+$, and that B is finite. Then B is a direct summand of $\mathbb{Z}^+_n$ for some $n \in \mathbb{N}$.*

Proof. By the same argument for Corollary 2.3 B must have the form (2.1) or (2.2), depending on whether $1 \in B$. So B must be a direct summand of $\mathbb{Z}^+_n$ for some $n \in \mathbb{N}$. □

Call a polynomial a $0-1$ *polynomial* if each of its coefficients is either 0 or 1. We associate each finite $A \subseteq \mathbb{Z}^+$ with the following $0-1$ polynomial

$$A(x) := \sum_{a \in A} x^a,$$

called the *characteristic polynomial* of A. Clearly every $0-1$ polynomial is the characteristic polynomial of the set of exponents corresponding to its non-zero coefficients. If A, B, $C \subseteq \mathbb{Z}^+$ are finite, then $A \oplus B = C$ if and only if $A(x)B(x) = C(x)$. We call a $0-1$ polynomial *c-irreducible* if $A(x) \neq A_1(x)A_2(x)$ for any $0-1$ polynomials $A_1(x) \not\equiv 1$, $A_2(x) \not\equiv 1$. The following result was first stated in [CM66] (simple examples, however, show that Lemma 1 in [CM66] is false).

Theorem 2.5. *Let $n > 1$. Then every factorization of $\frac{x^n-1}{x-1}$ into c-irreducible $0-1$ polynomials has the form*

$$\frac{x^n - 1}{x - 1} = F_{p_1}(x)F_{p_2}(x^{p_1})F_{p_3}(x^{p_1p_2}) \cdots F_{p_k}(x^{p_1p_2\cdots p_{k-1}}),$$

where $F_m(x) := \frac{x^m-1}{x-1}$, all p_j are primes (not necessarily distinct) and $n = p_1p_2\cdots p_k$.

Proof. This is a direct consequence of Corollary 2.3, by observing that

$$\mathbb{Z}^+_{p_1p_2\cdots p_k} = \mathbb{Z}^+_{p_1} \oplus p_1\mathbb{Z}^+_{p_2} \oplus p_1\cdots p_{k-1}\mathbb{Z}_{p_k}.$$

Note that each term in the factorization is c-irreducible, because it contains a prime number of terms.

□

3. Tiling the Non-Negative Real Line

Let $\Omega \subset \mathbb{R}$ be a tile with finite and positive Lebesgue measure that tiles $\mathbb{R}^+$ by translates of $\mathcal{T}$. In this case we will write $\Omega \oplus \mathcal{T} = \mathbb{R}^+$. In this section we derive the structure of tiles $\Omega \subset \mathbb{R}$ that tile $\mathbb{R}^+$ by translation.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}$ with finite positive Lebesgue measure. Suppose that Ω tiles $\mathbb{R}^+$ (and hence $\mathbb{R}$) by translation. Then there exists an affine map $\varphi(x) = ax + b$ such that*

$$\varphi(\Omega) = [0,1] + B$$

for some finite subset $B \subset \mathbb{Z}^+$ with $0 \in B$. Furthermore, B is a direct summand of $\mathbb{Z}_n^+$ for some $n \in \mathbb{N}$.

Proof. In this proof, all set relations involving the tile Ω will be interpreted as up to measure zero sets.

Let $\mathcal{T} \subset \mathbb{R}$ such that $\Omega \oplus \mathcal{T} = \mathbb{R}^+$. We first examine the special case $\mathcal{T} = \{0, 1, t_2, t_3, \dots\}$ where $t_j > 1$ for all $j \geq 2$. In this special case we prove that $\Omega = [0,1] + B$ for some $B \subset \mathbb{Z}^+$ and $0 \in B$. Let $\mathcal{T}_n = \mathcal{T} \cap [0, n-1]$ and $\Omega_n = \Omega \cap [0,n]$. We claim that $\mathcal{T}_n \subset \mathbb{Z}^+$ and $\Omega_n = [0,1] + B_n$ for some $B_n \subset \mathbb{Z}^+$, by induction on n.

Since $t_j > 1$, we must have $[0,1] \subseteq \Omega$. So the claim is clearly true for $n = 1$. Assume that the claim is true for all $n < k$. We show that the claim is also true for $n = k$. We divide the proof into two cases: $\Omega_{k-1} \subsetneq \Omega_k$ and $\Omega_{k-1} = \Omega_k$. Suppose that $\Omega_{k-1} \subsetneq \Omega_k$. Then $\Omega \cap (k-1, k] \neq \emptyset$. If $\Omega_k \neq [0,1] + B_k$ for any $B_k \subset \mathbb{Z}^+$, then $\Omega \cap (k-1,k] \subsetneq (k-1,k]$. Hence there exists a $t \in \mathcal{T}$ such that $(\Omega + t) \cap (k-1,k] \neq \emptyset$. Note that $t \in \mathcal{T}_{k-1}$, so $t \in \mathbb{Z}^+$. It follows that

$$\emptyset \subsetneq \Omega \cap (k-1-t, k-t] \subsetneq (k-1-t, k-t],$$

contradicting the inductive hypothesis. So $\Omega_k = [0,1] + B_k$ for some $B_k \subset \mathbb{Z}^+$. The assumption that $\Omega_{k-1} \subsetneq \Omega_k$ now implies that $B_k = B_{k-1} \cup \{k-1\}$, so $\mathcal{T}_k = \mathcal{T}_{k-1}$. This proves the claim for $n = k$ in the first case. Suppose that $\Omega_{k-1} = \Omega_k$. Then $\Omega_k = [0,1] + B_k$ with $B_k = B_{k-1}$. Therefore $\mathcal{T}_k = \mathcal{T}_{k-1} \cup \{k-1\}$. This completes the induction steps and proves the claim. So we have shown that $B, \mathcal{T} \subseteq \mathbb{Z}^+$, and clearly $0 \in B$.

It remains to show that B is a direct summand of $\mathbb{Z}_n^+$ for some $n \in \mathbb{N}$. Observe that $B \oplus \mathcal{T} = \mathbb{Z}^+$. Therefore B is a direct summand of $\mathbb{Z}_n^+$ for some $n \in \mathbb{N}$ by Corollary 2.4.

In general, suppose that Ω tiles $\mathbb{R}^+$ by translates of $\mathcal{T}$ where the elements in $\mathcal{T}$ are $t_0 < t_1 < t_2 < \cdots$. Let $\varphi(x) = \frac{1}{t_1 - t_0}(x - t_0)$ and $t'_j = \varphi(t_j)$. Then

$$\varphi(\Omega) \oplus \{0, 1, t'_2, t'_3, \dots\} = \mathbb{R}^+.$$

Hence $\varphi(\Omega) = [0,1] + B$ for some $B \subset \mathbb{Z}^+$ with $0 \in B$. $\square$

4. Proofs of Main Theorems

To prove our main theorems we first introduce some notation. For any finite set $A \subset \mathbb{Z}$ we denote $f_A(\xi) := A\left(e^{i2\pi\xi}\right)$ where $A(z)$ is the characteristic (Laurent) polynomial of A. We will use $\mathcal{Z}_A$ to denote the set of zeros of f_A. For a subset $\Omega \subset \mathbb{R}$ with positive and finite measure we will use $\mathcal{Z}_\Omega$ to denote the set of zeros of $\widehat{\chi}_\Omega(\xi)$.

Observe that for any finite $A \subset \mathbb{Z}$, $\xi \in \mathcal{Z}_A$ implies $\xi + m \in \mathcal{Z}_A$ for all $m \in \mathbb{Z}$. So $\mathcal{Z}_A = \mathbb{Z} \oplus X$ for some finite $X \subset \mathbb{R}$. If in addition A is a direct summand of $\mathbb{Z}_n^+$ for some $n \in \mathbb{N}$, then $n\mathcal{Z}_A \subseteq \mathbb{Z}$.

Lemma 4.1. *Let $A \subset \mathbb{Z}^+$ be a direct summand of $\mathbb{Z}_n^+$ for some $n \in \mathbb{N}$. Then there exists a direct summand A^* of $\mathbb{Z}_n^+$ with the same cardinality such that*

$$A - A \subseteq n\mathcal{Z}_{A^*} \cup \{0\}, \qquad A^* - A^* \subseteq n\mathcal{Z}_A \cup \{0\}. \tag{4.1}$$

Proof. We procced by induction on n. For $n = 1, 2$ it is easy to check that the lemma holds. Assume that the lemma holds for all $n < k$, where $k \geq 3$. We show that it holds for $n = k$.

Case 1. $1 \notin A$.

Then $A = rA_1$ for some $r > 1$, $r \mid k$ and direct summand A_1 of $\mathbb{Z}_{\frac{k}{r}}^+$. By the hypothesis there exists a direct summand A_1^* of $\mathbb{Z}_{\frac{k}{r}}^+$ such that (4.1) holds for A_1, A_1^* and $n = k/r$. Now $f_A(\xi) = f_{A_1}(r\xi)$ yields $\mathcal{Z}_A = \frac{1}{r}\mathcal{Z}_{A_1}$. Set $A^* = A_1^*$. Clearly A^* is a direct summand of $\mathbb{Z}_k^+$ because it is a direct summand of $\mathbb{Z}_{\frac{k}{r}}^+$, and we have

$$A - A = r(A_1 - A_1) \subseteq r \cdot \frac{k}{r}\mathcal{Z}_{A_1^*} \cup \{0\} = k\mathcal{Z}_{A^*} \cup \{0\},$$

and

$$A^* - A^* = A_1^* - A_1^* \subseteq \frac{k}{r}\mathcal{Z}_{A_1} \cup \{0\} = k\mathcal{Z}_A \cup \{0\}.$$

Case 2. $1 \in A$.

Then $A = \mathbb{Z}_r^+ \oplus rA_1$ for some $r > 1$, $r \mid k$ and direct summand A_1 of $\mathbb{Z}_{\frac{k}{r}}^+$. By the hypothesis there exists a direct summand A_1^* of $\mathbb{Z}_{\frac{k}{r}}^+$ such that (4.1) holds for A_1, A_1^* and $n = k/r$. Set $A^* = A_1^* \oplus \frac{k}{r}\mathbb{Z}_r^+$. A^* is a direct summand of $\mathbb{Z}_k^+$ because $A^* \oplus B_1^* = \mathbb{Z}_k^+$ where $A_1^* \oplus B_1^* = \mathbb{Z}_{\frac{k}{r}}$. We have

$$f_A(\xi) = f_{\mathbb{Z}_r^+}(\xi) f_{A_1}(r\xi), \qquad f_{A^*}(\xi) = f_{A_1^*}(\xi) f_{\mathbb{Z}_r^+}\left(\tfrac{k}{r}\xi\right).$$

It follows from $\mathcal{Z}_{\mathbb{Z}_r^+} = \frac{1}{r}\mathbb{Z} \setminus \mathbb{Z}$ that

$$\mathcal{Z}_A = \frac{1}{r}(\mathbb{Z} \cup \mathcal{Z}_{A_1}) \setminus \mathbb{Z}, \qquad \mathcal{Z}_{A^*} = \mathcal{Z}_{A_1^*} \cup \frac{r}{k}\left(\tfrac{1}{r}\mathbb{Z} \setminus \mathbb{Z}\right). \tag{4.2}$$

Let $m = a + \frac{k}{r}j$ and $m = a' + \frac{k}{r}j'$ be two distinct elements in A^*, where $a, a' \in A_1^*$ and $j, j' \in \mathbb{Z}_r^+$. If $a = a'$ then

$$m - m' = \frac{k}{r}(j - j') \in k\left(\tfrac{1}{r}\mathbb{Z} \setminus \mathbb{Z}\right) \subseteq k\mathcal{Z}_A.$$

If $a \neq a'$ then $a - a' \in \frac{k}{r}\mathcal{Z}_{A_1}$. Hence $a - a' + \frac{k}{r}l \in \frac{k}{r}\mathcal{Z}_{A_1}$ for all $l \in \mathbb{Z}$. Since $m - m' \notin k\mathbb{Z}$, we have

$$m - m' \in \tfrac{k}{r}\mathcal{Z}_{A_1} \setminus k\mathbb{Z} \subseteq k\mathcal{Z}_A.$$

Hence $A^* - A^* \subseteq k\mathcal{Z}_A \cup \{0\}$.

Now let $m = j + ra$, $m' = j' + ra'$ be two distinct elements in A, where $a, a' \in A_1$ and $j, j' \in \mathbb{Z}_r^+$. If $j = j'$ then $a \neq a'$, and by the hypothesis $a - a' \in \frac{k}{r}\mathcal{Z}_{A_1^*}$. So $m - m' = r(a - a') \in k\mathcal{Z}_{A_1^*}$. If $j \neq j'$ then $j - j' \notin r\mathbb{Z}$, so

$$m - m' = j - j' + r(a - a') \in \mathbb{Z} \setminus r\mathbb{Z} = \frac{r}{k}\left(\tfrac{1}{r}\mathbb{Z} \setminus \mathbb{Z}\right) \subseteq \mathcal{Z}_{A^*}.$$

Hence $A - A \subseteq \mathcal{Z}_{A^*}$.

We have now completed the induction steps and proven the lemma. $\square$

We will call two direct summand A and A^* satisfying (4.1) a *conjugate pair*, and A^* a *conjugate* of A. The proof of Lemma 4.1 leads to an explicit construction of conjugate pairs. Let $A \subset \mathbb{Z}^+$ be a direct summand of $\mathbb{Z}_n^+$. Then by Corollary 2.3 there exists a unique sequence $r_0, r_1, \ldots, r_{2k+1}$ in $\mathbb{N}$ with $\prod_{j=0}^{2k+1} r_j = n$, $r_j > 1$ for $0 < j < 2k+1$ and $r_0, r_{2k+1} \geq 1$, such that

$$A = \bigoplus_{j=0}^{k} d_{2j}\mathbb{Z}_{r_{2j+1}}^+, \qquad \text{where } d_m := \prod_{j=0}^{m} r_j. \tag{4.3}$$

Define the map ϑ_n on the set of direct summand of $\mathbb{Z}_n^+$ by

$$\vartheta_n(A) = \bigoplus_{j=0}^{k} \frac{n}{d_{2j+1}}\mathbb{Z}_{r_{2j+1}}^+. \tag{4.4}$$

Then $\vartheta_n(A)$ is exactly the conjugate set A^* constructed inductively in the proof of Lemma 4.1.

Lemma 4.2. *Suppose that $A \subset \mathbb{Z}^+$ is a direct summand of $\mathbb{Z}_n^+$. Then A and $\vartheta_n(A)$ form a conjugate pair, and $\vartheta_n(\vartheta_n(A)) = A$. Furthermore, if $A, B \subset \mathbb{Z}^+$ such that $A \oplus B = \mathbb{Z}_n^+$, then $\vartheta_n(A) \oplus \vartheta_n(B) = \mathbb{Z}_n^+$*

Proof. The proof of Lemma 4.1 already implies that A, $\vartheta_n(A)$ form an conjugate pair. It is easy to see that $\vartheta_n(\vartheta_n(A)) = A$ by directly applying (4.3) and (4.4). Now, suppose that A is given by (4.3) and $B \subset \mathbb{Z}^+$ satisfies $A \oplus B = \mathbb{Z}_n^+$. Then there are several cases: $r_0 = 1$ or $r_0 > 1$, and $r_{2k+1} = 1$ or $r_{2k+1} > 1$. If $r_0 = 1$, $r_{2k+1} > 1$ then

$$B = \bigoplus_{j=1}^{k+1} d_{2j-1}\mathbb{Z}^+_{r_{2j}}, \qquad \text{where} \quad r_{2k+2} := 1. \tag{4.5}$$

So

$$\vartheta_n(B) = \bigoplus_{j=1}^{k+1} \frac{n}{d_{2j}}\mathbb{Z}^+_{r_{2j}}. \tag{4.6}$$

It is now straightforward to check from (4.4) and (4.6) that $\vartheta_n(A) \oplus \vartheta_n(B) = \mathbb{Z}_n^+$. Other cases can be checked similarly. □

Definition 4.3. *Let $\Lambda, \mathcal{T} \subset \mathbb{R}$ be strongly periodic sets. We say that $\mathcal{T}$ is a* dual *of Λ if there exist a non-zero $\alpha \in \mathbb{R}$ and $A, B \subset \mathbb{Z}^+$ with $A \oplus B = \mathbb{Z}_n^+$ for some $n \in \mathbb{N}$ such that*

$$\Lambda = \alpha(A \oplus n\mathbb{Z}), \qquad \mathcal{T} = \frac{1}{n\alpha}\big(\vartheta_n(B) \oplus n\mathbb{Z}\big).$$

By Lemma 4.2 if $\mathcal{T}$ is a dual of Λ then Λ is a dual of $\mathcal{T}$.

Lemma 4.4. *Let $\Omega \subset \mathbb{R}$ satisfy $\mu(\Omega) = n \in \mathbb{N}$. Suppose that $\Lambda = L \oplus \mathbb{Z}$ where L is a finite subset of $\mathbb{R}$ such that $\Lambda - \Lambda \subseteq \mathcal{Z}_\Omega \cup \{0\}$. Then (Ω, Λ) is a spectral pair if and only if $|L| = n$.*

Proof. See [Ped96], Theorem 1, or [LW97], Theorem 2.1. □

We shall establish the following result, which is a stronger version of our main theorem.

Theorem 4.5. *Suppose that $\Omega \subset \mathbb{R}$ has positive and finite Lebesgue measure. Let $\Lambda, \mathcal{T} \subset \mathbb{R}$ be strongly periodic sets such that $\mathcal{T}$ is a dual of Λ. Then (Ω, Λ) is a spectral pair if and only if Ω tiles $\mathbb{R}$ by translates of $\mathcal{T}$.*

Proof. Without loss of generality we may assume that $\Lambda = \frac{1}{n}(A \oplus n\mathbb{Z})$ and $\mathcal{T} = \vartheta_n(B) \oplus n\mathbb{Z}$ for some $n \in \mathbb{N}$ and $A, B \subset \mathbb{Z}^+$ with $A \oplus B = \mathbb{Z}_n^+$.

($\Leftarrow$) The set $\Omega' = \Omega \oplus \vartheta_n(B)$ tiles $\mathbb{R}$ by translates of $n\mathbb{Z}$, so it is a fundamental domain of the lattice $n\mathbb{Z}$. Hence

$$\mathcal{Z}_{\Omega'} = \mathcal{Z}_\Omega \cup \mathcal{Z}_{\vartheta_n(B)} \supseteq \frac{1}{n}\mathbb{Z} \setminus \{0\}.$$

Since $\vartheta_n(A) \oplus \vartheta_n(B) = \mathbb{Z}_n^+$ we have

$$\mathcal{Z}_{\vartheta_n(A)} \cup \mathcal{Z}_{\vartheta_n(B)} = \mathcal{Z}_{\mathbb{Z}_n^+} = \frac{1}{n}\mathbb{Z} \setminus \mathbb{Z}.$$

Furthermore, $\mathcal{Z}_{\vartheta_n(A)} \cap \mathcal{Z}_{\vartheta_n(B)} = \emptyset$ because $f_{\vartheta_n(A)}(\xi) f_{\vartheta_n(B)}(\xi)$ has no multiple roots. Hence

$$\mathcal{Z}_\Omega \supseteq \mathcal{Z}_{\vartheta_n(A)} \cup \mathbb{Z} \setminus \{0\}.$$

Now, for any distinct $\lambda, \lambda' \in \Lambda$ we have $\lambda - \lambda' = \frac{1}{n}k + j$ for some $k \in A - A$, $j \in \mathbb{Z}$. If $k \neq 0$ then $\frac{k}{n} \in \mathcal{Z}_{\vartheta_n(A)}$ by (4.1), which implies that $\lambda - \lambda' = \frac{k}{n} + j \in \mathcal{Z}_{\vartheta_n(A)} \subseteq \mathcal{Z}_\Omega$. Otherwise $\lambda - \lambda' = j \in \mathbb{Z} \setminus \{0\} \subseteq \mathcal{Z}_\Omega$. By Lemma 4.4 (Ω, Λ) is a spectral pair.

$(\Rightarrow)$ Suppose that (Ω, Λ) is a spectral pair. For any $x \in [0,1)$ let $D_x := \Omega \cap (\mathbb{Z} + x)$. It follows from [Ped96], Theorem 2, that

$$|D_x| = |A|, \qquad D_x - D_x \subseteq n\mathcal{Z}_A \cup \{0\} \tag{4.7}$$

for almost all $x \in [0,1)$. We show that $(D_x - x) + \vartheta_n(B)$ is a complete residue system $\pmod n$ for every D_x satisfying (4.7). Note that $\vartheta_n(B) - \vartheta_n(B) \subseteq n\mathcal{Z}_B \cup \{0\}$, and observe that $k \not\equiv m \pmod n$ for any $k \in n\mathcal{Z}_A$ and $m \in n\mathcal{Z}_B$. Thus for any $k_1, k_2 \in D_x - x$ and $m_1, m_2 \in \vartheta_n(B)$ we must have $k_1 - k_2 \not\equiv m_2 - m_1 \pmod n$ unless $k_1 = k_2$ and $m_1 = m_2$. Hence $k_1 + m_1 \not\equiv k_2 + m_2 \pmod n$. Since $|D_x - x| \cdot |\vartheta_n(B)| = n$ it follows that $(D_x - x) + \vartheta_n(B) = (D_x - x) \oplus \vartheta_n(B)$ contains n distinct residue classes $\pmod n$, and hence is a complete residue system $\pmod n$. Therefore

$$D_x + \mathcal{T} = D_x \oplus \mathcal{T} = x + \mathbb{Z}$$

for almost all $x \in [0,1)$. This implies that Ω tiles $\mathbb{R}$ by translates of $\mathcal{T}$. $\square$

Theorem 1.1 is a simple consequence of Theorem 3.1 and Theorem 4.5.

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