# CONSTRUCTION OF COMPACTLY SUPPORTED SYMMETRIC SCALING FUNCTIONS 

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#### Abstract

In this paper we study scaling functions of a given regularity for arbitrary dilation factor $q$. We classify symmetric scaling functions and study the smoothness of some of them. We also introduce a new class of continuous symmetric scaling functions, the "Batman" functions, that have very small support. Their smoothness is established.


## 1. Introduction

Compactly supported wavelet functions are typically constructed from multiresolution analyses whose scaling functions are compactly supported, see [6] or [15]. It is an important problem to construct scaling functions (and hence wavelets) that possess desirable properties. These properties usually include high regularity, symmetry and small supports.

Recall that a multiresolution analysis with dilation factor $q$, where $q \in \mathbb{Z}$ and $|q|>1$, is a sequence of nested subspaces of $L^{2}(\mathbb{R})$

$$
\begin{equation*}
\cdots \subset V_{-2} \subset V_{-1} \subset V_{0} \subset V_{1} \subset V_{2} \subset \cdots \tag{1.1}
\end{equation*}
$$

such that

$$
\begin{equation*}
V_{j}=\operatorname{span}\left\{f\left(q^{j} x-k\right): k \in \mathbb{Z}\right\} \tag{1.2}
\end{equation*}
$$

for some $f(x) \in L^{2}(\mathbb{R})$, and

$$
\begin{equation*}
\overline{\bigcup_{j \in \mathbb{Z}} V_{j}}=L^{2}(\mathbb{R}) \tag{1.3}
\end{equation*}
$$

The function $f(x)$ is called the scaling function of the multiresolution analysis.
We shall mostly deal with multiresolution analyses whose scaling functions $f(x)$ are compactly supported and their integer translates are orthogonal in $L^{2}(\mathbb{R})$. Any

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such multiresolution analysis will allow us to construct an orthonormal wavelet basis for $L^{2}(\mathbb{R})$.

Let $f(x) \in L^{2}(\mathbb{R})$ be a compactly supported scaling function of a multiresolution analysis with dilation $q$. Then $\int_{\mathbb{R}} f(x) d x \neq 0$ and $f(x)$ satisfies a dilation equation (see [7])

$$
\begin{equation*}
f(x)=\sum_{k \in \mathbb{Z}} c_{k} f(q x-k), \quad \sum_{k \in \mathbb{Z}} c_{k}=q \tag{1.4}
\end{equation*}
$$

where $c_{k}$ are real and $c_{k} \neq 0$ for only finitely many $k \in \mathbb{Z}$. Suppose that $f(x-k)$, $k \in \mathbb{Z}$ are orthogonal in $L^{2}(\mathbb{R})$. Then the coefficients $\left\{c_{k}\right\}$ must satisfy

$$
\begin{equation*}
\sum_{k \in \mathbb{Z}} c_{k} c_{k+q j}=|q| \delta_{j}, \quad \forall j \in \mathbb{Z} \tag{1.5}
\end{equation*}
$$

where $\delta_{j}=1$ if $j=0$ and $\delta_{j}=0$ otherwise. The converse is not true though. For the integer translates of $f(x)$ to be orthogonal, there are additional conditions besides (1.5), see [11]. Often those conditions are overlooked in the study of wavelets.

Orthogonal scaling functions for dilation $q=2$ have been explicitly constructed by Daubechies [6]. For any given regularity, Daubechies has constructed the minimal support orthogonal scaling function and has studied the smoothness of those scaling functions using Fourier analytic methods. Construction of minimal support as well as non-minimal support orthogonal scaling functions for an arbitrary dilation $q$ has been presented by Heller [12]. Heller's constructions of minimal support orthogonal scaling functions are explicit, but not completely rigorous, see $\S 3$.

In applications, it is often desirable to use scaling functions that are symmetric. Construction of symmetric scaling functions for arbitrary dilation $|q|>2$ is the main concern of this paper. It is already shown by Daubechies [6] that, when $q=2$, the only symmetric orthogonal scaling function is the Haar function. However when $|q|>2$, symmetric orthogonal scaling functions with arbitrary regularity do exist. They were constructed for $q=3$ by Chui and Lian [2].

The contents of this paper are arranged as follows: In $\S 2$ we introduce the definitions and the basic results on scaling functions. We classify the scaling functions for a given dilation and regularity. In $\S 3$ we derive an explicit formula for all scaling sequences,
making rigorous the previous work by Heller. In $\S 4$ we give explicit construction of symmetric orthogonal scaling functions for any arbitray dilation $|q|>2$. We establish necessary and sufficient conditions for scaling functions to be symmetric, based on the modulus of symbols. And finally in $\S 5$ we discuss a new family of symmetric orthogonal scaling functions (the "Batman" functions) and find their smoothness by using the joint spectral radius of matrices.

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## 2. Basic Results

Let $\mathcal{S}_{q}(\mathbb{R})$ denote the set of all real sequences $\mathbf{c}=\left\{c_{k}: k \in \mathbb{Z}\right\}$ such that $\sum_{k \in \mathbb{Z}} c_{k}=$ $|q|$ and $c_{k}=0$ for all but finitely many $k \in \mathbb{Z}$, where $|q|>1$ is an integer. It is known that for each $\mathbf{c}=\left\{c_{k}\right\} \in \mathcal{S}_{q}(\mathbb{R})$ there exists a unique compactly supported $\Phi_{\mathbf{c}}(x)$ (in the sense of tempered distribution) satisfying

$$
\begin{equation*}
\Phi_{\mathbf{c}}(x)=\sum_{k \in \mathbb{Z}} c_{k} \Phi_{\mathbf{c}}(q x-k), \quad \text { for almost all } x \in \mathbb{R} \quad \text { and } \quad \hat{\Phi}_{\mathbf{c}}(0)=1 \tag{2.1}
\end{equation*}
$$

Moreover, any compactly supported solution $\phi(x)$ (in the sense of tempered distribution) of the dilation equation (2.1) must be a scalar multiple of $\Phi_{\mathbf{c}}(x)$, see Daubechies and Lagarias [8]. We call $\Phi_{\mathbf{c}}(x)$ the refinement function corresponding to $\mathbf{c}$.

Definition 2.1. The symbol of $\mathbf{c}=\left\{c_{k}: k \in \mathbb{Z}\right\} \in \mathcal{S}_{q}(\mathbb{R})$ is the trigonometric polynomial $M_{\mathbf{c}}(\omega)=\frac{1}{q} \sum_{k \in \mathbb{Z}} c_{k} e^{i k \omega}$. A sequence $\mathbf{c}=\left\{c_{k}: k \in \mathbb{Z}\right\}$ in $\mathcal{S}_{q}(\mathbb{R})$ is a $q$-scaling sequence if

$$
\sum_{k \in \mathbb{Z}} c_{k} c_{k+q j}=\left\{\begin{array}{cc}
|q| & \text { if } j=0  \tag{2.2}\\
0 & \text { if } j \neq 0
\end{array}\right.
$$

The following proposition summarizes the basic properties of symbols and $q$-scaling sequences.

Proposition 2.1. (i) $\mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$ is a q-scaling sequence if and only if

$$
\begin{equation*}
\sum_{k=0}^{q-1}\left|M_{\mathbf{c}}\left(\omega+\frac{2 \pi k}{q}\right)\right|^{2}=1, \quad \text { all } \omega \in \mathbb{R} \tag{2.3}
\end{equation*}
$$

(ii) Suppose that $\mathbf{c}=\left\{c_{k}: k \in \mathbb{Z}\right\}$ is a $q$-scaling sequence. Then $\sum_{j \in \mathbb{Z}} c_{k+q j}=1$ for all $k \in \mathbb{Z}$.

Proof. (i) is well-known, and a proof can be found in Gröchenig [11] (ii) is proved in Chui and Lian [2].

We define two elementary transformations on $\mathcal{S}_{q}(\mathbb{R})$, the translation $\tau_{m}$ for a given $m \in \mathbb{Z}$ and the reflection $\gamma$. They are defined as

$$
\tau_{m}\left(\left\{c_{k}\right\}\right)=\left\{c_{k-m}\right\}, \quad \text { and } \quad \gamma\left(\left\{c_{k}\right\}\right)=\left\{c_{-k}\right\}
$$

The refinement functions respectively satisfy

$$
\begin{equation*}
\Phi_{\gamma(\mathbf{c})}(x)=\Phi_{\mathbf{c}}(-x), \quad \Phi_{\tau_{m}(\mathbf{c})}(x)=\Phi_{\mathbf{c}}\left(x+\frac{m}{q-1}\right), \quad m \in \mathbb{Z} \tag{2.4}
\end{equation*}
$$

We also define the convolution of $\mathbf{b}=\left\{b_{k}\right\}$ and $\mathbf{c}=\left\{c_{k}\right\}$ of $\mathcal{S}_{q}(\mathbb{R})$ by $\mathbf{b} * \mathbf{c}:=$ $\left\{\frac{1}{q} \sum_{i} b_{i} c_{k-i}: k \in \mathbb{Z}\right\}$. Please note the extra factor $\frac{1}{q}$. Now it is easy to check that $\mathbf{b} * \mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$ and $M_{\mathbf{b} * \mathbf{c}}(\omega)=M_{\mathbf{b}}(\omega) M_{\mathbf{c}}(\omega)$.

Definition 2.2. We say that $\mathbf{b}, \mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$ are equivalent, and denote it by $\mathbf{b} \sim \mathbf{c}$, if $\mathbf{c}=\tau_{m}(\mathbf{b})$ or $\mathbf{c}=\tau_{m} \circ \gamma(\mathbf{b})$ for some $m \in \mathbb{Z}$.

Theorem 2.2. Let $\mathbf{b}, \mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$. Then $\left|M_{\mathbf{b}}(\omega)\right|^{2}=\left|M_{\mathbf{c}}(\omega)\right|^{2}$ if and only if there exist $\mathbf{a}, \mathbf{e}, \mathbf{e}^{\prime} \in \mathcal{S}_{q}(\mathbb{R})$ where $\mathbf{e} \sim \mathbf{e}^{\prime}$ such that

$$
\begin{equation*}
\mathbf{b}=\mathbf{a} * \mathbf{e}, \quad \mathbf{c}=\mathbf{a} * \mathbf{e}^{\prime} \tag{2.5}
\end{equation*}
$$

Proof. Suppose (2.5) holds. Then $M_{\mathbf{e}^{\prime}}(\omega)=e^{i m \omega} M_{\mathbf{e}}(\omega)$ if $\mathbf{e}^{\prime}=\tau_{m}(\mathbf{e})$ and $M_{\mathbf{e}^{\prime}}(\omega)=$ $e^{i m \omega} M_{\mathbf{e}}(-\omega)$ if $\mathbf{e}^{\prime}=\tau_{m} \circ \gamma(\mathbf{e})$. In either case $\left|M_{\mathbf{b}}(\omega)\right|^{2}=\left|M_{\mathbf{c}}(\omega)\right|^{2}$.

Conversely, suppose that $\left|M_{\mathbf{b}}(\omega)\right|^{2}=\left|M_{\mathbf{c}}(\omega)\right|^{2}$. Without loss of generality we will assume that $b_{0} \neq 0$ and $b_{i}=0$ for all $i<0$, for if it isn't so, we'll consider an equvalent (shifted) sequence with this property. We will assume the same for c. We make the substitution $z=e^{i \omega}$, let $B(z)=M_{\mathbf{b}}(\omega)$ and $C(z)=M_{\mathbf{c}}(\omega)$, and define $\tilde{B}(z)=z^{m} B(1 / z)$, where $m=\operatorname{deg}(B)$. Now the assumption reads $B(z) \tilde{B}(z)=$ $C(z) \tilde{C}(z)$. Notice that $B(z)$ and $C(z)$ must have the same degree and hence the same number of zeros (counted with their multiplicity). Let $A(z)=\operatorname{gcd}(B(z), C(z))$. Then $B(z)=A(z) E(z)$ and $C(z)=A(z) E^{\prime}(z)$ for some $E(z), E^{\prime}(z) \in \mathbb{R}[z]$. By assumption, $A(z) E(z) \tilde{A}(z) \tilde{E}(z)=A(z) E^{\prime}(z) \tilde{A}(z) \tilde{E}^{\prime}(z)$ and since $\operatorname{gcd}\left(E(z), E^{\prime}(z)\right)=$ 1 we obtain that $E^{\prime}(z)=\tilde{E}(z)$. Now (2.5) follows immediately by letting $M_{\mathbf{a}}(\omega)=$ $A\left(e^{i \omega}\right), M_{\mathbf{e}}(\omega)=E\left(e^{i \omega}\right), M_{\mathbf{e}^{\prime}}(\omega)=E^{\prime}\left(e^{i \omega}\right)$, and observing that $E^{\prime}(z)=\tilde{E}(z)$ implies $\mathbf{e}^{\prime} \sim \mathbf{e}$.

Theorem 2.2 essentially implies that to classify $q$-scaling sequences we need only to classify the square of the modulus of their symbols.

The following theorem is due to Mallat [15], who proves it for $q=2$. However, his proof generalizes easily to all $|q|>1$.

Theorem 2.3. Let $\mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$ be a $q$-scaling sequence. Then $\Phi_{\mathbf{c}}(x) \in L^{2}(\mathbb{R})$.
Remark. We call $\Phi_{\mathbf{c}}(x)$ the scaling function corresponding to $\mathbf{c}$.

## 3. Scaling Sequences of Arbitrary Regularity

In this section we shall classify all $q$-scaling sequences for any given regularity $r \geq 1$. A sequence $\mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$ is $r$-regular, or having regularity $r$, if $M_{\mathbf{c}}(\omega)=H_{q}^{r}(\omega) P(\omega)$ for some trigonometic polynomial $P(\omega)$, where $H_{q}(\omega):=\frac{1}{|q|} \sum_{k=0}^{q-1} e^{i k \omega}$. $H_{q}(\omega)$ is the symbol of the Haar sequence $\mathbf{h}=\left\{h_{k}\right\}$ with $h_{k}=1$ for $0 \leq h<|q|$ and $h_{k}=0$ otherwise. Proposition 2.1 (ii) shows that every $q$-scaling sequence is at least 1 regular.

Let $\Omega(\mathbb{C})$ denote the space of all trigonometric polynomials of complex coefficients, $\Omega(\mathbb{C})=\mathbb{C}\left[e^{i \omega}, e^{-i \omega}\right]$. Define the transfer operator $\mathcal{C}_{q, r}: \Omega(\mathbb{C}) \longrightarrow \Omega(\mathbb{C})$ by

$$
\begin{equation*}
\mathcal{C}_{q, r}[f(\omega)]=\sum_{k=0}^{q-1}\left|H_{q}\left(\omega+\frac{2 \pi k}{q}\right)\right|^{2 r} f\left(\omega+\frac{2 \pi k}{q}\right) . \tag{3.1}
\end{equation*}
$$

By Proposition 2.1, $Q(\omega)=H_{q}^{r}(\omega) P(\omega)$ is the symbol of a $q$-scaling sequence if and only if

$$
\begin{equation*}
\mathcal{C}_{q, r}\left[|P(\omega)|^{2}\right]=1, \quad P(0)=1 \tag{3.2}
\end{equation*}
$$

Lemma 3.1. Let $g(\omega) \in \Omega(\mathbb{C})$. Then $\mathcal{C}_{q, r}[g(\omega)]=0$ if and only if

$$
\begin{equation*}
g(\omega)=(1-\cos \omega)^{r} \sum_{n \neq q k} c_{n} e^{i n \omega} \tag{3.3}
\end{equation*}
$$

where $c_{n}=0$ for all but finitely many $n$.
Proof. By $\mathcal{C}_{q, r}[g(\omega)]=0$,

$$
\begin{equation*}
\left|H_{q}(\omega)\right|^{2 r} g(\omega)=-\sum_{k=1}^{q-1}\left|H_{q}\left(\omega+\frac{2 \pi k}{q}\right)\right|^{2 r} g\left(\omega+\frac{2 \pi k}{q}\right) . \tag{3.4}
\end{equation*}
$$

Note that $\left|e^{i \omega}-e^{i a}\right|^{2}=2-2 \cos (\omega-a)$. So

$$
\begin{equation*}
\left|H_{q}(\omega)\right|^{2}=\prod_{j=1}^{q-1}\left|e^{i \omega}-e^{i \frac{2 \pi j}{q}}\right|^{2}=2^{q-1} \prod_{j=1}^{q-1}\left(1-\cos \left(\omega-\frac{2 \pi j}{q}\right)\right) . \tag{3.5}
\end{equation*}
$$

As a result, $(1-\cos \omega)^{r}$ is a factor of $\left|H_{q}\left(\omega+\frac{2 \pi k}{q}\right)\right|^{2 r}$ for all $1 \leq k \leq q-1$. So $(1-\cos \omega)^{r}$ must be a factor of $g(\omega)$,

$$
g(\omega)=(1-\cos \omega)^{r} g_{1}(\omega) .
$$

Write $g_{1}(\omega) \in \Omega(\mathbb{C})$ as $g_{1}(\omega)=\sum_{n \in \mathbb{Z}} c_{n} e^{i n \omega}$ where $c_{n}=0$ for all but finitely many $n$. Notice that

$$
\begin{equation*}
\left|H_{q}(\omega)\right|^{2}=\frac{1}{q^{2}}\left|\frac{e^{i q \omega}-1}{e^{i \omega}-1}\right|^{2}=\frac{1-\cos q \omega}{q^{2}(1-\cos \omega)} \tag{3.6}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\left|H_{q}(\omega)\right|^{2 r} g(\omega)=\frac{1}{q^{2 r}}(1-\cos q \omega)^{r} \sum_{n \in \mathbb{Z}} c_{n} e^{i n \omega} . \tag{3.7}
\end{equation*}
$$

Applying (3.7) to $\mathcal{C}_{q, r}[g(\omega)]=0$, using the fact that $\left(1-\cos q\left(\omega+\frac{2 \pi k}{q}\right)\right)^{r}=(1-\cos q \omega)^{r}$,

$$
\sum_{k=0}^{q-1} \frac{1}{q^{2 r}}(1-\cos q \omega)^{r} \sum_{n \in \mathbb{Z}} c_{n} e^{i n\left(\omega+\frac{2 \pi k}{q}\right)}=0 \quad \text { for all } \omega .
$$

As a result,

$$
\begin{equation*}
\sum_{k=0}^{q-1} \sum_{n \in \mathbb{Z}} c_{n} e^{i n\left(\omega+\frac{2 \pi k}{q}\right)}=0 \quad \text { for all } \omega \tag{3.8}
\end{equation*}
$$

The above is equivalent to $c_{n}=0$ for all $n \equiv 0(\bmod q)$, because

$$
\sum_{k=0}^{q-1} e^{i n\left(\omega+\frac{2 \pi k}{q}\right)}=\left\{\begin{array}{cl}
0 & n \neq q k  \tag{3.9}\\
q e^{i n \omega} & n=q k
\end{array}\right.
$$

This proves the lemma.

Lemma 3.2. For any $r \geq 1$ there exists a unique $g_{r}(\omega) \in \Omega(\mathbb{C})$ in the form of $g_{r}(\omega)=\sum_{n=0}^{r-1}\left(a_{n}+b_{n} \sin \omega\right)(1-\cos \omega)^{n}$ where $a_{n}, b_{n} \in \mathbb{C}$ such that $\mathcal{C}_{q, r}\left[g_{r}(\omega)\right]=1$.
Proof. First, we show that there exists a $g(\omega) \in \Omega(\mathbb{C})$ satisfying $\mathcal{C}_{q, r}[g(\omega)]=1$. For all $0 \leq k \leq q-1$ write $\left|H_{q}\left(\omega+\frac{2 \pi k}{q}\right)\right|^{2 r}=h_{k}\left(e^{i \omega}\right)$ with each $h_{k}(z) \in \mathbb{C}\left[z, \frac{1}{z}\right]$, where

$$
\mathbb{C}\left[z, \frac{1}{z}\right]:=\left\{z^{-m} f(z): f(z) \in \mathbb{C}[z], m \in \mathbb{N}\right\}
$$

Now each $h_{k}(z)=z^{-N} \tilde{h}_{k}(z)$ for some $\tilde{h}_{k}(z) \in \mathbb{C}[z]$, and we choose $\tilde{h}_{k}(z)$ so that $\tilde{h}_{k}(0) \neq 0$ for at least one of the $0 \leq k \leq q-1$. Then $\tilde{h}_{0}(z), \ldots, \tilde{h}_{q-1}(z)$ are relatively prime because they have no common root. Since $\mathbb{C}[z]$ is a principal ideal domain, there exist $f_{0}(z), \ldots, f_{q-1}(z) \in \mathbb{C}[z]$ such that

$$
\sum_{k=0}^{q-1} \tilde{h}_{k}(z) f_{k}(z)=z^{N}, \quad \text { or } \quad \sum_{k=0}^{q-1} h_{k}(z) f_{k}(z)=1
$$

Let $u_{k}(\omega)=f_{k}\left(e^{i \omega}\right)$. Then

$$
\begin{equation*}
\sum_{k=0}^{q-1}\left|H_{q}\left(\omega+\frac{2 \pi k}{q}\right)\right|^{2 r} u_{k}(\omega)=1 \quad \text { for all } \omega \tag{3.10}
\end{equation*}
$$

We now define $g(\omega)=\frac{1}{q} \sum_{k=0}^{q-1} u_{k}\left(\omega-\frac{2 \pi k}{q}\right)$. Then for all $\omega$,

$$
\begin{aligned}
\sum_{k=0}^{q-1} \mid H_{q}(\omega & \left.+\frac{2 \pi k}{q}\right)\left.\right|^{2 r} g\left(\omega+\frac{2 \pi k}{q}\right) \\
& =\frac{1}{q} \sum_{k=0}^{q-1}\left|H_{q}\left(\omega+\frac{2 \pi k}{q}\right)\right|^{2 r} \sum_{j=0}^{q-1} u_{j}\left(\omega+\frac{2 \pi k}{q}-\frac{2 \pi j}{q}\right) \\
& =\frac{1}{q} \sum_{l=0}^{q-1} \sum_{j=0}^{q-1}\left|H_{q}\left(\omega+\frac{2 \pi l}{q}+\frac{2 \pi j}{q}\right)\right|^{2 r} u_{j}\left(\omega+\frac{2 \pi l}{q}\right) \\
& =\frac{1}{q} \sum_{l=0}^{q-1} \sum_{j=0}^{q-1}\left|H_{q}\left(\eta_{l}+\frac{2 \pi j}{q}\right)\right|^{2 r} u_{j}\left(\eta_{l}\right) \\
& =\frac{1}{q} \sum_{l=0}^{q-1} 1=1
\end{aligned}
$$

where $\eta_{l}=\omega+\frac{2 \pi l}{q}$. This proves the existence of a $g(\omega) \in \Omega(\mathbb{C})$ such that $\mathcal{C}_{q, r}[g(\omega)]=1$.

Now, we write $g(\omega)$ as

$$
\begin{aligned}
g(\omega) & =\sum_{n=0}^{r-1}\left(a_{n}+b_{n} \sin \omega\right)(1-\cos \omega)^{n}+(1-\cos \omega)^{r} \sum_{n=-N}^{N} c_{n} e^{i n \omega} \\
& =: g_{r}(\omega)+(1-\cos \omega)^{r} \sum_{n=-N}^{N} c_{n} e^{i n \omega} .
\end{aligned}
$$

Then it follows from (3.6) and (3.9) that

$$
\begin{equation*}
1=\mathcal{C}_{q, r}[g(\omega)]=\mathcal{C}_{q, r}\left[g_{r}(\omega)\right]+(1-\cos q \omega)^{r} \sum_{j} c_{q j} e^{i q j \omega} \tag{3.11}
\end{equation*}
$$

But if $c_{q j} \neq 0$ for some $j \geq 0$ then (3.11) is not possible, for as a trigonometric polynomial the order of $(1-\cos q \omega)^{r} \sum_{j} c_{q j} e^{i q j \omega}$ is at least $q r$ while the order of $\sum_{k=0}^{q-1}\left|H_{q}\left(\omega+\frac{2 \pi k}{q}\right)\right|^{2 r} g_{r}\left(\omega+\frac{2 \pi k}{q}\right)$ is at most $q r-1$. Therefore $c_{q j}=d_{q j}=0$, and hence $\mathcal{C}_{q, r}\left[g_{r}(\omega)\right]=1$.

Finally, suppose there is another $\tilde{g}_{r}(\omega)=\sum_{n=0}^{r-1}\left(a_{n}^{\prime}+b_{n}^{\prime} \sin \omega\right)(1-\cos \omega)^{n}$ satisfying $\mathcal{C}_{q, r}\left[\tilde{g}_{r}(\omega)\right]=1$ and $\tilde{g}_{r}(\omega) \neq g_{r}(\omega)$. Then $\mathcal{C}_{q, r}\left[\tilde{g}_{r}(\omega)-g_{r}(\omega)\right]=0$. Contradicting Lemma 3.1.

We can actually write down the explicit expression for $g_{r}(\omega)$. First we note that

$$
\begin{equation*}
\left(1-\cos \left(\omega-\frac{2 \pi j}{q}\right)\right)\left(1-\cos \left(\omega-\frac{2 \pi(q-j)}{q}\right)\right)=\left(\cos \omega-\cos \frac{2 \pi j}{q}\right)^{2} . \tag{3.12}
\end{equation*}
$$

Hence by (3.5),

$$
\left|H_{q}(\omega)\right|^{2}= \begin{cases}2^{q-1} q^{-2} \prod_{j=1}^{q_{1}}\left(\cos \omega-\cos \frac{2 \pi j}{q}\right)^{2} & q=2 q_{1}+1,  \tag{3.13}\\ 2^{q-1} q^{-2}(1+\cos \omega) \prod_{j=1}^{q_{1}}\left(\cos \omega-\cos \frac{2 \pi j}{q}\right)^{2} & q=2 q_{1}+2\end{cases}
$$

Now again by (3.5),

$$
\left|H_{q}\left(\omega+\frac{2 \pi k}{q}\right)\right|^{2 r}=O\left((1-\cos \omega)^{r}\right)
$$

for $1 \leq k \leq q-1$. Therefore by (3.11) $\left|H_{q}(\omega)\right|^{2 r} g_{r}(\omega)=1+O\left((1-\cos \omega)^{r}\right)$, or equivalently

$$
\begin{equation*}
g_{r}(\omega)=\left|H_{q}(\omega)\right|^{-2 r}+O\left((1-\cos \omega)^{r}\right) \tag{3.14}
\end{equation*}
$$ $\sum_{n=0}^{\infty} p_{n}(1-\cos \omega)^{n}$, namely,

$$
\begin{equation*}
g_{r}(\omega)=\sum_{n=0}^{r-1} p_{n}(1-\cos \omega)^{n} . \tag{3.15}
\end{equation*}
$$

Remark. In Heller [12] the existence of $g_{r}(\omega)$ as a polynomial of $1-\cos \omega$ of degree less than $r$ is not proven. The explicit formula for $p_{n}$ is derived by assuming this is the case.

Theorem 3.3. Let $A(\omega) \in \Omega(\mathbb{C})$. Then $A(\omega)=\left|M_{\mathbf{c}}(\omega)\right|^{2}$ for some $q$-scaling sequence $\mathbf{c}$ of regularity at least $r$ if and only if $A(\omega) \geq 0$ for all $\omega \in \mathbb{R}$ and $A(\omega)=$ $\left|H_{q}(\omega)\right|^{2 r} g(\omega)$ for some

$$
\begin{equation*}
g(\omega)=g_{r}(\omega)+(1-\cos \omega)^{r} \sum_{n \neq q k} c_{n} \cos n \omega, \tag{3.16}
\end{equation*}
$$

where $g_{r}(\omega)=\sum_{n=0}^{r-1} p_{n}(1-\cos \omega)^{n}$ is give by

$$
\begin{equation*}
p_{n}=\frac{q^{2 r}}{2^{r(q-1)}} \sum_{k_{1}+\cdots+k_{q_{1}}=n} \prod_{j=1}^{q_{1}}\binom{k_{j}+2 r-1}{2 r-1}\left(1-\cos \frac{2 \pi k_{j}}{q}\right)^{-k_{j}-2 r} \tag{3.17}
\end{equation*}
$$

for $q=2 q_{1}+1$, and

$$
\begin{equation*}
p_{n}=\frac{q^{2 r}}{2^{r(q-1)}} \sum_{k_{0}+k_{1} \cdots+k_{q_{1}}=n}\binom{k_{0}+r-1}{r-1} \prod_{j=1}^{q_{1}}\binom{k_{j}+2 r-1}{2 r-1}\left(1-\cos \frac{2 \pi k_{j}}{q}\right)^{-k_{j}-2 r} \tag{3.18}
\end{equation*}
$$

for $q=2 q_{1}+2$.
Proof. We only need to show that $p_{n}$ in (3.16) are indeed given by (3.17) and (3.18). We use the fact that for any $m \geq 1$

$$
(1-\omega)^{-m}=\sum_{n=0}^{\infty}\binom{n+m-1}{m-1} \omega^{n} .
$$

Hence for $a \neq 1$,

$$
(\omega-a)^{-m}=(1-a)^{-m}\left(1-\frac{1-\omega}{1-a}\right)^{-m}=(1-a)^{-m} \sum_{n=0}^{\infty}\binom{n+m-1}{m-1}(1-a)^{-n}(1-\omega)^{n} .
$$

It follows that

$$
\begin{aligned}
(\cos \omega & \left.-\cos \frac{2 \pi j}{q}\right)^{-2 r} \\
& =\left(1-\cos \frac{2 \pi j}{q}\right)^{-2 r} \sum_{n=0}^{\infty}\binom{n+2 r-1}{2 r-1}\left(1-\cos \frac{2 \pi j}{q}\right)^{-n}(1-\cos \omega)^{n}
\end{aligned}
$$

Similarly,

$$
\begin{equation*}
(1+\cos \omega)^{-r}=2^{-r} \sum_{n=0}^{\infty}\binom{n+r-1}{r-1} 2^{-n}(1-\cos \omega)^{n} . \tag{3.19}
\end{equation*}
$$

Now for $q=2 q_{1}+1$, (3.13) gives

$$
\begin{aligned}
\left|H_{q}(\omega)\right|^{-2 r} & =\frac{q^{2 r}}{2^{r(q-1)}} \prod_{j=1}^{q_{1}}\left(\cos \omega-\cos \frac{2 \pi j}{q}\right)^{-2 r} \\
& =\frac{q^{2 r}}{2^{r(q-1)}} \prod_{j=1}^{q_{1}} \sum_{n=0}^{\infty}\binom{n+2 r-1}{2 r-1}\left(1-\cos \frac{2 \pi k_{j}}{q}\right)^{-n-2 r}(1-\cos \omega)^{n}
\end{aligned}
$$

which yields (3.17). For $q=2 q_{1}+2$, (3.18) is proved similarly, only this time there is an extra factor $(1+\cos \omega)^{r}$ and (3.19) needs to be used.

Finally suppose that $g(\omega) \geq 0$ satisfies (3.16). Then

$$
\sum_{k=0}^{q-1}\left|H_{q}\left(\omega+\frac{2 \pi k}{q}\right)\right|^{2 r} g\left(\omega+\frac{2 \pi k}{q}\right)=1 \quad \text { for all } \omega .
$$

Moreover, by the Riesz Lemma (see [7], pp. 172), there exists a trigonometric polynomial $B(\omega)=\sum_{n} c_{n} e^{i n \omega}$ such that $|B(\omega)|^{2}=\left|H_{q}(\omega)\right|^{2 r} g(\omega)$. Clearly, $B(0)=1$. So $B(\omega)=M_{\mathbf{c}}(\omega)$ for some $q$-scaling sequence $\mathbf{c}$ of regularity at least $r$.

We observe that $p_{n} \geq 0$ for all $n$ and $p_{0}=1$. Therefore $g_{r}(\omega)>0$ and hence there exists a $q$-scaling sequence $\mathbf{c}$ such that $\left|M_{\mathbf{c}}(\omega)\right|^{2}=\left|H_{q}(\omega)\right|^{2 r} g_{r}(\omega)$.

Corollary 3.4. Let $\mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$ such that

$$
\left|M_{\mathbf{c}}(\omega)\right|^{2}=\left|H_{q}(\omega)\right|^{2 r} g_{r}(\omega) .
$$

Then all integer translates $\Phi_{\mathbf{c}}(\omega-k), k \in \mathbb{Z}$, are orthogonal in $L^{2}(\mathbb{R})$.

Proof. Suppose the corollary is false. Then there exists a $\omega_{0} \in(0,2 \pi)$ with the property that $q^{N} \omega_{0} \equiv \omega_{0}(\bmod 2 \pi)$ for some $N \in \mathbb{N}$ such that $M_{\mathbf{c}}\left(\omega_{0}+\frac{2 \pi k}{q}\right)=0$ for all $1 \leq k \leq q-1$, see [11]. But since $g_{r}(\omega)>0, M_{\mathbf{c}}(\omega)=0$ if and only if $H_{q}(\omega)=0$, which leads to $\omega=\frac{2 \pi j}{q}$ for $j \neq q j^{\prime}$. So for any $1 \leq k \leq q-1, \omega_{0}+\frac{2 \pi k}{q}=\frac{2 \pi j}{q}$ for some $j \neq q j^{\prime}$. But if so then $q^{N} \omega_{0} \not \equiv \omega_{0}(\bmod 2 \pi)$ for any $N \in \mathbb{N}$. This is a contradiction.

## 4. Symmetric Scaling Functions

We call $\mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$ symmetric if $\mathbf{c}=\tau_{m} \circ \gamma(\mathbf{c})$ for some $m \in \mathbb{Z}$. A function $f(x)$ is called symmetric if $f(x)=f(a-x)$ for some $a \in \mathbb{R}$.

Lemma 4.1. Let $\mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$. Then $\mathbf{c}$ is symmetric if and only if $\Phi_{\mathbf{c}}(x)$ is.
Proof. We prove the lemma for $q>0$. For $q<0$ the proof is similar
Suppose that $\mathbf{c}$ is symmetric. Then $\mathbf{c}=\tau_{m} \circ \gamma(\mathbf{c})$ for some $m \in \mathbb{Z}$. Therefore

$$
\Phi_{\mathbf{c}}(x)=\Phi_{\tau_{m} \circ \gamma(\mathbf{c})}(x)=\Phi_{\mathbf{c}}\left(\frac{m}{q-1}-x\right),
$$

and so $\Phi_{\mathbf{c}}(x)$ is symmetric.
Conversley, suppose that $\Phi_{\mathbf{c}}(x)$ is symmetric and $\Phi_{\mathbf{c}}(x)=\Phi_{\mathbf{c}}(a-x)$. Without loss of generality we let $\mathbf{c}=\left\{c_{k}: k \in \mathbb{Z}\right\}$ such that $c_{k}=0$ for all $k \notin[0, m]$ while $c_{0} c_{m} \neq 0$. Then $\Phi_{\mathbf{c}}(x)$ is supported exactly on $\left[0, \frac{m}{q-1}\right]$. Now this means $a=\frac{m}{q-1}$. Hence $\Phi_{\mathbf{c}}(x)=\Phi_{\tau_{m} \circ \gamma(\mathbf{c})}(x)$. We argue that $\mathbf{c}=\tau_{m} \circ \gamma(\mathbf{c})$. Note that

$$
\hat{\Phi}_{\mathbf{c}}(\omega)=M_{\mathbf{c}}\left(\frac{\omega}{q}\right) \hat{\Phi}_{\mathbf{c}}\left(\frac{\omega}{q}\right), \quad \hat{\Phi}_{\tau_{m} \circ \gamma(\mathbf{c})}(\omega)=M_{\tau_{m} \circ \gamma(\mathbf{c})}\left(\frac{\omega}{q}\right) \hat{\Phi}_{\tau_{m} \circ \gamma(\mathbf{c})}\left(\frac{\omega}{q}\right) .
$$

Therefore

$$
M_{\mathbf{c}}\left(\frac{\omega}{q}\right)=M_{\tau_{m} \circ \gamma(\mathbf{c})}\left(\frac{\omega}{q}\right)
$$

for all $\omega \in \mathbb{R}$, which gives $\mathbf{c}=\tau_{m} \circ \gamma(\mathbf{c})$.
Theorem 4.2. (i) Suppose that $\mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$ is symmetric. Let $\left|M_{\mathbf{c}}(\omega)\right|^{2}=P(\cos \omega)$ where $P(z) \in \mathbb{R}[z]$. Then

$$
\begin{equation*}
P(z)=g^{2}(z) \quad \text { or } \quad P(z)=\left(\frac{1+z}{2}\right) g^{2}(z) \tag{4.1}
\end{equation*}
$$

for some $g(z) \in \mathbb{R}[z], g(1)=1$.
(ii) Conversely, for any $P(z)=g^{2}(z)$ or $P(z)=\left(\frac{1+z}{2}\right) g^{2}(z)$ where $g(z) \in \mathbb{R}[z]$ and $g(1)=1$, there exists up to equivalence a unique symmetric $\mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$ such that $\left|M_{\mathbf{c}}(\omega)\right|^{2}=P(\cos \omega)$.
Proof. We first prove (i). Suppose that $\mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$ is symmetric. Then $\mathbf{c}=\tau_{m} \circ \gamma(\mathbf{c})$ for some $m \in \mathbb{Z}, M_{\mathbf{c}}(\omega)=e^{i m \omega} M_{\mathbf{c}}(-\omega)$. Hence

$$
\left|M_{\mathbf{c}}(\omega)\right|^{2}=M_{\mathbf{c}}(\omega) \cdot M_{\mathbf{c}}(-\omega)=e^{-i m \omega} M_{\mathbf{c}}^{2}(\omega)=\left(e^{-i \frac{m \omega}{2}} M_{\mathbf{c}}(\omega)\right)^{2}
$$

Since $\left|M_{\mathbf{c}}(\omega)\right|^{2} \geq 0$ is real, the imaginary part of $e^{-i \frac{m \omega}{2}} M_{\mathbf{c}}(\omega)$ must be 0 .
Now, if $m=2 k$ then

$$
e^{-i \frac{m \omega}{2}} M_{\mathbf{c}}(\omega)=\operatorname{Re}\left(e^{-i k \omega} M_{\mathbf{c}}(\omega)\right)=g(\cos \omega)
$$

for some $g(z) \in \mathbb{R}[z]$. Hence $\left|M_{\mathbf{c}}(\omega)\right|^{2}=g^{2}(\cos \omega)$. If $m=2 k+1$ then

$$
e^{-i \frac{m \omega}{2}} M_{\mathbf{c}}(\omega)=\operatorname{Re}\left(e^{-i \frac{(2 k+1) \omega}{2}} M_{\mathbf{c}}\left(2 \cdot \frac{\omega}{2}\right)\right)=\tilde{g}\left(\cos \frac{\omega}{2}\right)
$$

for some $\tilde{g}(z) \in \mathbb{R}[z]$. Hence

$$
\begin{equation*}
\left|M_{\mathbf{c}}(\omega)\right|^{2}=\tilde{g}^{2}\left(\cos \frac{\omega}{2}\right) \tag{4.2}
\end{equation*}
$$

But $\cos ^{2}\left(\frac{\omega}{2}\right)=\frac{1}{2}(1+\cos \omega)$, so

$$
\begin{equation*}
\tilde{g}\left(\cos \frac{\omega}{2}\right)=g_{1}(\cos \omega)+\cos \left(\frac{\omega}{2}\right) \cdot g_{2}(\cos \omega) \tag{4.3}
\end{equation*}
$$

where $g_{1}(z), g_{2}(z) \in \mathbb{R}[z]$. However, by (4.2) $\left|M_{\mathbf{c}}(\omega)\right|^{2}=\tilde{g}^{2}\left(\cos \frac{\omega}{2}\right)=P(\cos \omega)$. It follows that either $g_{1}(z)=0$ or $g_{2}(z)=0$. Since $g_{2}(z) \neq 0$, we must have $g_{1}(z)=0$. So

$$
\left|M_{\mathbf{c}}(\omega)\right|^{2}=\cos ^{2}\left(\frac{\omega}{2}\right) \cdot g_{2}^{2}(\cos \omega)=\frac{1}{2}(1+\cos \omega) \cdot g_{2}^{2}(\cos \omega)
$$

proving (i).
We next prove (ii). The existence is quite straighforward. Suppose that $P(z)=$ $g^{2}(z)$. Then $M_{\mathbf{c}}(\omega)=g(\cos \omega)$ defines a symmetric $\mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$ as $M_{\mathbf{c}}(-\omega)=M_{\mathbf{c}}(\omega)$. Suppose that $P(z)=\left(\frac{1+z}{2}\right) g^{2}(z)$. Then

$$
\begin{equation*}
M_{\mathbf{c}}(\omega)=e^{i \frac{\omega}{2}} \cos \frac{\omega}{2} \cdot g(\cos \omega)=\frac{e^{i \omega}+1}{2 \sqrt{2}} \cdot g\left(\frac{e^{i \omega}+e^{-i \omega}}{2}\right) \tag{4.4}
\end{equation*}
$$

defines a symmetric $\mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$ because $M_{\mathbf{c}}(\omega)=e^{i \omega} M_{\mathbf{c}}(-\omega)$.
We show that the symmetric $\mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$ is unique up to equivalence by contradiction. Assume that there is another symmetric $\mathbf{c}^{\prime} \in \mathcal{S}_{q}(\mathbb{R})$ such that $\left|M_{\mathbf{c}^{\prime}}(\omega)\right|^{2}=P(\cos \omega)$. By Theorem 2.2, there exist $\mathbf{a}, \mathbf{e}, \mathbf{e}^{\prime} \in \mathcal{S}_{q}(\mathbb{R})$ such that $\mathbf{e}$ is equivalent to $\mathbf{e}^{\prime}$ and

$$
\mathbf{c}=\mathbf{a} * \mathbf{e}, \quad \mathbf{c}^{\prime}=\mathbf{a} * \mathbf{e}^{\prime}
$$

Therefore there exists some $k \in \mathbb{Z}$ such that

$$
M_{\mathbf{c}^{\prime}}(\omega)=e^{i k \omega} M_{\mathbf{a}}(\omega) M_{\mathbf{e}}(\omega) \quad \text { or } \quad M_{\mathbf{c}^{\prime}}(\omega)=e^{i k \omega} M_{\mathbf{a}}(\omega) M_{\mathbf{e}}(-\omega)
$$

depending on the equivalence relation of $\mathbf{e}$ and $\mathbf{e}^{\prime}$. In the first case we must have $\mathbf{c}^{\prime}=\tau_{k}(\mathbf{c})$, and so $\mathbf{c}$ and $\mathbf{c}^{\prime}$ are equivalent. In the second case, because $\mathbf{c}$ and $\mathbf{c}^{\prime}$ are both symmetric,

$$
M_{\mathbf{a}}(\omega) M_{\mathbf{e}}(\omega)=e^{i m_{1} \omega} M_{\mathbf{a}}(-\omega) M_{\mathbf{e}}(-\omega), \quad M_{\mathbf{a}}(\omega) M_{\mathbf{e}}(-\omega)=e^{i m_{2} \omega} M_{\mathbf{a}}(-\omega) M_{\mathbf{e}}(\omega)
$$

Hence $M_{\mathbf{a}}^{2}(\omega)=e^{i\left(m_{1}+m_{2}\right) \omega} M_{\mathbf{a}}^{2}(-\omega)$. This implies that

$$
M_{\mathbf{a}}(\omega)= \pm e^{i m \omega} M_{\mathbf{a}}(-\omega)
$$

where $m=\left(m_{1}+m_{2}\right) / 2$ is clearly an integer. But $M_{\mathbf{a}}(0)=1$, so $M_{\mathbf{a}}(\omega)=$ $e^{i m \omega} M_{\mathbf{a}}(-\omega)$. The equivalence of $\mathbf{c}$ and $\mathbf{c}^{\prime}$ follows from

$$
M_{\mathbf{c}^{\prime}}(\omega)=e^{i k \omega} M_{\mathbf{a}}(\omega) M_{\mathbf{e}}(-\omega)=e^{i(k+m) \omega} M_{\mathbf{c}}(-\omega)
$$

Remark. It is possible for a nonsymmetric $\mathbf{c} \in \mathcal{S}_{q}(\mathbb{R})$ to satisfy (4.1). A simple example is to let $c_{0}=4 q, c_{1}=-4 q, c_{2}=q$ and all other $c_{k}=0$. Then $\mathbf{c}$ is nonsymmetric, but nevertheless $\left|M_{\mathbf{c}}(\omega)\right|^{2}=(5-4 \cos \omega)^{2}$.

Example 4.1. For regularity $r=1$ and arbitrary $q>3$, by Theorem 3.3 any scaling sequences c satisfy $\left|M_{\mathbf{c}}(\omega)\right|^{2}=\left|H_{q}(\omega)\right|^{2} g(\omega)$ where

$$
g(\omega)=1+(1-\cos \omega) \sum_{n \neq q k} c_{n} \cos n \omega .
$$

Choosing $g(\omega)=1+(1-\cos \omega)\left(c_{1} \cos \omega+c_{2} \cos 2 \omega\right)$, and applying (4.1) and (4.4) we obtain two scaling sequences $\mathbf{c}_{1}$ and $\mathbf{c}_{2}$ given by

$$
\begin{align*}
& M_{\mathbf{c}_{1}}(\omega)=\frac{1}{2} H_{q}(\omega)\left(\alpha+(1-\alpha) e^{i \omega}+(1-\alpha) e^{i 2 \omega}+\alpha e^{i 3 \omega}\right),  \tag{4.5}\\
& M_{\mathbf{c}_{2}}(\omega)=\frac{1}{2} H_{q}(\omega)\left(\beta+(1-\beta) e^{i \omega}+(1-\beta) e^{i 2 \omega}+\beta e^{i 3 \omega}\right), \tag{4.6}
\end{align*}
$$

where $\alpha=\frac{1}{2}-\frac{\sqrt{6}}{4}$ and $\beta=\frac{1}{2}+\frac{\sqrt{6}}{4}$. The scaling sequence $\mathbf{c}_{1}$ corresponds to the continuous "Batman" scaling function (Figure 1), while $\mathbf{c}_{2}$ corresponds to a discontinuous scaling function (Figure 2). For $q=3$, the corresponding two wavelets (symmetric and antisymmetric) are shown in Figure 3. We shall study the "Batman" function in detail in $\S 5$.


Figure 1. The "Batman" scaling function of dilation $q=3$ (Example 4.1)

Example 4.2. Consider scaling sequences for $q=5$ and regularity $r=2$. Choose

$$
\begin{aligned}
g(\omega) & =1+8(1-\cos \omega)+(1-\cos \omega)^{2}\left(a_{1} \cos \omega+a_{2} \cos 2 \omega\right) \\
& =1+8 z+z^{2}+\left(a_{1}+a_{2}\right) z^{2}-\left(a_{1}+4 a_{2}\right) z^{3}+2 a_{1} z^{4}
\end{aligned}
$$

where $z=1-\cos \omega$. By Theorem 3.3 any $\mathbf{c} \in \mathcal{S}_{5}(\mathbb{R})$ satisfying $\left|M_{\mathbf{c}}(\omega)\right|^{2}=\left|H_{q}(\omega)\right|^{4} g(\omega)$ is a scaling sequence. Solving for $a_{1}, a_{2}$ to complete the square for $g(\omega)$ we obtain two solutions,

$$
g(\omega)=\left(1+4 z-4 z^{2}\right)^{2}, \quad \text { or } \quad g(\omega)=\left(1+4 z-\frac{8}{3} z^{2}\right)^{2} .
$$

These lead to two symmetric scaling sequences

$$
\begin{align*}
& \mathbf{c}_{1}=\frac{1}{5}\{-1,0,0,2,3,6,5,6,3,2,0,0,-1\}  \tag{4.7}\\
& \mathbf{c}_{2}=\frac{1}{15}\{-2,-2,1,6,9,16,19,16,9,6,1,-2,-2\} . \tag{4.8}
\end{align*}
$$



Figure 2. The other (discontinuous) scaling function of dilation $q=3$ (Example 4.1)


The corresponding scaling functions are shown in Figure 4 and Figure 5, respectively. Both are continuous but only $\Phi_{\mathbf{c}_{2}}(x)$ is differentiable, see $\S 5$.

Example 4.3. As $r$ grows, it becomes increasingly harder to find symmetric scaling sequences by hand. Fortunately, Theorem 4.2 can be used in conjunction with


Figure 3. Two "Batman" wavelets (long and short) for dilation $q=3$
standard software tools such as Mathematica. Figure 6 shows two minimal support symmetric scaling functions for $q=4$ and $r=3$. The polynomial $P(z)$ (defined in Theorem 4.2) has the form $P(z)=\frac{1}{2}(1+z) g^{2}(z)$.

## 5. The "Batman" Scaling Function

In Example 4.1 (Figure 1) we have introduced the "Batman" scaling function (of dilation $q \geq 3$ ), which is given by the $q$-scaling sequence

$$
\begin{equation*}
\mathbf{c}=\{\alpha, \frac{1}{2}, 1-\alpha, \underbrace{1, \ldots, 1}_{q-3}, 1-\alpha, \frac{1}{2}, \alpha\} \tag{5.1}
\end{equation*}
$$

where $\alpha=\frac{1}{2}-\frac{\sqrt{6}}{4}$. The corresponding refinement equation is

$$
\begin{aligned}
f(x)= & \alpha f(q x)+\frac{1}{2} f(q x-1)+(1-\alpha) f(q x-2)+f(q x-3)+\cdots \\
& +f(q x-q+1)+(1-\alpha) f(q x-q)+\frac{1}{2} f(q x-q-1)+\alpha f(q x-q-2)
\end{aligned}
$$

Let $\phi_{q}(x)$ denote the "Batman" scaling function corresponding to the "Batman" scaling sequence for dilation $q$ given by (5.1). The support of $\phi_{q}(x)$ is precisely $\left[0, \frac{q+2}{q-1}\right]$, which gives $[0,2.5]$ for $q=3$ and $[0,2]$ for $q=4$. In what follows, we show that $\phi_{q}(x)$ are continuous for all $q$ and determine the Hölder exponent of $\phi_{q}(x)$ using


Figure 4. Continuous scaling function of dilation 5, $r=2$ (Example 4.2)
the joint spectral radius of matrices. Detailed discussions on the joint spectral radius can be found in Daubechies and Lagarias [10], Berger and Wang [1], and Lagarias and Wang [13].

Consider the general two-scale dilation equation

$$
\begin{equation*}
f(x)=\sum_{n=0}^{N} c_{n} f(q x-n) \tag{5.2}
\end{equation*}
$$

where $c_{0} c_{N} \neq 0$ and $q \geq 2$. If $f(x)$ is a compactly supported solution to (5.2) then its support is $\left[0, \frac{N}{q-1}\right]$. The regularity of $f(x)$ can be obtained by rewriting (5.2) into the equivalent form of product of matrices. Let $L=\left\lceil\frac{N}{q-1}\right\rceil$ and $\mathbf{v}(x)$ be the $L$-dimensional vector

$$
\mathbf{v}(x)=[f(x), f(x+1), \ldots, f(x+L-1)]^{T}, \quad 0 \leq x \leq 1
$$

Define the $L \times L$ matrices $P_{k} \in M_{L}(\mathbb{R}), 0 \leq k \leq q-1$,

$$
\begin{equation*}
P_{k}=\left[c_{q(i-1)-(j-1)+k}\right] . \tag{5.3}
\end{equation*}
$$



Figure 5. "Smooth hat" differentiable scaling function of dilation 5, $r=2$ (Example 4.2)

Then (5.2) is equivalent to

$$
\begin{equation*}
\mathbf{v}(x)=P_{d_{1}} \mathbf{v}\left(\sigma_{q} x\right) \tag{5.4}
\end{equation*}
$$

where $x \in[0,1]$ has the base $q$ expansion

$$
x=0 . d_{1} d_{2} d_{3} \cdots, \quad \text { all } 0 \leq d_{j} \leq q-1
$$

and $\sigma_{q} x$ is the fractional part of $q x$,

$$
\begin{equation*}
\sigma_{q} x \equiv q x \quad(\bmod 1) \tag{5.5}
\end{equation*}
$$

Iterating (5.4) we obtain

$$
\begin{equation*}
\mathbf{v}(x)=P_{d_{1}} P_{d_{2}} \cdots P_{d_{m}} \mathbf{v}\left(\sigma_{q}^{m} x\right) \tag{5.6}
\end{equation*}
$$

All $P_{k}$ are column stochastic, i.e. the entries of each column sum up to one. Therefore, the vector $[1,1, \ldots, 1]$ is a common left 1 -eigenvector of all $P_{k}$. Hence by taking any


Figure 6. Two scaling functions of dilation $4, r=3$ (Example 4.3)
nonsingular $Q \in M_{L}(\mathbb{R})$ such that the first row of $Q$ consists all 1 's, all $P_{k}$ can be simultaneously block triangularized

$$
Q P_{k} Q^{-1}=\left[\begin{array}{cc}
1 & 0  \tag{5.7}\\
* & A_{k}
\end{array}\right], \quad 0 \leq k \leq q-1 .
$$

Proposition 5.1. Suppose that the joint spectral radius $\hat{\rho}=\hat{\rho}\left(A_{0}, A_{1}, \ldots, A_{q-1}\right)<$ 1. Then the refinement function satisfying (5.2) is continuous. Furthermore, let
$r=\log _{q}(1 / \hat{\rho})$. Then $f(x) \in C^{r-\epsilon}$ but $f(x) \notin C^{r+\epsilon}$ for all $\epsilon>0$, and $f(x) \in C^{r}$ if and only if the semigroup of matrices generated by $A_{k} / \hat{\rho}, 0 \leq k \leq q-1$, is bounded. Proof. See Wang [16] or Collela and Heil [5].

Theorem 5.2. The"Batman" scaling function $\phi_{q}(x)$ is continuous, with Hölder exponent $\log _{q}(4 / \sqrt{6})$.
Proof. We first consider the case $q \geq 4$. We have $L=\left\lceil\frac{q+2}{q-1}\right\rceil=2$ and the $q$ matrices $P_{0}, \ldots, P_{q-1}$ are all 2 by 2 matrices. Let $Q=\left[\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right]$. Then

$$
Q P_{k} Q^{-1}=\left[\begin{array}{cc}
1 & 0 \\
* & A_{k}
\end{array}\right]
$$

All $A_{k}$ are scalars. Now by (5.3)

$$
P_{k}=\left[\begin{array}{cc}
c_{k} & c_{k-1} \\
c_{q+k} & c_{q+k-1}
\end{array}\right]
$$

and it is straightforward to check that

$$
\hat{\rho}\left(A_{0}, A_{1}, \ldots, A_{q-1}\right)=\max \left(\left|A_{0}\right|,\left|A_{1}\right|, \ldots,\left|A_{q-1}\right|\right)=\frac{\sqrt{6}}{4}
$$

The theorem follows immediately from Proposition 5.1.
In the case $q=3$ the support of $\phi_{3}(x)$ is $[0,2.5]$ so $L=3$ and the matrices $P_{k}$ are 3 by 3. We have

$$
Q P_{k} Q^{-1}=\left[\begin{array}{cc}
1 & 0 \\
* & A_{k}
\end{array}\right] \quad \text { by taking } \quad Q=\left[\begin{array}{ccc}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{array}\right]
$$

where

$$
A_{0}=\left[\begin{array}{cc}
\alpha & 0  \tag{5.8}\\
\alpha & \frac{1}{2}-\alpha
\end{array}\right], \quad A_{1}=\left[\begin{array}{cc}
\frac{1}{2}-\alpha & \alpha \\
0 & \alpha
\end{array}\right], \quad A_{2}=\left[\begin{array}{cc}
\frac{1}{2}-\alpha & \frac{1}{2}-\alpha \\
0 & 0
\end{array}\right]
$$

Note that for each $A=\left[a_{i j}\right] \in M_{2}(\mathbb{R})$,

$$
\begin{equation*}
\|A\|_{1}=\max \left\{\left|a_{11}\right|+\left|a_{21}\right|,\left|a_{12}\right|+\left|a_{22}\right|\right\} \tag{5.9}
\end{equation*}
$$

defines a matrix norm on $M_{2}(\mathbb{R})$. ( $\|.\|_{1}$ is actually the induced operator norm from the $L^{1}$-norm on $\mathbb{R}^{2}$.) Since

$$
2|\alpha|=\frac{\sqrt{6}}{2}-1<\frac{1}{2}-\alpha=\frac{\sqrt{6}}{4}
$$

it follows that all $\left\|A_{k}\right\|_{1}=\frac{1}{2}-\alpha=\frac{\sqrt{6}}{4}$. This forces $\hat{\rho}\left(A_{0}, A_{1}, A_{2}\right) \leq \frac{\sqrt{6}}{4}$ and the semigroup generated by $\left\{A_{k} / \frac{\sqrt{6}}{4}\right\}$ be bounded (cf. Berger and Wang [1], Lemma II). However $\hat{\rho}\left(A_{0}, A_{1}, A_{2}\right) \geq \frac{\sqrt{6}}{4}$ because $\frac{\sqrt{6}}{4}$ is an eignevalue of $A_{0}$. Therefore

$$
\begin{equation*}
\hat{\rho}=\hat{\rho}\left(A_{0}, A_{1}, A_{2}\right)=\frac{\sqrt{6}}{4} \tag{5.10}
\end{equation*}
$$

proving the theorem for $q=3$.
The same technique can be applied to show that the "Smooth hat" scaling function in Figure 5, corresponding to the 5 -scaling sequence defined in (4.8), is differentiable.

Theorem 5.3. The symmetric scaling function $\Phi_{\mathbf{c}}(x)$ where

$$
\mathbf{c}=\frac{1}{15}\{-2,-2,1,6,9,16,19,16,9,6,1,-2,-2\}
$$

is differentiable.
To prove the above theorem we first obtain the five matrices $P_{k}, 0 \leq k \leq 4$. It is then straightforward to check that they have a common left $\frac{1}{5}$-eigenvector $[1,-2,3]$ in addition to the common left 1 -eignevector $[1,1,1]$. The rest follows by simultaneously triangularizing $P_{k}$ and applying results in Daubechies and Lagarias [9]. We omit the details here.

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