Self-Affine Tiles

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June 10, 1998

Abstract

A *self-affine tile* in $\mathbf{R}^n$ is a set T of positive Lebesgue measure satisfying $\mathbf{A}(T) = \cup_{d\in\mathcal{D}}(T+d)$, where $\mathbf{A}$ is an expanding $n\times n$ real matrix with $|\det(\mathbf{A})| = m$ an integer, and $\mathcal{D} = \{d_1,\ldots,d_m\} \subset \mathbf{R}^n$ a set of m digits. Self-affine tiles arise in many contexts, including radix expansions, fractal geometry, and the construction of compactly supported orthonormal wavelet bases of $L^2(\mathbf{R}^n)$. They are also studied as interesting tiles. In this article we survey the fundamental properties of self-affine tiles. We examine necessary and sufficient conditions for digit sets $\mathcal{D}$ to give rise to self-affine tiles. A special class of self-affine tiles is the *integeral self-affine tiles*, in which $\mathbf{A}$ is an integer matrix and $\mathcal{D} \subset \mathbf{Z}^n$. We study the tiling properties and the measures of integeral self-affine tiles. We also compute the Hausdorff dimensions of the boundaries of integeral self-affine tiles.

1 Introduction

Let $\mathbf{A}$ be an expanding matrix in $M_n(\mathbf{R})$, that is, one with all eigenvalues $|\lambda_i| > 1$, and suppose that $|\det(\mathbf{A})| = m$ for some integer $m > 1$. Let $\mathcal{D} = \{d_1, d_2, \ldots, d_m\} \subset \mathbf{R}^n$ be a finite set of vectors. A result of Hutchinson [27] states that there exists a unique nonempty compact set $T := T(\mathbf{A},\mathcal{D})$ such that

$$T = \bigcup_{j=1}^{m} \mathbf{A}^{-1}(T+d_j). \tag{1.1}$$

More precisely, T is the *attractor* of the *iterated function system* $\{\phi_j(x) = \mathbf{A}^{-1}x + \mathbf{A}^{-1}d_j : 1 \le j \le m\}$. In fact, T is given explicitly by

$$T = \left\{ \sum_{k=1}^{\infty} \mathbf{A}^{-k} d_k : \text{ each } d_k \in \mathcal{D} \right\}. \tag{1.2}$$

For most pairs $(\mathbf{A},\mathcal{D})$ the set $T(\mathbf{A},\mathcal{D})$ has Lebesgue measure $\mu(T) = 0$. If $T(\mathbf{A},\mathcal{D})$ has positive Lebesgue measure we call $T(\mathbf{A},\mathcal{D})$ a *self-affine tile.*

*Research supported in part by the National Science Foundation, grant DMS–9307601.

The name "self-affine tile" refers to the fact that

$$\mathbf{A}(T) = \bigcup_{j=1}^{m} (T + d_j) = T + \mathcal{D}; \tag{1.3}$$

geometrically it means that the affinely dilated set $\mathbf{A}(T)$ is perfectly tiled by the m translates $T+d_j$ of T. A simple example of a self-affine tile is the unit square $T = [0,1]^2$, which satisfies $\mathbf{A}(T) = T + \mathcal{D}$ for

$$\mathbf{A} = 2I, \qquad \mathcal{D} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Self-affine tiles have been studied as "exotic" tiles and as tiles giving interesting tilings of $\mathbf{R}^n$ ([1], [2] [11], [12], [13] [22], [24], [29], [34], [37], [52]). Furthermore, they arise in many other contexts, particularly in fractal geometry ([14], [15], [16], [51]), compactly supported wavelet bases ([23], [33], [36]), radix expansions ([19]), and in Markov partitions ([28]). The current interests in self-affine tiles come largely from these applications.

Most of the studies on self-affine tiles employ one or both of the two approaches: algebraic and Fourier analytic. It is rather easy to see the role of algebraic methods. For example, given an expanding matrix $\mathbf{A}$ and a digit set $\mathcal{D}$, by iterating (1.3) we obtain $\mathbf{A}^k(T) = T + \mathcal{D}_{\mathbf{A},k}$ where

$$\mathcal{D}_{\mathbf{A},k} = \left\{ \sum_{j=0}^{k-1} \mathbf{A}^j d_j \, : \, \text{ each } d_k \in \mathcal{D} \right\}. \tag{1.4}$$

As we shall see, many properties of $T(\mathbf{A}, \mathcal{D})$ depend fundamentally on the algebraic properties of $\mathcal{D}_{\mathbf{A},k}$. Of course, this is but one of the many instances where algebraic methods can be employed.

But harmonic analysis can be a powerful tool in the study of self-affine tiles as well. Let $T := T(\mathbf{A}, \mathcal{D})$ be a self-affine tile. The set-valued equation $\mathbf{A}(T) = T + \mathcal{D}$ can be written as

$$\chi_T(x) = \sum_{d \in \mathcal{D}} \chi_T(\mathbf{A}x - d). \tag{1.5}$$

Let $m_{\mathcal{D}}(\xi) = \frac{1}{|\mathcal{D}|} \sum_{d \in \mathcal{D}} e^{i2\pi\langle d, \xi \rangle}$. Taking the Fourier transform in (1.5) results in

$$\widehat{\chi}_T(\xi) = m_{\mathcal{D}}(\mathbf{B}^{-1}\xi)\widehat{\chi}_T(\mathbf{B}^{-1}\xi), \text{ where } \mathbf{B} := \mathbf{A}^T. \tag{1.6}$$

This yields

$$\widehat{\chi}_T(\xi) = c \prod_{j=1}^{\infty} m_{\mathcal{D}}(\mathbf{B}^{-j}\xi), \text{ where } c := \widehat{\chi}_T(0) = \mu(T). \tag{1.7}$$

By analyzing $m_{\mathcal{D}}(\xi)$ and the infinite product (1.7) a number of nontrivial results on the tile T and its tilings can be proved ([7], [22], [29], [35], [37]).

We shall provide a glimpse of both approaches in this overview. The fundamental question this paper addresses is this: for a given matrix $\mathbf{A}$ and digit set $\mathcal{D}$, under what conditions will $T(\mathbf{A}, \mathcal{D})$ be a tile? We derive several necessary and sufficient conditions in §2, and later in §4. In §3 we introduce integral self-affine tiles and prove some basic results concerning their measures and tilings. Some of these results are then used in §5 to study Haar-type wavelet bases. In §6 we show a method for finding the exact Hausdorff dimension of the boundaries of self-affine tiles.

Due to the restriction on the length of the paper, We have limited the discussions of this overview mostly to self-affine tiles as sets. In doing so we have made several conspicuous, and perhaps unjustified, omissions. In particular, we have left out the study on self-replicating tilings and on the topological properties of the tiles entirely. We apologize in advance for our inability to include these results and shall refer the readers to [2], [28], [30], [34], [53] for more details.

We are greatly indebt to Professor Ka-Sing Lau and the mathematics department of the Chinese University of Hong Kong for the kind invitation to visit. We would also like to thank Jeff Lagarias, Ka-Sing Lau, Rick Kenyon, Sze-Man Ngai and Bob Strichartz for encouraging and helpful discussions.

2 Conditions For A Tile

As mentioned in the introduction, for a given pair $(\mathbf{A}, \mathcal{D})$ where $\mathbf{A} \in M_n(\mathbf{R})$ is expanding and $\mathcal{D} \subset \mathbf{R}^n$ has cardinality $|\mathcal{D}| = |\det(\mathbf{A})|$, the corresponding attractor $T(\mathbf{A}, \mathcal{D})$ is usually not a tile. A fundamental question is thus: under what condition(s) is $T(\mathbf{A}, \mathcal{D})$ a tile? To gain some insight into this question we first look at the following example.

Example 2.1. Let $\mathbf{A} = [3]$ and $\mathcal{D} = \{0, 1, 4\}$. We show that $T = T(\mathbf{A}, \mathcal{D})$ is not a tile by showing that $\mu(T) = 0$, where μ denotes the Lebesgue measure. Note that

$$3T = T + \mathcal{D} = T + \{0, 1, 4\}.$$

Hence

$$\begin{aligned} 9T &= 3T + 3\mathcal{D} \\ &= T + \{0, 1, 4\} + \{0, 3, 12\} \\ &= T + \{0, 1, 3, 4, 7, 12, 13, 16\}. \end{aligned}$$

It follows by taking the Lebesgue measure that

$$9\mu(T) = \mu(T + \{0, 1, 3, 4, 7, 12, 13, 16\}) \leq 8\mu(T),$$

and hence $\mu(T) = 0$. ∎

For integral $\mathbf{A}$ and $\mathcal{D}$, the following theorem was established by Bandt [1]:

Theorem 2.1 *Let $\mathbf{A} \in M_n(\mathbf{Z})$ be an expanding matrix and let $\mathcal{D} \subset \mathbf{Z}^n$ be a set of complete coset representatives of $\mathbf{Z}^n/\mathbf{A}(\mathbf{Z}^n)$. Then $T = T(\mathbf{A}, \mathcal{D})$ has nonempty interior. Furthermore, T is the closure of its interior and $\mu(\partial T) = 0$.*

Proof. We present a new proof here. We first show that $\mu(T) > 0$. Let $T_0 = [0, 1]^n$ and

$$T_k = \bigcup_{d \in \mathcal{D}} \mathbf{A}^{-1}(T_{k-1} + d), \quad k \geq 1. \tag{2.1}$$

It is easy to check, by induction on k, that the unions in (2.1) are measure-disjoint and $\mu(T_k) = 1$ for all k. Since $T_k \longrightarrow T$ in the Haudorff metric (c.f. Hutchinson [27]), it follows that $\mu(T) \geq 1$.

Now let $\pi_n : \mathbf{R}^n \longrightarrow \mathbf{T}^n$ be the canonical covering map, where $\mathbf{T}^n := \mathbf{R}^n/\mathbf{Z}^n$ is the n-torus. Then $\mathbf{A}_* := \pi_n \circ \mathbf{A} \circ \pi_n^{-1}$ is a well defined endomorphism on $\mathbf{T}^n$. Clearly,

$$\mathbf{A}_*(\pi_n(T)) = \pi_n(\mathbf{A}(T)) = \pi_n(T + \mathcal{D}) = \pi_n(T).$$

So $\pi_n(T)$ is invariant under $\mathbf{A}_*$. But $\mathbf{A}_*$ is ergodic because $\mathbf{A}$ is expanding (c.f. Walters [54]). Hence $\pi_n(T) = \mathbf{T}^n$. This means that

$$\bigcup_{\alpha \in \mathbf{Z}^n} (T + \alpha) = \mathbf{R}^n.$$

To see that $T^o \neq \emptyset$, let $\mathcal{J} \subset \mathbf{Z}^n$ be the smallest set such that $T + \mathcal{J} \supseteq (0,1)^n$. Suppose that $T^o = \emptyset$. Fix an $\alpha_0 \in \mathcal{J}$. Then any $x \in (0,1)^n \cap (T + \alpha_0)$ must belong to another $T + \beta$ for some $\beta \in \mathcal{J}$. Hence $T + (\mathcal{J} \setminus \{\alpha_0\}) \supseteq (0,1)^n$, contradicting the minimality assumption of $\mathcal{J}$. So $T^o \neq \emptyset$. Now

$$\mathbf{A}(\overline{T^o}) = \overline{T^o} + \mathcal{D}.$$

By the uniqueness we must have $T = \overline{T^o}$.

Finally we prove that $\mu(\partial T) = 0$. Let $x_0 \in T^o$. For sufficiently large $k \geq 0$ the interior of $\mathbf{A}^k(T - x_0)$ will contain T. But

$$\mathbf{A}^k(T - x_0) = T + \mathcal{J}', \quad \text{where } \mathcal{J}' = \mathcal{D} + \mathbf{A}\mathcal{D} + \cdots + \mathbf{A}^{k-1}\mathcal{D} - \mathbf{A}^k x_0,$$

and the union $T + \mathcal{J}'$ is measure-disjoint. Since ∂T is contained in the overlapps in the union, it follows that $\mu(\partial T) = 0$. ∎

For any $\mathbf{A}$ and digit set $\mathcal{D}$ we denote

$$\mathcal{D}_{\mathbf{A},k} := \mathcal{D} + \mathbf{A}\mathcal{D} + \cdots + \mathbf{A}^{k-1}\mathcal{D}.$$

Note that if $0 \in \mathcal{D}$ then $\mathcal{D}_{\mathbf{A},k} \subseteq \mathcal{D}_{\mathbf{A},k+1}$. In this case we denote

$$\mathcal{D}_{\mathbf{A},\infty} := \bigcup_{k=1}^{\infty} \mathcal{D}_{\mathbf{A},k}\,, \quad \Delta(\mathbf{A},\mathcal{D}) := (\mathcal{D} - \mathcal{D})_{\mathbf{A},\infty}.$$

The above theorem is a special case of the following more general theorem, due to Kenyon [29] and Lagarias and Wang [34]:

Theorem 2.2 *Let $\mathbf{A} \in M_n(\mathbf{R})$ be an expanding matrix such that $|\det(\mathbf{A})| = m \in \mathbf{Z}$. Suppose that $\mathcal{D} \subset \mathbf{R}^n$ has cardinality m, with $0 \in \mathcal{D}$. Let $T = T(\mathbf{A},\mathcal{D})$. Then the following conditions are equivalent:*

(a) T has positive Lebesgue measure.

(b) T has nonempty interior.

(c) T is the closure of its interior, and its boundary ∂T has Lebesgue measure zero.

(d) For each $k \geq 1$ all m^k expansions in $\mathcal{D}_{\mathbf{A},k}$ are distinct, and $\mathcal{D}_{\mathbf{A},\infty}$ is uniformly discrete.

Although not difficult, the proof is rather tedious. A proof can be found in Lagarias and Wang [34].

One other question we naturally ask is how does a self-affine tile $T(\mathbf{A}, \mathcal{D})$ tile $\mathbf{R}^n$. We show below that T tiles by translation.

Theorem 2.3 *Let $\mathbf{A} \in M_n(\mathbf{R})$ be an expanding matrix and $\mathcal{D} \subset \mathbf{R}^n$ with $|\mathcal{D}| = |\det(\mathbf{A})|$. Suppose that $T = T(\mathbf{A}, \mathcal{D})$ has nonempty interior. Then there exists a set of translations $\mathcal{J} \subseteq \Delta(\mathbf{A}, \mathcal{D})$ such that $T + \mathcal{J}$ is a tiling of $\mathbf{R}^n$.*

Proof. The fundamental idea here is to repeatedly inflate the tile T at some interior point. Since $T^o \neq \emptyset$, by (1.2) there exists an interior point $x_0 \in T^o$ that has a finite radix expansion

$$x_0 = \sum_{j=1}^{N} \mathbf{A}^{-j} d_j^*, \quad \text{each } d_j^* \in \mathcal{D}.$$

Let $\tilde{T} = T - x_0$ and $\tilde{\mathcal{D}} := \mathcal{D}_{\mathbf{A},N} - \mathbf{A}^N x_0$. Then $0 \in \tilde{\mathcal{D}}$ and $\mathbf{A}^N(\tilde{T}) = \tilde{T} + \tilde{D}$. Iterations yield that for all $k \geq 1$,

$$\mathbf{A}^{Nk}(\tilde{T}) = \tilde{T} + \tilde{\mathcal{D}}_{\mathbf{A}^N,k}. \tag{2.2}$$

Because 0 is in the interior of $\tilde{T}$, any ball $B_r(0)$ will be covered by $\mathbf{A}^{Nk}(\tilde{T})$ for sufficiently large k. Furthermore, $\tilde{\mathcal{D}}_{\mathbf{A}^N,k} \subseteq \tilde{\mathcal{D}}_{\mathbf{A}^N,k+1}$ because $0 \in \tilde{\mathcal{D}}$. Hence $\tilde{T}$ tiles $\mathbf{R}^n$ by translates of $\mathcal{J} := \tilde{\mathcal{D}}_{\mathbf{A}^N,\infty}$, which implies that T tiles $\mathbf{R}^n$ by translates of $\mathcal{J}$. Now clearly we have $\mathcal{J} \subseteq \Delta(\mathbf{A}, \mathcal{D})$, proving the theorem. ∎

An immediate corollary of Theorem 2.3 is that if $\mathbf{A} \in M_n(\mathbf{Z})$ and $\mathcal{D} \subset \mathbf{Z}^n$, then we may find a $\mathcal{J} \subseteq \mathbf{Z}^n$ such that $T(\mathbf{A}, \mathcal{D}) + \mathcal{J}$ is a tiling of $\mathbf{R}^n$, provided that $T(\mathbf{A}, \mathcal{D})$ has nonempty interior.

3 Integral Self-Affine Tiles

A particular class of self-affine tiles is the so-called *integral self-affine tiles*, where $\mathbf{A} \in M_n(\mathbf{Z})$ and $\mathcal{D} \in \mathbf{Z}^n$. The integrality allows us to establish many more properties about the tile $T(\mathbf{A}, \mathcal{D})$. In some applications, such as orthonormal wavelet bases, one encounters only integral self-affine tiles. Moreover a large class of self-affine tiles, including all self-affine tiles in the one dimension, are affinely equivalent to integral self-affine tiles, see Kenyon [29], and Lagarias and Wang [34].

Let $\mathbf{A}$ be an expanding matrix in $M_n(\mathbf{Z})$ and $\mathcal{D} \subset \mathbf{Z}^n$ with $|\mathcal{D}| = |\det(\mathbf{A})|$. Associated to the pair $(\mathbf{A}, \mathcal{D})$ is the smallest $\mathbf{A}$-invariant sublattice of $\mathbf{Z}^n$ containing the difference set $\mathcal{D} - \mathcal{D}$, which we denote by $\mathbf{Z}[\mathbf{A}, \mathcal{D}]$. If $0 \in \mathcal{D}$ then

$$\mathbf{Z}[\mathbf{A}, \mathcal{D}] = \mathbf{Z}[\mathcal{D}, \mathbf{A}(\mathcal{D}), \ldots, \mathbf{A}^{n-1}(\mathcal{D})]. \tag{3.1}$$

This follows from the Hamilton-Cayley Theorem that $\mathbf{A}^n \in \mathbf{Z}[\mathbf{A}^0, \mathbf{A}^1, \ldots, \mathbf{A}^{n-1}]$.

We call a digit set $\mathcal{D}$ *primitive* (with respect to $\mathbf{A}$) if $\mathbf{Z}[\mathbf{A}, \mathcal{D}] = \mathbf{Z}^n$, and we also call the associated tile $T(\mathbf{A}, \mathcal{D})$ a *primitive tile.* Most of the questions we consider here can be reduced to the case of primitive tiles.

Lemma 3.1 *Let* $\mathbf{A}$ *be an expanding matrix in* $M_n(\mathbf{Z})$ *and* $\mathcal{D} \subset \mathbf{Z}^n$ *with* $|\mathcal{D}| = |\det(\mathbf{A})|$. *Suppose that* $0 \in \mathcal{D}$ *and* $\mathbf{Z}[\mathbf{A}, \mathcal{D}] = \mathbf{B}(\mathbf{Z}^n)$ *for some* $\mathbf{B} \in M_n(\mathbf{Z})$. *Then there is an expanding matrix* $\tilde{\mathbf{A}} \in M_n(\mathbf{Z})$ *and a primitive digit set* $\tilde{\mathcal{D}} \subset \mathbf{Z}^n$ *with respect to* $\tilde{\mathbf{A}}$, $|\tilde{\mathcal{D}}| = |\det(\tilde{\mathbf{A}})|$, *such that*

$$T(\mathbf{A}, \mathcal{D}) = \mathbf{B}(T(\tilde{\mathbf{A}}, \tilde{\mathcal{D}})). \tag{3.2}$$

Proof. Since $\mathbf{Z}[\mathbf{A}, \mathcal{D}] = \mathbf{B}(\mathbf{Z}^n)$ is $\mathbf{A}$-invariant, $\mathbf{A}\mathbf{B}(\mathbf{Z}^n) \subseteq \mathbf{B}(\mathbf{Z}^n)$. Hence $\mathbf{A}\mathbf{B} = \mathbf{B}\tilde{\mathbf{A}}$ for some $\tilde{\mathbf{A}} \in M_n(\mathbf{Z})$. $\tilde{\mathbf{A}}$ is expanding because $\tilde{\mathbf{A}} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}$. Now $\mathcal{D} \subseteq \mathbf{B}(\mathbf{Z}^n)$, so $\mathcal{D} = \mathbf{B}(\tilde{\mathcal{D}})$ for some $\tilde{\mathcal{D}} \subset \mathbf{Z}^n$. Let $\tilde{T} := T(\tilde{\mathbf{A}}, \tilde{\mathcal{D}})$. It satisfies $\tilde{\mathbf{A}}(\tilde{T}) = \tilde{T} + \tilde{\mathcal{D}}$, so

$$\mathbf{A}(\mathbf{B}(\tilde{T})) = \mathbf{B}\tilde{\mathbf{A}}(\tilde{T}) = \mathbf{B}(\tilde{T} + \tilde{\mathcal{D}}) = \mathbf{B}(\tilde{T}) + \mathcal{D}.$$

The uniqueness yields $\mathbf{B}(\tilde{T}) = T$. ∎

Theorem 3.2 *Let* $\mathbf{A} \in M_n(\mathbf{Z})$ *be expanding and* $\mathcal{D} \subset \mathbf{Z}^n$ *with* $|\mathcal{D}| = |\det(\mathbf{A})|$. *Then* $k := \mu(T(\mathbf{A}, \mathcal{D})) \in \mathbf{Z}$. *Furthermore,* $T(\mathbf{A}, \mathcal{D}) + \mathbf{Z}^n$ *is a perfect covering of* $\mathbf{R}^n$ *of multiplicity* k.

Proof. As before let $\pi_n : \mathbf{R}^n \longrightarrow \mathbf{T}^n$ be the canonical covering map. The integer matrix $\mathbf{A}$ induces an endormorphism $\mathbf{A}_* : \mathbf{T}^n \longrightarrow \mathbf{T}^n$ defined by $\mathbf{A}_* := \pi_n \circ \mathbf{A} \circ \pi_n^{-1}$. Let $\nu : \mathbf{T}^n \longrightarrow \mathbf{Z}$ denote the function $\nu(z) := |\pi_n^{-1}(z) \cap T|$ where $T := T(\mathbf{A}, \mathcal{D})$. Since T is compact, there exists a finite $k \in \mathbf{Z}$ such that

$$k = \max\left\{l \in \mathbf{Z} : \ \nu(z) \geq l \text{ for almost all } z \in \mathbf{T}^n\right\}.$$

Now there exist disjoint (up to measure zero sets) fundamental domains $F_1, F_2, \ldots, F_k$ of the lattice $\mathbf{Z}^n$ such that each $F_j \subseteq T$. Denote $F = \cup_{j=1}^k F_j$ and $\Omega = T \setminus F$. We show that $\Omega_* := \pi_n(\Omega)$ is invariant under $\mathbf{A}_*$.

To see this, note that

$$\mathbf{A}_*(\Omega_*) = \pi_n \circ \mathbf{A}(\Omega) = \pi_n\Big((T + \mathcal{D}) \setminus \bigcup_{j=1}^{k} \mathbf{A}(F_j)\Big). \tag{3.3}$$

Let $z_0 \in \mathbf{T}^n \setminus \Omega_*$. Then z_0 is covered exactly k times under $\pi_n : T \longrightarrow \mathbf{T}^n$; so it is covered exactly $k|\mathcal{D}|$ times under $\pi_n : T + \mathcal{D} \longrightarrow \mathbf{T}^n$. However, z_0 is also covered $|k\det(\mathbf{A})| = k|\mathcal{D}|$ times under $\cup_{j=1}^k \mathbf{A}(F_j)$ times because each F_j is a fundamental domain of $\mathbf{Z}^n$ and $\mathbf{A} \in M_n(\mathbf{Z})$. So $z_0 \notin \mathbf{A}_*(\Omega_*)$ by (3.3). This yields $\mathbf{A}(\Omega_*) \subseteq \Omega_*$. By the ergodicty of $\mathbf{A}_*$ the set Ω_* have measure zero or is all of $\mathbf{T}^n$. But the latter is ruled out by the definition of k. Therefore $\mu(\Omega) = 0$, and $\nu(z) = k$ for almost all $z \in \mathbf{T}^n$. This proves $\mu(T) = k \in \mathbf{Z}$ and $T + \mathbf{Z}^n$ is a perfect covering of $\mathbf{R}^n$ of multiplicity k. ∎

As we will see in §4, a Haar-type orthonormal wavelet basis can be constructed from an integral self-affine tile $T(\mathbf{A}, \mathcal{D})$ with $\mu(T(\mathbf{A}, \mathcal{D}) = 1$. In this case $T(\mathbf{A}, \mathcal{D})$ tiles $\mathbf{R}^n$ by $\mathbf{Z}^n$-translations. The following is a necessary condition for it to hold.

Theorem 3.3 *Let* $\mathbf{A} \in M_n(\mathbf{Z})$ *be expanding and* $\mathcal{D} \subset \mathbf{Z}^n$ *with* $|\mathcal{D}| = |\det(\mathbf{A})|$. *Suppose that* $\mu(T(\mathbf{A}, \mathcal{D})) = 1$. *Then* $\mathcal{D}$ *is primitive and is a complete set of coset representatives of* $\mathbf{Z}^n/\mathbf{A}(\mathbf{Z}^n)$.

Proof. We project $T := T(\mathbf{A}, \mathcal{D})$ onto the n-torus $\mathbf{T}^n$ by π_n. By Theorem 2.3 there exists a $\mathcal{J} \subseteq \mathbf{Z}^n$ such that $T + \mathcal{J}$ is a tiling of $\mathbf{R}^n$. Since $\mu(T) = 1$, $\mathcal{J} = \mathbf{Z}^n$. Hence $\pi_n(T) = \mathbf{T}^n$. Now $\mathbf{A}(T) = T + \mathcal{D}$ yields

$$\mathbf{T}^n = \pi_n(T) = \bigcup_{d \in \mathcal{D}} \left(\tilde{T}_* + \pi_n(\mathbf{A}^{-1} d) \right), \quad \text{where } \tilde{T}_* := \pi_n(\mathbf{A}^{-1}(T)).$$

Since the measure of $\tilde{T}_*$ is at most $1/|\mathcal{D}|$, all $\pi_n(\mathbf{A}^{-1} d)$ must be distinct in $\mathbf{T}^n$. This shows that $\mathcal{D}$ must be a complet set of coset representatives of $\mathbf{Z}^n / \mathbf{A}(\mathbf{Z}^n)$.

The primitiveness of $\mathcal{D}$ follows directly from Lemma 3.1. ∎

The converse of Theorem 3.3 is true in the one dimension (§4) but is false for $n \geq 2$. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad \mathcal{D} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}.$$

Then $\mathcal{D}$ is a primitive complete set of coset representatives of $\mathbf{Z}^2 / \mathbf{A}(\mathbf{Z}^2)$. However, $\mu(T(\mathbf{A}, \mathcal{D}))$ has Lebesgue measure 3, see [37].

In the above example the tile $T(\mathbf{A}, \mathcal{D})$ tiles $\mathbf{R}^2$ by lattice translates, using the lattice $3\mathbf{Z} \oplus \mathbf{Z}$. In general we have:

Theorem 3.4 *Let $\mathbf{A} \in M_n(\mathbf{Z})$ be expanding and $\mathcal{D}$ be a complete set of coset representatives of $\mathbf{Z}^n / \mathbf{A}(\mathbf{Z}^n)$. Then there exists a full rank lattice $\mathcal{L} \subseteq \mathbf{Z}^n$ such that $T(\mathbf{A}, \mathcal{D})$ tiles $\mathbf{R}^n$ by $\mathcal{L}$-translations.*

For the proof of Theorem 3.4 we refer the readers to Conz, Hervè and Raugi [7] or Lagarias and Wang [37].

4 Digit Sets of Integral Self-Affine Tiles

Although Theorem 2.2 (d) provides a necessary and sufficient condition for $T(\mathbf{A}, \mathcal{D})$ to be a tile, the condition itself is rather difficult to verify. In this section we explicitly classify integral digit sets $\mathcal{D}$ that result in tiles for certain types of expanding matrices $\mathbf{A} \in M_n(\mathbf{Z})$.

Theorem 4.1 *Let p be a prime and $\mathcal{D} \subset \mathbf{Z}$ be a primitive digit set with $|\mathcal{D}| = |p|$. Then $T(\mathbf{A}, \mathcal{D})$ is a tile if and only if $\mathcal{D}$ is a complete set of residues (mod p).*

Proof. The sufficency is already established. We prove the necessity. Without loss of generality we assume that $0 \in \mathcal{D}$ and $d \geq 0$ for all $d \in \mathcal{D}$. Let $f_{\mathcal{D}}(z)$ denote the characteristic polynomial $f_{\mathcal{D}}(z) := \frac{1}{|p|} \sum_{d \in \mathcal{D}} z^d$. We prove that there exists a $k \geq 1$ such that the cyclotomic polynomial $F_{p^k}(z) := \frac{z^{p^k} - 1}{z^{p^{k-1}} - 1}$ divides $f_{\mathcal{D}}(z)$.

Let $m_{\mathcal{D}}(\xi) := f_{\mathcal{D}}(e^{i 2\pi \xi})$. Note that the characteristic function of $T := T(\mathbf{A}, \mathcal{D})$ satisfies

$$\chi_T(x) = \sum_{d \in \mathcal{D}} \chi_T(px - d). \tag{4.1}$$

Taking the Fourier transform yields $\widehat{\chi}_T = m_{\mathcal{D}}(p^{-1}\xi)\widehat{\chi}_T(p^{-1}\xi)$. By iteration,

$$\widehat{\chi}_T(\xi) = c\prod_{j=1}^{\infty} m_{\mathcal{D}}(p^{-j}\xi), \qquad \text{where } c = \widehat{\chi}_T(0) = \mu(T). \tag{4.2}$$

The convergence of the infinite product (4.2) is well known. By Theorem 3.2, $T + \mathbf{Z}$ is a perfect covering of $\mathbf{R}$ of multiplicity $\mu(T) \in \mathbf{Z}$, so $\widehat{\chi}_T(l) = 0$ for all nonzero integer l. In particular $\widehat{\chi}_T(1) = 0$. By (4.2) there exists some integer $k \geq 1$ such that $m_{\mathcal{D}}(p^{-k}) = 0$. Hence $f_{\mathcal{D}}(e^{i2\pi p^{-k}}) = 0$, proving that $F_k(z)|f_{\mathcal{D}}(z)$ and hence $(z^{p^k} - 1)|f_{\mathcal{D}}(z)(z^{p^{k-1}} - 1)$.

Observe that if two integers satisfy $j_1 \equiv j_2 \pmod{p^k}$ then $z^{j_1} \equiv z^{j_2} \pmod{(z^{p^k} - 1)}$. Because

$$f_{\mathcal{D}}(z)(z^{p^{k-1}} - 1) = \sum_{d\in\mathcal{D}+p^{k-1}} z^d - \sum_{d\in\mathcal{D}} z^d \equiv 0 \pmod{(z^{p^k} - 1)},$$

and because a nonzero polynomial of degree less than p^k can never be divisible by $z^{p^k} - 1$, we must have

$$\mathcal{D} + p^{k-1} \pmod{p^k} = \mathcal{D} \pmod{p^k}. \tag{4.3}$$

$p^{k-1} \in \mathcal{D} + p^{k-1}$, so $d \equiv p^{k-1} \pmod{p^k}$ for some $d \in \mathcal{D}$. Similarly now $2p^{k-1} \in \mathcal{D} + p^{k-1} \pmod{p^k}$, so $2p^{k-1} \in \mathcal{D} \pmod{p^k}$. This argument yields

$$\mathcal{D} \equiv \{0, p^{k-1}, 2p^{k-1}, \ldots, (p-1)p^{k-1}\} \pmod{p^k}.$$

But $\mathcal{D}$ is primitive, so $\gcd\{d : d \in \mathcal{D}\} = 1$. Therefore $k = 1$ and $\mathcal{D}$ is a complete set of residues $\pmod{p}$. ∎

The above theorem was due to Kenyon [29]. The same argument can be used to prove the following generalization, a proof of which can be found in Lagarias and Wang [35].

Theorem 4.2 *Let $\mathbf{A} \in M_n(\mathbf{Z})$ be expanding such that $|\det(\mathbf{A})| = p$ is a prime and $p\mathbf{Z}^n \not\subseteq \mathbf{A}^2(\mathbf{Z}^n)$. Let $\mathcal{D} \subset \mathbf{Z}^n$ with $|\mathcal{D}| = |\det(\mathbf{A})|$ be primitive. Then $T(\mathbf{A}, \mathcal{D})$ is a tile if and only if $\mathcal{D}$ is a set of complete coset representatives of $\mathbf{Z}^n/\mathbf{A}(\mathbf{Z}^n)$.*

It should be pointed out that the classification of digit sets for a given matrix $\mathbf{A}$ is in general very difficult, even in the integral case. This is evident from the fact that even for $\mathbf{A} = [6]$ in the one dimension it is not completely known what digit sets $\mathcal{D}$ result in self-affine tiles. The only other cases in which all digit sets resulting in self-affine tiles are classified are $\mathbf{A} = 2I$ for $n = 2$ ([29]) and $\mathbf{A} = [p^k]$ for $n = 1$, where p is a prime ([35]).

So far we have discussed mostly digit sets that are complete set of coset representatives of $\mathbf{Z}^n/\mathbf{A}(\mathbf{Z}^n)$. Naturally one may ask whether there are other types of digit sets $\mathcal{D}$ that also give self-affine tiles. Here is a simple example:

Example 4.1. Let $\mathbf{A} = [4]$ and $\mathcal{D} = \{0, 1, 8, 9\}$. Clearly $\mathcal{D}$ is primitive and is not a complete set of residues $\pmod{4}$. But one may check directly that $T(\mathbf{A}, \mathcal{D}) = [0, 1] \cup [2, 3]$. ∎

Example 4.1 is an example of a class of digit sets called *product form digit sets*. Suppose that $0 \in \mathcal{E}$ is a set of complete coset representatives of $\mathbf{Z}^n/\mathbf{A}(\mathbf{Z}^n)$, and suppose that it has a factorization

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_r, \qquad \text{where } 0 \in \mathcal{E}_i \text{ and } |\mathcal{E}| = \prod_{i=1}^{r} |\mathcal{E}_i|. \tag{4.4}$$

A digit set $\mathcal{D}$ has the product-form if

$$\mathcal{D} := \mathbf{A}^{f_1}(\mathcal{E}_1) + \mathbf{A}^{f_2}(\mathcal{E}_2) + \cdots + \mathbf{A}^{f_r}(\mathcal{E}_r) \tag{4.5}$$

for some integers $0 \le f_1 \le f_2 \le \cdots \le f_r$.

Theorem 4.3 *Let $\mathbf{A} \in M_n(\mathbf{Z})$ be expanding and $\mathcal{D}$ is the product-form digit set defined in (4.5). Then $T(\mathbf{A}, \mathcal{D})$ is a measure-disjoint union of translates of $T(\mathbf{A}, \mathcal{E})$, and*

$$\mu(T(\mathbf{A}, \mathcal{D})) = \mu(T(\mathbf{A}, \mathcal{E})) \prod_{i=1}^{r} |\mathcal{E}_i|^{f_i}. \tag{4.6}$$

Proof. Let $\mathcal{A}_{i,k} := \{\sum_{j=0}^{k-1} \mathbf{A}^j e_{i,j} : \text{all } e_{i,j} \in \mathcal{E}_i\}$ with $\mathcal{A}_{i,0} = \{0\}$. We prove that $T(\mathbf{A}, \mathcal{D}) = T(\mathbf{A}, \mathcal{E}) + \mathcal{A}$ where

$$\mathcal{A} := \mathcal{A}_{1,f_1} + \mathcal{A}_{2,f_2} + \cdots + \mathcal{A}_{r,f_r}.$$

$T(\mathbf{A}, \mathcal{D}) = \{\sum_{j=0}^{\infty} \mathbf{A}^{-j} d_j : \text{all } d_j \in \mathcal{D}\}$ from (1.2), and by assumption $d_j = \sum_{i=0}^{r} \mathbf{A}^{f_i} e_{i,j}$ where $e_{i,j} \in \mathcal{E}_i$. So

$$\begin{aligned}
\sum_{j=1}^{\infty} \mathbf{A}^{-j} d_j &= \sum_{j=1}^{\infty} \mathbf{A}^{-j} \sum_{i=0}^{r} \mathbf{A}^{f_i} e_{i,j} \\
&= \sum_{i=0}^{r} \Big(\sum_{j=f_i}^{\infty} \mathbf{A}^{-j} \mathbf{A}^{f_i} e_{i,j} + \sum_{j=1}^{f_i} \mathbf{A}^{-j} \mathbf{A}^{f_i} e_{i,j} \Big) \\
&= \sum_{j=0}^{\infty} \mathbf{A}^{-j} \Big(\sum_{i=0}^{r} e_{i,j+f_i} \Big) \sum_{i=0}^{r} \sum_{j=1}^{f_i} \mathbf{A}^{f_i - j} e_{i,j}.
\end{aligned} \tag{4.7}$$

Since $\sum_{i=0}^{r} e_{i,j+f_i} \in \mathcal{E}$ and $\sum_{i=0}^{r} \sum_{j=1}^{f_i} \mathbf{A}^{f_i-j} e_{i,j} \in \mathcal{A}$, we have $\sum_{j=0}^{\infty} \mathbf{A}^{-j} d_j \in T(\mathbf{A}, \mathcal{E}) + \mathcal{A}$; hence $T(\mathbf{A}, \mathcal{D}) \subseteq T(\mathbf{A}, \mathcal{E}) + \mathcal{A}$.

Conversely, one verifies that any element in $T(\mathbf{A}, \mathcal{E}) + \mathcal{A}$ must be in $T(\mathbf{A}, \mathcal{D})$ be reversing (4.7) (we omit the details here), yielding $T(\mathbf{A}, \mathcal{E}) + \mathcal{A} \subseteq T(\mathbf{A}, \mathcal{D})$. Therefore, $T(\mathbf{A}, \mathcal{D}) = T(\mathbf{A}, \mathcal{E}) + \mathcal{A}$.

We still need to show that the translates of $T(\mathbf{A}, \mathcal{E})$ in $T(\mathbf{A}, \mathcal{E}) + \mathcal{A}$ are measure-disjoint. For any $m \ge 1$ we have

$$\mathbf{A}^m(T(\mathbf{A}, \mathcal{E})) = T(\mathbf{A}, \mathcal{E}) + \mathcal{E}_{\mathbf{A},m}, \tag{4.8}$$

where $\mathcal{E}_{\mathbf{A}<m} := \{\sum_{k=0}^{m-1} \mathbf{A}^k e_k : \text{all } e_k \in \mathcal{E}\}$. Since each $\mathcal{E}_i \subseteq \mathcal{E}$ and $0 \in \mathcal{E}$, $\mathcal{A} \subseteq \mathcal{E}_{\mathbf{A},m}$ whenever $m \ge f_r$. But the translates of $T(\mathbf{A}, \mathcal{E})$ in (4.8) are measure-disjoint, it follows that the translates of $T(\mathbf{A}, \mathcal{E})$ in $T(\mathbf{A}, \mathcal{E}) + \mathcal{A}$ must be measure-disjoint.

Finally, all expansions $\sum_{i=0}^{r} \sum_{j=0}^{f_i-1} \mathbf{A}^j e_{i,j}$ where $e_{i,j} \in \mathcal{E}_i$ in $\mathcal{A}$ are distinct because $\mathcal{E} = \mathcal{E}_1 + \cdots + \mathcal{E}_r$ is a direct sum by (4.4). The measure-disjointness of $T(\mathbf{A}, \mathcal{E}) + \mathcal{A}$ yields (4.6). ∎

The digit set $\mathcal{D} = \{0, 1, 8, 9\}$ in Example 4.1 is a product-form digit set, with $\mathcal{E} = \{0, 1, 2, 3\} = \{0, 1\} + \{0, 2\}$ and $\mathcal{D} = \{0, 1\} + 4\{0, 2\}$. There are integral self-affine tiles whose digit sets are not product-form digit sets, see [35]. One simple such example is $\mathbf{A} = [4]$, $\mathcal{D} = \{0, 1, 8, 25\}$. Can you prove that $T(\mathbf{A}, \mathcal{D})$ is a tile?

5 Haar-Type Wavelet Bases of $L^2(\mathbf{R}^n)$

Let $\psi_1(x), \ldots, \psi_r(x) \in L^2(\mathbf{R}^n)$ and $\mathbf{A} \in M_n(\mathbf{Z})$ be expanding. Suppose that

$$\left\{ |\det(\mathbf{A})|^{\frac{m}{2}} \psi_i(\mathbf{A}^m x - \alpha) : \ \alpha \in \mathbf{Z}^n, 1 \le i \le r, \ m \in \mathbf{Z} \right\}$$

is an orthonormal basis of $L^2(\mathbf{R}^n)$. Then we call this basis a *wavelet basis* of $L^2(\mathbf{R}^n)$, and $\psi_1(x), \ldots, \psi_r(x)$ *wavelets*. The simpliest wavelet is the wavelet basis of $L^2(\mathbf{R})$ constructed by A. Haar [25], which has $\mathbf{A} = [2]$ and consists of a single wavelet

$$\psi(x) = \begin{cases} 1 & 0 \le x < 1/2 \\ -1 & 1/2 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

A popular way to construct wavelet bases is by *multiresolution analysis*. We shall not discuss the details here; a comprehensive discussion can be found in Daubechies [9]. Let $\mathbf{A} \in M_n(\mathbf{Z})$ be expanding. A *scaling function* (of a multiresolution analysis), from which a wavelet basis can be constructed, is a function $\phi(x) \in L^2(\mathbf{R}^n)$ such that

(i) $\phi(x)$ satisfies a dilation equation

$$\phi(x) = \sum_{\alpha \in \mathbf{Z}^n} c_\alpha \phi(\mathbf{A}x - \alpha). \tag{5.1}$$

(ii) $\{\phi(x - \alpha) : \alpha \in \mathbf{Z}^n\}$ is an orthonormal set in $L^2(\mathbf{R}^n)$, and $\int_{\mathbf{R}^n} \phi(x)\, dx \neq 0$.

A *Haar-type* wavelet basis is a one constructed from a scaling function of the form $\phi(x) = c\chi_\Omega(x)$ for some compact set $\Omega \subset \mathbf{R}^n$ and constant c. Gröchenig and Madych [23] established the following relation between Haar-type wavelet bases and self-affine tiles:

Theorem 5.1 *Let $\mathbf{A} \in M_n(\mathbf{Z})$ be expanding and $\Omega \subset \mathbf{R}^n$ be compact. Then the following are equivalent:*

(a) *$\phi(x) = c\chi_\Omega(x)$ is a scaling function with respect to $\mathbf{A}$ for some constant c.*

(b) *There exists a set of complete coset representatives of $\mathbf{Z}^n/\mathbf{A}(\mathbf{Z}^n)$ such that $\mathbf{A}(\Omega) = \Omega + \mathcal{D}$ up to a measure zero set, and $\mu(\Omega) = 1$.*

Proof. (a)$\Rightarrow$(b). By assumption $\phi(x)$ satisfies some dilation equation

$$\phi(x) = \sum_{\alpha \in \mathbf{Z}^n} c_\alpha \phi(\mathbf{A}x - \alpha). \tag{5.2}$$

The orthonormality of $\{\phi(x-\alpha) : \alpha \in \mathbf{Z}^n\}$ implies that $\Omega+\alpha$, $\alpha \in \mathbf{Z}^n$, are measure-disjoint. By letting $y = \mathbf{A}x$ and rewritting (5.2) as

$$\chi_{\mathbf{A}(\Omega)}(y) = \sum_{\alpha \in \mathbf{Z}^n} c_\alpha \chi_{\Omega+\alpha}(y), \tag{5.3}$$

it yields $c_\alpha = 0$ or $c_\alpha = 1$.

Let $\mathcal{D} = \{\alpha : c_\alpha = 1\}$. Integrating (5.3) yields $|\mathcal{D}| = |\det(\mathbf{A})|$, and the measure-disjointness of $\Omega + \alpha$ in (5.3) implies that

$$\mathbf{A}(\Omega) = \bigcup_{d \in \mathcal{D}} (\Omega + d) = \Omega + \mathcal{D} \tag{5.4}$$

up to a measure zero set.

To show that $\mu(\Omega) = 1$, let $\pi_n : \mathbf{R}^n \longrightarrow \mathbf{T}^n$ be the canonical covering map and $\mathbf{A}_* := \pi_n \circ \mathbf{A} \circ \pi_n^{-1}$. By (5.4), $\mathbf{A}_*(\pi_n(\Omega)) = \pi_n(\Omega)$ up to a measure zero set. It follows from the ergodicity of $\mathbf{A}_*$ that $\pi_n(\Omega) = \mathbf{T}^n$ up to a measure zero set. Hence $\mu(\Omega) \geq 1$. But $\mu(\Omega) \leq 1$ because $\Omega + \alpha$, $\alpha \in \mathbf{Z}^n$, are measure-disjoint. Therefore $\mu(\Omega) = 1$.

(b)$\Rightarrow$(a). By the ergodicity argument above, $\Omega + \mathbf{Z}^n$ is a covering of $\mathbf{R}^n$ up to a measure zero set. Since $\mu(\Omega) = 1$, all $\Omega + \alpha$, $\alpha \in \mathbf{Z}^n$, are measure-disjoint. Hence $\phi(x) := \chi_\Omega(x)$ satisfies $\phi(x) = \sum_{d \in \mathcal{D}} \phi(\mathbf{A}x - d)$ and $\{\phi(x - \alpha) : \alpha \in \mathbf{Z}^n\}$ is an orthonormal system in $L^2(\mathbf{R}^n)$. So $\phi(x) = \chi_\Omega(x)$ is a scaling function. ∎

Remark. It can be shown that if a compact set Ω satisfies $A(\Omega) = \Omega + \mathcal{D}$ up to a measure zero set, then $\Omega \supseteq T(\mathbf{A}, \mathcal{D})$ and $\Omega = T(\mathbf{A}, \mathcal{D})$ up to a measure zero set. We omit the proof here.

Naturally we would like to know when will $\mu(T(\mathbf{A}, \mathcal{D})) = 1$ for any given $\mathbf{A}$ and $\mathcal{D}$. Theorem 3.3 states that $\mathcal{D}$ must be a primitive set of complete coset representatives of $\mathbf{Z}^n / \mathbf{A}(\mathbf{Z}^n)$. A counterexample is given in §3 to show that the converse of the theorem is false for $n \geq 2$. The converse, however, is true in the one dimension.

Theorem 5.2 *Let $q \in \mathbf{Z}$, $|q| > 1$ and $\mathcal{D}$ be a complete set of residues $\pmod q$. Suppose that $\mathcal{D}$ is primitive. Then $\mu(T([q], \mathcal{D})) = 1$.*

Proof. We present the following Fourier analytic proof, due to Gröchenig and Haas [22]. The technique here is valuable for studying other type of scaling functions as well.

Without loss of generality we assume that $0 \in \mathcal{D}$. The primitiveness of $\mathcal{D}$ is equivalent to $\gcd\{d : d \in \mathcal{D}\} = 1$.

Let $m_{\mathcal{D}}(\xi) := \frac{1}{|q|} \sum_{d \in \mathcal{D}} e^{i2\pi d\xi}$. Key to the proof is the following linear transition operator

$$C_{\mathcal{D}}(f)(\xi) = \sum_{l=0}^{|q|-1} \left| m_{\mathcal{D}}(q^{-1}(\xi + l)) \right|^2 f(q^{-1}(\xi + l)) \tag{5.5}$$

defined on the space of $\mathbf{Z}$-periodic functions. Let

$$g_{\mathcal{D}}(\xi) := \sum_{k \in \mathbf{Z}} \mu(T \cap (T + k)) e^{i2\pi k\xi} \tag{5.6}$$

where $T := T([q], \mathcal{D})$. One easily checks (see Gröchenig [21]), using the assumption that $\mathcal{D}$ is a complete set of residues $\pmod q$, that

$$C_{\mathcal{D}}(1) = 1, \qquad C_{\mathcal{D}}(g_{\mathcal{D}}) = g_{\mathcal{D}}. \tag{5.7}$$

Assume that $\mu(T) > 1$. Then $g_{\mathcal{D}}(\xi)$ is not a constant, hence the set

$$Z_{\mathcal{D}} := \left\{ \xi : \ g_{\mathcal{D}}(\xi) = \min_{\eta \in \mathbf{R}} g_{\mathcal{D}}(\eta) \right\}$$

is a nonempty discrete $\mathbf{Z}$-periodic set, and $Z_{\mathcal{D}} \cap \mathbf{Z} = \emptyset$ by (5.6). Fix an $\xi_0 \in Z_{\mathcal{D}}$. By (5.7)

$$g_{\mathcal{D}}(\xi_0) = \sum_{l=0}^{|q|-1} \left| m_{\mathcal{D}}(q^{-1}(\xi_0 + l)) \right|^2 g_{\mathcal{D}}(q^{-1}(\xi_0 + l)). \tag{5.8}$$

But $\sum_{l=0}^{|q|-1} |m_{\mathcal{D}}(q^{-1}(\xi_0 + l))|^2 = 1$ by $C_{\mathcal{D}}(1) = 1$, so $g_{\mathcal{D}}(q^{-1}(\xi_0 + l)) = g_{\mathcal{D}}(\xi_0)$ whenever $m_{\mathcal{D}}(q^{-1}(\xi_0 + l)) \neq 0$. In particular there exists an l_1 such that $\xi_1 := q^{-1}(\xi_0 + l_1) \in Z_{\mathcal{D}}$. Note that $q\xi_1 \equiv \xi_0 \pmod 1$.

Now let $\widehat{Z}_{\mathcal{D}} := Z_{\mathcal{D}} \pmod 1$. So for any $\widehat{\xi}_0 \in \widehat{Z}_{\mathcal{D}}$ there exists a $\widehat{\xi}_1 \in \widehat{Z}_{\mathcal{D}}$ such that $q\widehat{\xi}_1 = \widehat{\xi}_0$. But $\widehat{Z}_{\mathcal{D}}$ is finite. Hence the map $\widehat{\xi} \mapsto q\widehat{\xi}$ is a permutation on $\widehat{Z}_{\mathcal{D}}$.

Back to (5.8). There exists no $l_2 \neq l_1$ in the sum such that $\xi_2 := q^{-1}(\xi_0 + l_2) \in Z_{\mathcal{D}}$ because otherwise $q\xi_1 \equiv q\xi_2 \pmod 1$ while $\xi_1 \not\equiv \xi_2 \pmod 1$, contradicting the fact that $\widehat{\xi} \mapsto q\widehat{\xi}$ is a permutation on $\widehat{Z}_{\mathcal{D}}$. Hence $m_{\mathcal{D}}(q^{-1}(\xi_0 + l)) = 0$ for all $0 \leq l \leq |q| - 1$, $l \neq l_1$. This means

$$\left| m_{\mathcal{D}}(q^{-1}(\xi_0 + l_1)) \right|^2 |m_{\mathcal{D}}(\xi_1)|^2 = 1. \tag{5.9}$$

Because $0 \in \mathcal{D}$, (5.9) is possible only if $e^{i2\pi d\xi_1} = 1$, and hence $d\xi_1 \in \mathbf{Z}$ for all $d \in \mathcal{D}$. But $\xi_1 \notin \mathbf{Z}$, it follows that $\gcd\{d : d \in \mathcal{D}\} > 1$, a contradiction. ∎

Theorem 5.2 generalizes to higher dimensions only in special cases. One such case is when $\mathbf{A} \in M_n(\mathbf{Z})$ is *irreducible*, which means that the characteristic polynomial of $\mathbf{A}$ is irreducible in $\mathbf{Q}[z]$.

Theorem 5.3 *Let $\mathbf{A} \in M_n(\mathbf{Z})$ be an expanding irreducible matrix, and $\mathcal{D}$ be a primitive set of complete coset representatives of $\mathbf{Z}^n/\mathbf{A}(\mathbf{Z}^n)$. Then $\mu(T(\mathbf{A}, \mathcal{D})) = 1$.*

A proof can be found in [37]. It uses a deep result of Cerveau, Conze, and Raugi [6] characterizing the set of zeros of certain trigonometric polynomials. In the case of reducible $\mathbf{A}$, $\mu(T(\mathbf{A}, \mathcal{D})) > 1$ only when the digit set has the so called *quasi-product form*, see [37].

Another interesting question is: For a given expanding $\mathbf{A} \in M_n(\mathbf{Z})$, is it always possible to construct Haar-type wavelet basis? In other words, is it always possible to find a digit set $\mathcal{D}$ such that $\mu(T(\mathbf{A}, \mathcal{D})) = 1$? The answer is clearly affirmative in the one dimension as a result of Theorem 5.2. The answer is known to be affirmative in dimensions $n = 2,\ 3$ ([22], [33], [36], [37]). In dimension n, Haar-type wavelet bases exist if $|\det(\mathbf{A})| > n$. But what if $|\det(\mathbf{A})| \leq n$, for example, $|\det(\mathbf{A})| = 2$? The question becomes intriguing, because in this case the digit set $\mathcal{D}$ consists of only two digits. Since we may assume that $0 \in \mathcal{D}$, we have in reality the freedom to choose for only one digit. If the dimension is large, it is not clear we can always choose this digit so that $\mathcal{D}$ is primitive, i.e. $\mathbf{Z}[\mathbf{A}, \mathcal{D}] = \mathbf{Z}^n$. Although no counterexample has been found yet, it is almost certain that they exist. This problem has a surprising connection to algebraic number theory, see Lagarias and Wang [36].

Figure 1: The Fractal Red Cross

6 Boundaries of Self-Affine Tiles

An important problem in fractal geometry is to find the Hausdorff dimension of a fractal set. Since a tile by definition has positive Lebesgue measure, its Hausdorff dimension is simply the dimension of the space in which it resides. A more interesting problem is to find the Hausdorff dimension of the boundary of a self-affine tile.

Getting the exact Hausdorff dimension of a fractal set is tricky in general. This had been the case for the boundaries of self-affine tiles. Boundaries of several well known tiles, such as the Gosper Flake (Gardner [18]) or the Fractal Red Cross (Strichartz [50]), were studied and their exact Hausdorff dimension derived. We illustrate how the Hausdorff dimensions of the boundaries of some self-affine tiles can be computed by the example of the Fractal Red Cross (Figure 1) in [50], which is the self-affine tile $T := T(\mathbf{A}, \mathcal{D})$ with

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad \mathcal{D} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}.$$

By Figure 1, the boundary of $\mathbf{A}(T)$ consists of 12 pieces, each congruent to a quarter of the boundary of T. Note that there are 5×4 quarter boundaries in the five translates of

T in $\mathbf{A}(T)$, but 8 of them are overlaps that become part of the interior of $\mathbf{A}(T)$. So in fact $\partial(\mathbf{A}(T)) = \mathbf{A}(\partial T)$ consists of $20-8=12$ quarter boundaries of T. Let $\mathcal{H}^s$ denote the s-dimensional Hausdorff measure. Then $\mathcal{H}^s(\mathbf{A}(\partial T)) = 3\mathcal{H}^s(\partial T)$. On the other hand, $\mathcal{H}^s(\mathbf{A}(\partial T)) = 5^{s/2}\mathcal{H}^s(\partial T)$ by the scaling property of $\mathcal{H}^s$. For $\mathcal{H}^s(\partial T)$ to be finite and nonzero we must have $5^{s/2}=3$. So $s = 2\log_5 3$ is the dimension of ∂T.

The above method can be made rigorous. The drawback is that it depends fundamentally on the visualization of the tiles, making it useful only on a case by case basis. For many self-affine tiles, the method either does not work, or requires ingenuity to work.

In this section we outline a method for finding the exact Hausdorff dimension of the boundaries of integral self-affine tiles. It employs the same basic idea, but requires no visualization of the tiles and works in *all* cases where the expanding matrix $\mathbf{A} \in M_n(\mathbf{Z})$ is similar to a similarity.

Let $T := T(\mathbf{A}, \mathcal{D})$ be an integral self-affine tile with $\mu(T) = 1$. Because T tiles $\mathbf{R}^n$ by $\mathbf{Z}^n$-translations and T is the closure of its interior,

$$\partial T = \bigcup_{\alpha \in \mathbf{Z}^n \setminus \{0\}} T \cap (T+\alpha). \tag{6.1}$$

Denote $B_\alpha := T \cap (T+\alpha)$ for all $\alpha \in \mathbf{Z}^n \setminus \{0\}$. Of course there are only finitely many nonemtpy B_α's. Let $\mathcal{K}_0 = \{\alpha \in \mathbf{Z}^n \setminus \{0\} : B_\alpha \neq \emptyset\}$. To find the Hausdorff dimension of ∂T we utilize the fact that $\{B_\alpha : \alpha \in \mathcal{K}_0\}$ form a *self-similar system*; more precisely,

$$\begin{aligned} \mathbf{A}(B_\alpha) &= \mathbf{A}(T) \cap \mathbf{A}(T+\alpha) \\ &= (T+\mathcal{D}) \cap (T+\mathcal{D}+\mathbf{A}\alpha) \\ &= \bigcup_{d,d' \in \mathcal{D}} \left(B_{\mathbf{A}\alpha+d'-d} + d\right). \end{aligned} \tag{6.2}$$

Now, label elements in $\mathcal{K}_0$ as $\mathcal{K}_0 = \{\alpha_1, \alpha_2, \ldots, \alpha_J\}$ and define

$$\mathcal{E}_{i,j} := \{d \in \mathcal{D} :\ \mathbf{A}\alpha_i + d' - d \in \mathcal{K}_0 \text{ for some } d' \in \mathcal{D}\}. \tag{6.3}$$

Let $B_i := B_{\alpha_i}$ for $1 \le i \le J$. Then we may rewrite (6.2) as

$$\mathbf{A}(B_i) = \bigcup_{j=1}^{J} (B_j + \mathcal{E}_{i,j}), \qquad 1 \le i \le J. \tag{6.4}$$

Theorem 6.1 *Let $T := T(\mathbf{A}, \mathcal{D})$ be an integral self-affine tile with $\mu(T(\mathbf{A}, \mathcal{D})) = 1$. Suppose that $\mathbf{A}$ is similar to a similarity. Then*

$$\underline{\dim}_B(\partial T) = \overline{\dim}_B(\partial T) = \dim_H(\partial T) = \frac{n \log \rho(\mathbf{M})}{\log|\det(\mathbf{A})|}, \tag{6.5}$$

where $\mathbf{M} := [|\mathcal{E}_{i,j}|]_{J\times J}$ and $\rho(\mathbf{M})$ is its specrtal radius.

We call the matrix $\mathbf{M} = [|\mathcal{E}_{i,j}|]$ the *substitution matrix* of the boundary of T.

To prove Theorem 6.1 we first observe that iterating (6.4) yields

$$\mathbf{A}^N(B_i) = \bigcup_{j=1}^{J} (B_j + \mathcal{E}^N_{i,j}), \qquad 1 \le i \le J, \tag{6.6}$$

where for all $1 \le i, j \le 1$,

$$\mathcal{E}_{i,j}^N = \bigcup_{k=1}^{J} \Big(\mathbf{A}(\mathcal{E}_{i,k}^{N-1}) + \mathcal{E}_{k,j} \Big), \qquad \mathcal{E}_{i,j}^1 := \mathcal{E}_{i,j}. \tag{6.7}$$

Lemma 6.2 $[|\mathcal{E}_{i,j}^N|] = \mathbf{M}^N$ *for all* $N \ge 1$.

Proof. We prove the lemma by induction on N. The lemma is clearly true for $N = 1$. Assume that it holds for $N - 1$; we show that it also holds for N.

Observe that $\mathcal{E}_{i,j} \subseteq \mathcal{D}$ and $\mathcal{D}$ is a complete set of coset representatives of $\mathbf{Z}^n / \mathbf{A}(\mathbf{Z}^n)$. Hence

$$\Big| \mathbf{A}(\mathcal{E}_{i,k}^{N-1}) + \mathcal{E}_{k,j} \Big| = |\mathcal{E}_{i,k}^{N-1}||\mathcal{E}_{k,j}|. \tag{6.8}$$

We shall establish that for all $k \ne l$,

$$\Big(\mathbf{A}(\mathcal{E}_{i,k}^{N-1}) + \mathcal{E}_{k,j} \Big) \cap \Big(\mathbf{A}(\mathcal{E}_{i,l}^{N-1}) + \mathcal{E}_{l,j} \Big) = \emptyset. \tag{6.9}$$

This is clear if we can show that $\mathcal{E}_{k,j} \cap \mathcal{E}_{l,j} = \emptyset$ because this will mean the two sets have no elements in a same coset of $\mathbf{Z}^n / \mathbf{A}(\mathbf{Z}^n)$. Assume that $d \in \mathcal{E}_{k,j} \cap \mathcal{E}_{l,j}$. Then there exist d_1, $d_2 \in \mathcal{D}$ such that

$$\mathbf{A}\alpha_k + d_1 - d = \mathbf{A}\alpha_l + d_2 - d = \alpha_j.$$

So $\mathbf{A}(\alpha_k - \alpha_l) = d_2 - d_1$, contradicting $k \ne l$. This proves (6.9).

It now follows from (6.7) and (6.8) that

$$|\mathcal{E}_{i,j}^N| = \sum_{k=1}^{J} |\mathcal{E}_{i,k}^{N-1}||\mathcal{E}_{k,j}|,$$

proving the lemma. ∎

Proof of Theorem 6.1. Let $\lambda := \rho(\mathbf{M})$. Since $\mathbf{M}$ is a nonnegative matrix, there is a nonnegative eignevector v associated to λ. Without loss of generality we assume that $v_1 > 0$ and that $\mathbf{A}$ is a similarity. Let $s := n \log \rho(\mathbf{M}) / \log |\det(\mathbf{A})|$. We divide the proof into three parts.

(I) $\underline{\dim}_B(\partial T) \ge s$.

Let $C_i(\varepsilon)$ denote the least number of ε-cubes needed to cover T_i. Observe that

$$\mathbf{A}^N(B_1) = \bigcup_{j=1}^{J} (B_j + \mathcal{E}_{1,j}^N) \supseteq B_1 + \mathcal{E}_{1,1}^N,$$

hence

$$B_1 \supseteq \mathbf{A}^{-N}(B_1) + \mathbf{A}^{-N}(\mathcal{E}_{1,1}^N). \tag{6.10}$$

This means at least $|\mathcal{E}_{1,1}^N|$ ε_N-cubes are needed to cover B_1, where $\varepsilon_N := |\det(\mathbf{A})|^{-\frac{N}{n}}$. Hence

$$C_1(\varepsilon_N) \ge |\mathcal{E}_{1,1}^N|.$$

Now for any sufficiently small $\varepsilon > 0$ there exists an $N > 0$ such that $\varepsilon_{N+1} < \varepsilon \le \varepsilon_N$. So

$$\frac{\log C_1(\varepsilon)}{-\log \varepsilon} \ge \frac{\log C_1(\varepsilon_N)}{-\log \varepsilon_{N+1}} \ge \frac{n \log |\mathcal{E}^N_{1,1}|}{(N+1)\log|\det(\mathbf{A})|}.$$

It is well known that $\lim_{N\to\infty} \frac{\log|\mathcal{E}^N_{1,1}|}{N} = \log \lambda$. This yields

$$\underline{\dim}_B(\partial T) \ge \underline{\dim}_B(B_1) \ge s.$$

(II) $\overline{\dim}_B(\partial T) \le s$.

Let $\delta_0 := \max\{2\mathrm{diam}(B_j) : 1 \le j \le J\}$ and $\delta_N := |\det(\mathbf{A})|^{-\frac{N}{n}}$. Observe that each B_j can be covered by a single δ_0-cube. The iteration

$$\mathbf{A}^N(B_i) = \bigcup_{j=1}^{J} (B_j + \mathcal{E}^N_{i,j}), \qquad 1 \le i \le J$$

yields

$$C_i(\delta_N) \le \sum_{j=1}^{J} \left|\mathcal{E}^N_{i,j}\right|.$$

Note that for each $1 \le i \le J$,

$$\limsup_{N\to\infty} \frac{\log\left(\sum_{j=1}^{J} \left|\mathcal{E}^N_{i,j}\right|\right)}{-\log \delta_N} \le \frac{n \log \lambda}{\log|\det(\mathbf{A})|} = s.$$

The same techniques employed in (I) immediately gives (II).

(III) $\dim_H(\partial T) = s$.

$\dim_H(\partial T) = s$ follows easily from Falconer [16], Theorem 3.1 and 3.2. Details can be found in [51]. ∎

Example 6.1 One of the best known self-affine tile is the Twin Dragon, which is given by

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \mathcal{D} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

In this example, one can show ([51]) that $\mathcal{K}_0 = \{e_1, -e_1, e_2, -e_2, e_1 - e_2, e_2 - e_1\}$ and the substitution matrix is

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial is $f(\lambda) = (\lambda^3 - \lambda^2 - 2)(\lambda^3 + \lambda^2 - 2)$. Hence by Theorem 6.1,

$$\dim_H(\partial T) = 2\log_2 \lambda_0$$

where λ_0 is the largest root of $\lambda^3 - \lambda^2 - 2$.

In general, the set $\mathcal{K}_0$ for any given self-affine tile can be found via a "pruning algorithm," see [51]. One can also obtain a priori a set $\mathcal{K}_1 \supseteq \mathcal{K}_0$ by estimating the diameter of the tile. It turns out that the substitution matrix obtained using $\mathcal{K}_1$ will have the exactly same spectral radius as the substitution matrix from $\mathcal{K}_0$.

It should be pointed out that Duvall and Keesling [14] have recently computed the exact Hausdorff dimension of the boundary of the well known Lévy Dragon, using a rather different approach. The method in [14] can handle more general self-similar tiles, although typically requires much larger matrices (in the case of the Lévy Dragon it is a 752×752 matrix).

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