ON THE STRUCTURES OF GENERATING ITERATED FUNCTION SYSTEMS OF CANTOR SETS

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ABSTRACT. A generating IFS of a Cantor set F is an IFS whose attractor is F. For a given Cantor set such as the middle-3rd Cantor set we consider the set of its generating IFSs. We examine the existence of a minimal generating IFS, i.e., every other generating IFS of F is an iterating of that IFS. We also study the structures of the semi-group of homogeneous generating IFSs of a Cantor set F in $\mathbb{R}$ under the open set condition (OSC). If $\dim_H F < 1$ we prove that all generating IFSs of the set must have logarithmic commensurable contraction factors. From this Logarithmic Commensurability Theorem we derive a structure theorem for the semi-group of homogeneous generating IFSs of F under the OSC. We also examine the impact of geometry on the structures of the semi-groups. Several examples will be given to illustrate the difficulty of the problem we study.

1. INTRODUCTION

In this paper, a family of maps $\Phi = \{\phi_j\}_{j=1}^N$ in $\mathbb{R}^d$ is called an *iterated function system* (written as IFS for brevity) if for each j, $\phi_j(x) = \rho_j R_j(x) + a_j$ where $\rho_j \in \mathbb{R}$ with $0 < |\rho_j| < 1$, R_j is an orthogonal matrix and $a_j \in \mathbb{R}^d$. According to Hutchinsin [11], there is a unique non-empty compact $F = F_\Phi \in \mathbb{R}^d$, which is called the *attractor of* Φ, such that $F = \bigcup_{j=1}^N \phi_j(F)$.

It is well known that the standard middle-third Cantor set C is the attractor of the IFS $\{\phi_0, \phi_1\}$ where

$$(1.1) \qquad \phi_0(x) = \frac{1}{3}x, \qquad \phi_1(x) = \frac{1}{3}x + \frac{2}{3}.$$

A natural question is: Is it possible to express C as the attractor of another IFS?

Surprisingly, the general question whether the attractor of an IFS can be expressed as the attractor of another IFS, which seems a rather fundamental question in fractal geometry, has

hardly been studied, even for some of the best known Cantor sets such as the middle-third Cantor set.

A closer look at this question reveals that it is not as straightforward as it may appear. It is easy to see that for any given IFS $\{\phi_j\}_{j=1}^N$ one can always iterate it to obtain another IFS with identical attractor. For example, the middle-third Cantor set C satisfies

$$
\begin{aligned}
C &= \phi_0(C) \cup \phi_1(C) \\
  &= \phi_0 \circ \phi_0(C) \cup \phi_0 \circ \phi_1(C) \cup \phi_1(C) \\
  &= \phi_0 \circ \phi_0(C) \cup \phi_0 \circ \phi_1(C) \cup \phi_1 \circ \phi_0(C) \cup \phi_1 \circ \phi_1(C).
\end{aligned}
$$

Hence C is also the attractor of the IFS $\{\phi_0 \circ \phi_0, \phi_0 \circ \phi_1, \phi_1\}$ and the IFS $\{\phi_0 \circ \phi_0, \phi_0 \circ \phi_1, \phi_1 \circ \phi_0, \phi_1 \circ \phi_1\}$, as well as infinitely many other iterations of the original IFS $\{\phi_0, \phi_1\}$. The complexity doesn't just stop here. Since C is centrally symmetric, $C = -C + 1$, we also have

$$
C = \left(-\frac{1}{3}C + \frac{1}{3}\right) \cup \left(-\frac{1}{3}C + 1\right).
$$

Thus C is also the attractor of the IFS $\{-\frac{1}{3}x + \frac{1}{3}, -\frac{1}{3}x + 1\}$, or even $\{-\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}x + \frac{2}{3}\}$.

Definition 1.1. Let $\Phi = \{\phi_i\}_{i=1}^N$ and $\Psi = \{\psi_j\}_{j=1}^M$ be two IFSs. We say that Ψ is *derived* from Φ if for each $1 \le j \le M$, $\psi_j = \phi_{i_1} \circ \cdots \circ \phi_{i_k}$ for some $1 \le i_1, \ldots, i_k \le N$. We say that Ψ is an *iteration* of Φ if Ψ is derived from Φ, and it has the same attractor as Φ. Let F be a compact set in $\mathbb{R}^d$. A *generating IFS of F* is an IFS Φ whose attractor is F. A *generating IFS family of F* is a set $\mathcal{I}$ of generating IFSs of F. A generating IFS family $\mathcal{I}$ of F is said to have a *minimal element* $\Phi_0 \in \mathcal{I}$ if every $\Psi \in \mathcal{I}$ is an iteration of Φ_0.

Example 1.1. In this example consider the question raised by Mattila: Is it true that any self-similar subset F of the middle-third Cantor set C is trivial, in the sense that F has a generating IFS that is derived from the generating IFS $\{\phi_0, \phi_1\}$ of C given in (1.1)?

We give a negative answer here by constructing a counterexample. For now, let $\Phi = \{\frac{1}{9}x, \frac{1}{9}(x + \frac{2}{27}), \frac{1}{9}(x + \frac{2}{3^6})\}$. Then by looking at the ternary expansion of the elements in F_Φ it is easy to see that $F_\Phi \subset C$. But clearly Φ cannot be derived from the original IFS given in (1.1).

The objective of this paper is to study the existence of a minimal IFS in a generating IFS family of a self-similar set $F \subset \mathbb{R}$. To see the complexity of this problem, we first give the following example.

Example 1.2. Let F be the attractor of the IFS $\Phi = \{\frac{1}{10}(x+a) : a \in \mathcal{A}\}$ where $\mathcal{A} = \{0, 1, 5, 6\}$. Let $\mathcal{G}_F$ denotes the set of all generating IFSs of F. We claim that $\mathcal{G}_F$ does not contain a minimal element.

To see the claim, note that any ϕ in a generating IFS of F must map either to the left or to the right part of F, because the hole in the middle (having length $\text{diam}(F)/2$) would be too large for a subset of F to be similar to F. Thus ϕ must have contraction factor $\leq 1/4$. Assume that $\mathcal{G}_F$ contains a minimal element Φ_0, towards a contradiction. Then $\Phi_0 = \Phi$, because each map in Φ (with contraction factors $> 1/16$) cannot be a composition of two maps in Φ_0. However we can find another generating IFS of F given by

$$\Psi := \left\{ \frac{x}{100}, \frac{x+1}{100}, \frac{x+1/2}{10}, \frac{x+15}{100}, \frac{x+16}{100}, \frac{x+5}{10}, \frac{x+6}{10} \right\},$$

which can not be derived from Φ_0 since the map $\frac{x+1/2}{10}$ is not the composition of elements in Φ_0. It leads to a contradiction. To see that Ψ is a generating IFS of F, one can check that F satisfies the following relation:

$$
\begin{aligned}
F &= \frac{F + \{0, 1, 5, 6\}}{10} = \frac{F + \{0, 1, 5, 6, 10, 11, 15, 16\}}{100} \cup \frac{F + \{5, 6\}}{10} \\
&= \frac{F + \{0, 1, 15, 16\}}{100} \cup \frac{F + 1/2}{10} \cup \frac{F + \{5, 6\}}{10}.
\end{aligned}
$$

Naturally, one cannot expect the existence of a minimal IFS in a generating IFS family $\mathcal{I}$ of a set F to be the general rule — not without first imposing restrictions on $\mathcal{I}$ and F. But what are these restrictions? A basic restriction is the *open set condition* (OSC). Without the OSC either the existence of a minimal IFS is hopeless, or the problem appears rather intractable. But even with the OSC a compact set may have generating IFSs that superficially seem to bear little relation to one another. One such example is the unit interval $F = [0, 1]$. It is evident that other restrictions will be needed. We study this issue in this paper.

While the questions we study in the paper appear to be rather fundamental questions of fractal geometry in themselves, our study is also motivated by several questions in related areas. One of the well known questions in tiling is whether there exists a 2-reptile that is also a 3-reptile in the plane ([5]). Another question comes from the application of fractal geometry to image compression, see Barnsley [2], Lu [14] and Deliu, Geronimo and Shonkwiler [7]. In this application, finding a generating IFS of a given set plays the central role, and naturally, better compressions are achieved by choosing a minimal generating IFS. The

other question concerns the symmetry of a self-similar set such as the Sierpinski Gasket, see e.g. Bandt and Retta [4], Falconer and O'Conner [10] and Strichartz [19].

For any IFS Φ we shall use F_Φ to denote its attractor. We call an IFS $\Phi = \{\rho_j x + a_j\}_{j=1}^N$ *homogeneous* if all contraction factors ρ_j are identical. In this case we use ρ_Φ to denote the homogeneous contraction factor. We call Φ *positive* if all $\rho_j > 0$. A fundamental result concerning the structures of generating IFSs of a self-similar set is the Logarithmic Commensurability Theorem stated below. It is the foundation of many of our results in this paper.

Theorem 1.1 (**The Logarithmic Commensurability Theorem**)**.** *Let F be the attractor of a homogeneous IFS $\Phi = \{\phi_i = \rho x + t_i\}_{i=1}^N$ in $\mathbb{R}^d$ satisfying the OSC. Suppose that $\dim_H F = s < 1$. Let $\psi(x) = \lambda x + d$ such that $\psi(E) \subseteq F$ for some Borel subset E of F with positive s-dimensional Hausdorff measure, i.e., $\mathcal{H}^s(E) > 0$. Then $\log |\lambda| / \log |\rho_\Phi| \in \mathbb{Q}$, that means $|\lambda| = r^k$ and $|\rho| = r^m$ for some $r > 0$ and positive integers k, m.*

Note that the set of all homogeneous generating IFSs of a self-similar set F forms a semi-group. Let $\Phi = \{\phi_i\}_{i=1}^N$ and $\Psi = \{\psi_j\}_{j=1}^M$ be two generating IFSs of F. We may define $\Phi \circ \Psi$ by $\Phi \circ \Psi = \{\phi_i \circ \psi_j : 1 \leq i \leq N, \ 1 \leq j \leq M\}$. Then clearly $\Phi \circ \Psi$ is also a generating IFS of F.

Definition 1.2. *Let F be any compact set in $\mathbb{R}^d$. We shall use $\mathcal{I}_F$ to denote the set of all homogeneous generating IFSs of form $\{\rho x + a_i\}_{i=1}^N$ in $\mathbb{R}^d$ of F satisfying the OSC, augmented by the "identity" $Id = \{id(x) := x\}$. We shall use $\mathcal{I}_F^+$ to denote the set of all positive homogeneous generating IFSs of F in $\mathcal{I}_F$, augmented by the identity Id.*

We augment the Identity into $\mathcal{I}_F$ and $\mathcal{I}_F^+$ so that they are not empty. Clearly both $\mathcal{I}_F$ and $\mathcal{I}_F^+$, equipped with the composition as product, are semi-groups. If F is not the attractor of a homogeneous IFS with OSC then $\mathcal{I}_F$ is trivial. The Logarithmic Commensurability Theorem leads to the following structure theorem for $\mathcal{I}_F$ and $\mathcal{I}_F^+$:

Theorem 1.2. *Let F be a compact set in $\mathbb{R}^d$ with $\dim_H F < 1$. Then Both $\mathcal{I}_F$ and $\mathcal{I}_F^+$ are finitely generated Abelian semi-groups. Furthermore assume that there exists a $\Phi = \{\phi_i\}_{i=1}^N \in \mathcal{I}_F$ such that $\Phi \neq Id$ and N is not a power of another positive integer. Then the following two statements hold.*

(i) *If $\rho_\Phi > 0$ then Φ is the minimal element for $\mathcal{I}_F^+$, namely $\mathcal{I}_F^+ = < \Phi >:= \{\Phi^m : m \geq 0\}$. If $\rho_\Phi < 0$ then either $\mathcal{I}_F^+$ has a minimal element, or $\mathcal{I}_F^+ = < \Phi^2, \Psi >$ for some Ψ with $\rho_\Psi = \rho_\Phi^q$ where $q \in \mathbb{N}$ is odd and $\Psi^2 = \Phi^{2q}$.*

(ii) *Either $\mathcal{I}_F = < \Phi >$ or $\mathcal{I}_F = < \Phi, \Psi >$ for some Ψ with $\rho_\Psi = -\rho_\Phi^q$ where $q \in \mathbb{N}$ and $\Psi^2 = \Phi^{2q}$.*

Definition 1.3. Let $\Phi = \{\phi_j\}_{j=1}^N$ be an IFS in $\mathbb{R}$. We say Φ satisfies the *separation condition* (SC) if $\phi_i(F_\Phi) \cap \phi_j(F_\Phi) = \emptyset$ for all $i \neq j$. We say Φ satisfies the *convex open set condition* (COSC) if Φ satisfies the OSC with a convex open set.

The following is another main theorem in this paper:

Theorem 1.3. *Let $F \subset \mathbb{R}$ be a compact set with $\dim_H F < 1$ such that F is the attractor of a homogeneous IFS satisfying the COSC. Let Φ be any generating homogeneous IFS of F with the OSC. Then Φ also satisfies the COSC. Furthermore we have:*

(i) *The semi-group $\mathcal{I}_F^+$ has a minimal element Φ_0, namely $\mathcal{I}_F^+ = < \Phi_0 >$.*

(ii) *Suppose that F is not symmetric. Then $\mathcal{I}_F$ has a minimal element Φ_0, $\mathcal{I}_F = < \Phi_0 >$.*

(iii) *Suppose that F is symmetric. Then there exist Φ_+ and Φ_- in $\mathcal{I}_F$ with $\rho_{\Phi_+} = -\rho_{\Phi_-} > 0$ such that every $\Psi \in \mathcal{I}_F$ can be expressed as $\Psi = \Phi_+^m$ if $\rho_\Psi > 0$ and $\Psi = \Phi_+^m \circ \Phi_-$ if $\rho_\Psi < 0$ for some $m \geq 0$.*

In the following we give an example to show that the condition COSC in Theorem 1.3 cannot be replaced with the SC.

Example 1.3. Let F be the attractor of the IFS $\Phi = \{\frac{1}{16}(x + a) : a \in \mathcal{A}\}$ where $\mathcal{A} = \{0, 1, 64, 65\}$. It is not difficult to check that Φ satisfies the SC but does not satisfy the COSC. We prove that $\mathcal{I}_F^+$ does not contain a minimal element by contradiction. Assume this is not true. Let $\Phi_0 = \{\rho x + c_i\}_{i=1}^N$ be the minimal element of $\mathcal{I}_F^+$. By the dimension formula and Theorem 1.1, $\log \rho / \log 16^{-1} = \log N / \log 4 \in \mathbb{Q}$. Therefore $N = 2$ and $\rho = \frac{1}{4}$ or $N = 4$ and $\rho = \frac{1}{16}$. But it is easy to check that if $N = 2$ then the IFS Φ_0 must satisfy the COSC, but Φ does not, a contradiction to Theorem 1.3. Hence we must have $N = 4$ and hence $\Phi_0 = \Phi$ by Proposition 2.1. Now let $\Psi = \{\frac{1}{64}(x + b) : a \in \mathcal{B}\}$ where $q = 64$ and $\mathcal{B} = \{0, 1, 16, 17, 256, 257, 272, 273\}$. One can check directly $\mathcal{B} + q\mathcal{B} = \mathcal{A} + p\mathcal{A} + p^2\mathcal{A}$. Thus $\Psi^2 = \Phi^3$, which implies $\Psi \in \mathcal{I}_F$. However Ψ is not derived from Φ, which leads to a contradiction. Hence $\mathcal{I}_F^+$ does not contain a minimal element.

We organize the paper as follows. Due to the technical nature of the proof of the Logarithmic Commensurability Theorem we shall postpone it until §3. In §2 we prove Theorem 1.2 and Theorem 1.3. In §4 we determine all the generating IFSs for the standard middle-third Cantor set.

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2. Structures of the Semi-groups and the Convex Open Set Condition

In this section we prove Theorem 1.2 and Theorem 1.3, and examine the impact of geometry to the structures of the semi-groups $\mathcal{I}_F$ and $\mathcal{I}_F^+$. We first give two essential propositions.

Proposition 2.1. *Let $\Phi = \{\phi_i(x) := \rho R x + a_i\}_{i=1}^N$ and $\Psi = \{\psi_j(x) := \rho R x + b_j\}_{j=1}^M$ be two homogeneous IFSs in $\mathbb{R}^d$ satisfying the OSC, where R is an orthogonal matrix. If $F_\Phi = F_\Psi$, then $\Phi = \Psi$.*

Proof. Denote $F = F_\Phi = F_\Psi$. It is easy to see $N = M$ by comparing the Hausdorff dimension of F_Φ and F_Ψ. Let ν be the normalized s-dimensional Hausdorff measure restricted to F, where $s = \dim_H F$, i.e. $\nu = \frac{1}{\mathcal{H}^s(F)}\mathcal{H}^s$. It is well known that ν is the self-similar measure defined by Φ (as well as Ψ) with equal weights, i.e.

$$\nu = \frac{1}{N}\sum_{j=1}^N \nu \circ \phi_j^{-1} = \frac{1}{N}\sum_{j=1}^N \nu \circ \psi_j^{-1}.$$

Now taking the Fourier transform of ν and applying the self-similarity yield

$$\widehat{\nu}(\xi) = A(\xi)\widehat{\nu}(\rho R^{-1}\xi) = B(\xi)\widehat{\nu}(\rho R^{-1}\xi)$$

where $A(\xi) := \frac{1}{N}\sum_{j=1}^N e^{2\pi i a_j \xi}$ and $B(\xi) := \frac{1}{N}\sum_{j=1}^N e^{2\pi i b_j \xi}$. Since $\nu(\rho R^{-1}\xi) \neq 0$ on a neighborhood V of 0, $A(\xi) = B(\xi)$ on V. It implies $\{a_j\} = \{b_j\}$, proving the lemma. ∎

Proposition 2.2. *Let $\Phi = \{\phi_i(x) := \rho R x + a_i\}_{i=1}^N$ and $\Psi = \{\phi_i(x) := \lambda S x + b_j\}_{j=1}^M$ be two homogeneous IFSs in $\mathbb{R}^d$ satisfying the OSC, where R, S are two orthogonal matrices. Then $F_\Phi = F_\Psi$ if $\Phi \circ \Psi = \Psi \circ \Phi$. Conversely, if $RS = SR$ and $F_\Phi = F_\Psi$, then $\Phi \circ \Psi = \Psi \circ \Phi$.*

Proof. Suppose that $\Phi \circ \Psi = \Psi \circ \Phi$ then $\Phi \circ \Psi^m = \Psi^m \circ \Phi$ for any $m \in \mathbb{N}$. Therefore $\Phi \circ \Psi^m(F_\Psi) = \Psi^m \circ \Phi(F_\Psi)$. But $\Psi^m(E) \longrightarrow F_\Psi$ as $m \longrightarrow \infty$ in the Hausdorff metric for any compact set E. Taking limit we obtain $\Phi(F_\Psi) = F_\Psi$. Therefore $F_\Phi = F_\Psi$.

Conversely, if $F_\Phi = F_\Psi$ and $RS = SR$, then both $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are generating IFSs of F with the identical linear part, and both satisfy the OSC. Hence $\Phi \circ \Psi = \Psi \circ \Phi$ by Proposition 2.1. ∎

Proof of Theorem 1.2. By Proposition 2.2, $\mathcal{I}_F$ is Abelian, and so is $\mathcal{I}_F^+$. To see that $\mathcal{I}_F$ and $\mathcal{I}_F^+$ are finitely generated, we assume that $\mathcal{I}_F$ is non-trivial and fix an arbitrary $\Gamma = \{\gamma_k\}_{k=1}^M \in I_F$ with $\Gamma \neq Id$. Write $M = L^n$ where L is not a power of another positive integer. Denote $\rho = |\rho_\Gamma|^{\frac{1}{n}}$.

Suppose that $\Psi = \{\psi_j\}_{j=1}^J \in \mathcal{I}_F$ and $\Psi \neq Id$. Then the dimension formula $M = |\rho_\Gamma|^{-s}$ and $J = |\rho_\Psi|^{-s}$ where $s = \dim_H F$ implies that $\log M / \log J = \log |\rho_\Gamma| / |\rho_\Psi| \in \mathbb{Q}$, by Theorem 1.1. It follows that $J = L^m$ and $\rho_\Psi = \pm|\rho_\Gamma|^{\frac{m}{n}} = \pm\rho^m$ for some $m \in \mathbb{N}$.

Define $\mathcal{P}^+ = \{m : \rho^m = \rho_\Psi \text{ for some } \Psi \in \mathcal{I}_F\}$ and $\mathcal{P}^- = \{m : \rho^m = -\rho_\Psi \text{ for some } \Psi \in \mathcal{I}_F\}$. We will show that $\mathcal{I}_F^+$ is finitely generated. Set $a = \gcd(\mathcal{P}^+)$. Let $\Psi_1, \ldots, \Psi_n \in \mathcal{I}_F$ with $\rho_{\Psi_j} = \rho^{m_j}$ such that $\gcd(m_1, m_2, \ldots, m_n) = a$. By a standard result in elementary number theory every sufficiently large integer $ma \geq N_0$ can be expressed as $ma = \sum_{j=1}^n p_j m_j$ with $p_j \geq 0$. Thus every $\Psi \in \mathcal{I}_F^+$ with $\rho_\Psi = \rho^{ma}$, $ma \geq N_0$, can be expressed as $\Psi = \Pi_{j=1}^n \Psi_j^{p_j}$ since the two IFSs have the same contraction factor. Let $\{\Psi_{n+1}, \ldots, \Psi_K\} \subseteq \mathcal{I}_F^+$ consist of all elements $\Psi \in \mathcal{I}_F^+$ with $\rho_\Psi \geq \rho^{N_0}$ that are not already in $\{\Psi_1, \ldots, \Psi_n\}$. Then $\mathcal{I}_F^+ = <\Psi_1, \Psi_2, \ldots, \Psi_K>$, and it is finitely generated. The proof that $\mathcal{I}_F$ is finitely generated is virtually identical, and we omit it.

Now we turn to the proof of (i). Assume that $\Phi = \{\phi_i\}_{i=1}^N \in \mathcal{I}_F$ such that $\Phi \neq Id$ and N is not a power of another positive integer. Let $\Psi = \{\psi_j\}_{j=1}^J \in \mathcal{I}_F^+$ with $\Psi \neq Id$. Since N is not a power of another integer and $|\rho_\Phi|/\rho_\Psi \in \mathbb{Q}$, we have $J = N^m$ for some m, which implies that $\rho_\Psi = |\rho_\Phi|^m$. If $\rho_\Phi > 0$ then $\Psi = \Phi^m$ via Proposition 2.1 because they have the same contraction factor. Thus $\mathcal{I}_F^+ = <\Phi>$. Suppose that $\rho_\Phi < 0$. We have two cases: Either every $\Psi \in \mathcal{I}_F^+$ has $\rho_\Psi = |\rho_\Phi|^{2m'}$ for some m', or there exists a $\Psi \in \mathcal{I}_F^+$ with $\rho_\Psi = |\rho_\Phi|^m$ for some odd m. In the first case every $\Psi \in \mathcal{I}_F^+$ has $\Psi = (\Phi^2)^m$ again by Proposition 2.1. Hence $\mathcal{I}_F^+ = <\Phi^2>$. In the second case, let q be the smallest odd integer such that $\rho_{\Psi_0} = |\rho_\Phi|^q$ for some $\Psi_0 \in \mathcal{I}_F^+$. For any $\Psi \in \mathcal{I}_F^+$ we have $\rho_\Psi = |\rho_\Phi|^m$. If $m = 2m'$ then $\Psi = (\Phi^2)^{m'}$. If m is odd then $m \geq q$ and $m - q = 2m'$. Thus $\rho_\Psi = \rho_{\Phi^{2m'} \circ \Psi_0}$, and hence $\Psi = \Phi^{2m'} \circ \Psi_0$. It follows that $\mathcal{I}_F^+ = <\Phi^2, \Psi_0>$ with $\Psi_0^2 = (\Phi^2)^q$. This proves (i).

We next prove (ii), which is rather similar to (ii). Again, any $\Psi \in \mathcal{I}_F$ must have $\rho_\Psi = \pm|\rho_\Phi|^m$ for some m. If $\mathcal{I}_F = <\Phi>$ we are done. Otherwise there exists a $\Psi_0 \in \mathcal{I}_F$ such that

$\Psi_0 \not\in < \Phi >$ and it has the largest contraction factor in absolute value. Since $\rho_{\Psi_0} = \pm|\rho_\Phi|^q$ for some q, and $\Psi_0 \neq \Phi^q$, we must have $\rho_{\Psi_0} = -\rho_\Phi^q$. We show that $\mathcal{I}_F =< \Phi^2, \Psi_0 >$. For any $\Psi \in \mathcal{I}_F$ either $\Psi = \Phi^m$ for some m or $\rho_\Psi = -\rho_\Phi^m$. In the latter case $m \geq q$. So $\rho_\Psi = \rho_{\Phi^{m-q} \circ \Psi_0}$, implying that $\Psi = \Phi^{m-q} \circ \Psi_0$. Also it is clear $\Psi_0^2 = \Phi^{2q}$ because they have the same contraction factor. We have proved (ii). This finishes the proof of Theorem 1.2. ∎

In the remaining part of this section we prove Theorem 1.3. We shall first prove several results about the COSC.

Lemma 2.3. *Let $\Phi = \{\phi_j\}$ be an IFS in $\mathbb{R}$. Then Φ satisfies the COSC if and only if for all $i \neq j$ we have $\phi_i(x) \leq \phi_j(y)$ for all $x, y \in F_\Phi$ or $\phi_i(x) \geq \phi_j(y)$ for all $x, y \in F_\Phi$.*

Proof. It follows immediately from the definition of the COSC. ∎

Lemma 2.4. *Let Φ and Ψ be two homogeneous IFSs in $\mathbb{R}$ satisfying the OSC such that $\rho_\Phi = -\rho_\Psi$ and $F_\Phi = F_\Psi$. Assume that Φ satisfies the COSC. Then F_Φ must be symmetric.*

Proof. Let $\Phi = \{\phi_i(x) := \rho x + a_i\}_{i=1}^N$ and $\Psi = \{\psi_j(x) := -\rho x + b_j\}_{j=1}^M$. Then Ψ also satisfies the COSC by Theorem 1.3 (See the proof below; the proof of that part does not depend on this lemma). Without loss generality we assume that $\rho > 0$ and $a_1 < a_2 < \cdots < a_N$, $b_1 < b_2 < \cdots < b_M$. Denote $\mathcal{A} = \{a_i\}$ and $\mathcal{B} = \{b_j\}$. The OSC for Φ and Ψ now implies $\Phi^2 = \Psi^2$. Observe that

$$\Phi^2 = \{\rho^2 x + a_i + \rho a_j\}_{i,j=1}^N, \qquad \Psi^2 = \{\rho^2 x + b_i - \rho b_j\}_{i,j=1}^M.$$

It follows from the COSC for Φ^2 that the lexicographical order for $\{a_i + \rho a_j\}_{i,j=1}^N$ also yields a strictly increasing order for the set. Similarly, the lexicographical order for $\{b_i - \rho b_{M+1-j}\}_{i,j=1}^M$ also yields a strictly increasing order for the set. Therefore $M = N$ and $a_i + \rho a_j = b_i - \rho b_{N+1-j}$ for all i, j. Fix $j = 1$ yields $a_i = b_i + c$ for some constant c. Fix $i = 1$ yields $a_j = -b_{N+1-j} + c'$ for some constant c'. Thus $a_j = a_{N+1-j} + c''$ for some constant c''. Hence $\mathcal{A}$ is symmetric, which implies that F_Φ is symmetric. ∎

Proof of Theorem 1.3. We prove that Ψ satisfies the COSC if Φ does. By Theorem 1.1 there exist integers m, n such that $\rho_\Phi^n = \rho_\Psi^m$. It follows from Proposition 2.1 that $\Phi^n = \Psi^m$. Assume that Ψ does not satisfy the COSC. Then there exist $\psi_i, \psi_j \in \Psi$ so that $\psi_i(x) < \psi_j(y)$ and $\psi_i(z) > \psi_j(w)$ for some $x, y, z, w \in F$. If ρ_Ψ^{m-1} is positive, then the same inequalities will hold if we replace ψ_i, ψ_j by $\psi_1^{m-1} \circ \psi_i$ and $\psi_1^{m-1} \circ \psi_j$, respectively; otherwise

if ρ_Ψ^{m-1} is negative then the reverse inequalities will hold. But this is impossible because both $\psi_1^{m-1} \circ \psi_i$ and $\psi_1^{m-1} \circ \psi_j$ are in Ψ^m, and hence in Φ^n, which satisfies the COSC.

To prove the rest of the theorem we first prove the following claim.

Claim. Let Φ, Ψ be any two elements in $\mathcal{I}_F$ with $|\rho_\Phi| > |\rho_\Psi|$. Then there exists a $\Gamma \in \mathcal{I}_F$ such that $\Psi = \Phi \circ \Gamma$, where $\Phi \circ \Gamma := \{\phi \circ \gamma : \phi \in \Phi, \gamma \in \Gamma\}$.

Proof of Claim: Let $\Phi = \{\phi_i(x)\}_{i=1}^N$ and $\Psi = \{\psi_j(x)\}_{j=1}^M$. Since both Φ and Ψ satisfy the COSC, we may without loss of generality assume that $\phi_1(F) \leq \cdots \leq \phi_M(F)$ and $\psi_1(F) \leq \cdots \leq \psi_N(F)$, where $X \leq Y$ for two sets X and Y means $x \leq y$ for all $x \in X$ and $y \in Y$. Set $e = \min F$, $f = \max F$ and $F_0 = [e, f]$. Clearly each $\phi_i(F_0)$ (resp. $\Psi_j(F_0)$) is a sub-interval of F_0, with end points $\phi_i(a)$ and $\phi_i(b)$ (resp. $\psi_i(a)$ and $\psi_i(b)$). The COSC for Φ and Ψ now imply that $\phi_1(F_0) \leq \cdots \leq \phi_M(F_0)$ and $\psi_1(F_0) \leq \cdots \leq \psi_N(F_0)$.

It follows from Theorem 1.1 that $\frac{\log|\rho_\Phi|}{\log|\rho_\Psi|} = \frac{\log N}{\log M} = \frac{n}{m}$ for some positive integers m and n with $\gcd(m, n) = 1$. Thus $N^m = M^n$, or $N = M^{\frac{n}{m}}$. This forces $K = M^{\frac{1}{m}}$ to be an integer, for otherwise the co-primeness of n, m makes $N = M^{\frac{n}{m}}$ an irrational number. Therefore $M = K^m$ and $N = K^n$. In particular, $\frac{M}{N} = L \in \mathbb{N}$.

Now $\Phi^q = \Psi^r$ by Proposition 2.1, where $q = 2m$ and $r = 2n$. For each $\mathbf{i} = i_1 i_2 \cdots i_q \in \{1, \ldots, N\}^q$ denote $\phi_{\mathbf{i}} := \phi_{i_1} \circ \cdots \circ \phi_{i_q}$, and similarly define $\psi_{\mathbf{j}}$ for $\mathbf{j} \in \{1, \ldots, M\}^r$. Then $\Phi^q = \{\phi_{\mathbf{i}} : \mathbf{i} \in \{1, \ldots, N\}^q\}$ and $\Psi^r = \{\psi_{\mathbf{j}} : \mathbf{j} \in \{1, \ldots, M\}^r\}$. It is clear that both Φ^q and Ψ^q satisfy the COSC. We order the maps in Φ^q and Ψ^r according to the orders of $\phi_{\mathbf{i}}(F_0)$ and $\psi_{\mathbf{j}}(F_0)$ respectively. Then the first N^{q-1} maps in Φ^q are $\mathcal{J}_1 = \{\phi_{1\mathbf{i}'} : \mathbf{i}' \in \{1, \ldots, N\}^{q-1}\}$, while the first N^{q-1} maps in Ψ^r are $\mathcal{J}_2 = \{\psi_{j_1\mathbf{j}'} : 1 \leq j_1 \leq L, \mathbf{j}' \in \{1, \ldots, M\}^{r-1}\}$. Therefore $\mathcal{J}_1 = \mathcal{J}_2$. Note that

$$\bigcup_{\varphi \in \mathcal{J}_1} \varphi(F) = \phi_1(F), \qquad \bigcup_{\varphi \in \mathcal{J}_2} \varphi(F) = \bigcup_{j=1}^L \psi_j(F).$$

It follows that $F = \bigcup_{j=1}^L \phi_1^{-1} \circ \psi_j(F)$, so $\Gamma_1 := \{\phi_1^{-1} \circ \psi_j\}_{j=1}^L$ is a generating IFS for F. It clearly satisfies the COSC.

We can continue the same argument by counting the next N^{q-1} elements in the two sequences. This yields $F = \bigcup_{j=L+1}^{2L} \phi_2^{-1} \circ \psi_j(F)$, so $\Gamma_2 := \{\phi_2^{-1} \circ \psi_j\}_{j=L+1}^{2L}$ is a generating IFS for F. Continuing to the end yields $\Gamma_1, \ldots, \Gamma_N$ in $\mathcal{I}_F$, with the property that

$$(2.1) \qquad \{\psi_j : (k-1)L + 1 \leq j \leq kL\} = \{\phi_k \circ \varphi : \varphi \in \Gamma_k\}.$$

But all Γ_k are equal because they have the same contraction factor. It follows from (2.1) that $\Psi = \Phi \circ \Gamma$, with $\Gamma := \Gamma_k$. This proves the Claim. ∎

To prove part (i) of the theorem, let $\Phi_0 \in \mathcal{I}_F^+$ have the largest contraction factor. Such a Φ_0 exists because for any $\Phi \in \mathcal{I}_F^+$ we must have $\rho_\Phi = N^{-1/\dim_H(F)}$ for some positive integer N. Now any $\Phi \neq \Phi_0$ in $\mathcal{I}_F^+$ we have $\rho_\Phi < \rho_{\Phi_0}$. By the Claim, $\Phi = \Phi_0 \circ \Gamma_1$ for some $\Gamma_1 \in \mathcal{I}_F^+$. If $\Gamma_1 = \Phi_0$ then $\Phi = \Phi_0^2$, and we finish the proof. If not then $\rho_\Gamma < \rho_{\Phi_0}$, yielding $\Gamma_1 = \Phi_0 \circ \Gamma_2$ for some $\Gamma_2 \in \mathcal{I}_F^+$. Apply the Claim recursively, and the process will eventually terminate. Hence $\Phi = \Phi_0^k$ for some k. The proof of part (i) is now complete.

To prove part (ii) of the theorem, if $\mathcal{I}_F = \mathcal{I}_F^+$ then there is nothing we need to prove. Assume that $\mathcal{I}_F \neq \mathcal{I}_F^+$. Let $\mathcal{I}_F^- \subset \mathcal{I}_F$ consisting of all homogeneous IFSs with negative contraction factors, and $\Phi_- \in \mathcal{I}_F^-$ have the largest contraction factor in absolute value. Let $\Phi_+ \in \mathcal{I}_F^+$ have the largest contraction factor in $\mathcal{I}_F^+$. If $|\rho_{\Phi_-}| = \rho_{\Phi_+}$ then F is symmetric by Lemma 2.4, a contradiction. So $|\rho_{\Phi_-}| \neq \rho_{\Phi_+}$. Note that $\Phi_-^2 = \Phi_+^m$ for some m by part (i). Thus $m = 1$ or $m > 2$. If $m > 2$ then $\rho_{\Phi_+} > |\rho_{\Phi_-}|$. Following the Claim we have $\Phi_- = \Phi_+ \circ \Gamma$ for some $\Gamma \in \mathcal{I}_F$. But $\rho_\Gamma < 0$ and $|\rho_\Gamma| > |\rho_{\Phi_-}|$. This is a contradiction. Therefore $m = 1$ and $\Phi_-^2 = \Phi_+$. Part (ii) of the theorem follows from part (i) and the Claim.

Finally we prove (iii). If F is symmetric, then for any IFS $\Psi \in \mathcal{I}_F$ there is another $\Psi' \in \mathcal{I}_F$ such that $\rho_\Psi = -\rho_{\Psi'}$ because $F = -F + c$ for some c. Let Φ_+ and Φ_- be the elements in $\mathcal{I}_F$ whose contraction factors have the largest absolute values, $\rho_{\Phi_+} = -\rho_{\Phi_-} > 0$. Proposition 2.1 and the same argument to prove part (i) now easily apply to prove that for any $\Psi \in \mathcal{I}_F$, $\Psi = \Phi_+^m$ if $\rho_\Psi > 0$ and $\Psi = \Phi_+^m \circ \Phi_-$ if $\rho_\Psi < 0$ for some $m \in \mathbb{N}$. ∎

3. Logarithmic Commensurability of Contraction Factors

In this section we prove Theorem 1.1. We first prove several lemmas.

Lemma 3.1. *Let F be the attractor of an IFS $\Phi = \{\phi_i\}_{i=1}^N$ in $\mathbb{R}^d$ satisfying the OSC. Denote $s = \dim_H F$. Let $E \subseteq F$ be a Borel set with $\mathcal{H}^s(E) > 0$. Then for any $\epsilon > 0$, there exist $k \in \mathbb{N}$ and $\mathbf{i} \in \{1, \ldots, N\}^k$ such that $\mathcal{H}^s(\phi_\mathbf{i}(F) \cap E) \geq (1 - \epsilon)\mathcal{H}^s(\phi_\mathbf{i}(F))$.*

Proof. By the classical density theorems for s-sets (see, e.g., [9, Corollary 2.5-2.6]),

$$\limsup_{r \to 0} \frac{\mathcal{H}^s(E \cap U_r(x))}{(2r)^s} = \limsup_{r \to 0} \frac{\mathcal{H}^s(F \cap U_r(x))}{(2r)^s} \geq 2^{-s} \quad \text{for } \mathcal{H}^s\text{-a.e. } x \in E,$$

where $U_r(x)$ denotes the open ball of center x and radius r. It together with $E \subseteq F$ yields

$$\limsup_{r \to 0} \frac{\mathcal{H}^s(E \cap U_r(x))}{\mathcal{H}^s(F \cap U_r(x))} = 1 \quad \text{for } \mathcal{H}^s\text{-a.e. } x \in E.$$

Hence for given $\epsilon > 0$, there exists an open ball $U_r(x)$ such that $U_r(x) \cap E \neq \emptyset$ and

$$(3.1) \qquad\qquad \mathcal{H}^s(E \cap U_r(x)) \geq (1 - \epsilon)\mathcal{H}^s(F \cap U_r(x)).$$

Set $\mathcal{A}_k = \{\mathbf{i} \in \{1, \ldots, N\}^k : \phi_{\mathbf{i}}(F) \subset U_r(x)\}$ for $k \in \mathbb{N}$. Then $F \cap U_r(x) = \lim_{k \to \infty} \bigcup_{\mathbf{i} \in \mathcal{A}_k} \phi_{\mathbf{i}}(F)$ and $E \cap U_r(x) = \lim_{k \to \infty} \bigcup_{\mathbf{i} \in \mathcal{A}_k} \phi_{\mathbf{i}}(F) \cap E$. Since Φ satisfies the OSC, we have

$$\mathcal{H}^s(E \cap U_r(x)) = \lim_{k \to \infty} \sum_{\mathbf{i} \in \mathcal{A}_k} \mathcal{H}^s(\phi_{\mathbf{i}}(F) \cap E), \quad \mathcal{H}^s(F \cap U_r(x)) = \lim_{k \to \infty} \sum_{\mathbf{i} \in \mathcal{A}_k} \mathcal{H}^s(\phi_{\mathbf{i}}(F)).$$

These two equalities together with (3.1) yield the desired existence result of the lemma. ∎

Lemma 3.2. *Let F be the attractor of a homogeneous IFS $\Phi = \{\phi_i = \rho x + t_i\}_{i=1}^N$ in $\mathbb{R}^d$ satisfying the OSC. Denote $s = \dim_H F$. Assume that $\psi(x) = \lambda x + e$ is a map such that $\psi(E) \subseteq F$ for some Borel set $E \subseteq F$ with $\mathcal{H}^s(E) > 0$. Then there exists a map $\xi(x) = \lambda \rho^m x + c$ such that $m \in \mathbb{N}$ and $\xi(F) \subseteq F$.*

Proof. By Lemma 3.1, for each $n \in \mathbb{N}$ we can choose an integer k_n and a word $\mathbf{i}_n \in \{1, \ldots, N\}^{k_n}$ such that

$$(3.2) \qquad\qquad \mathcal{H}^s(E \cap \phi_{\mathbf{i}_n}(F)) \geq (1 - 1/n)\mathcal{H}^s(\phi_{\mathbf{i}_n}(F)).$$

Since $\psi(E) \subseteq F$, we have $\psi(E \cap \phi_{\mathbf{i}_n}(F)) \subseteq F$ and thus

$$(3.3) \qquad \phi_{\mathbf{i}_n}^{-1}(\psi(E \cap \phi_{\mathbf{i}_n}(F)) \subseteq \phi_{\mathbf{i}_n}^{-1}(F) = \bigcup_{\mathbf{j} \in \{1, \ldots, N\}^{k_n}} \phi_{\mathbf{i}_n}^{-1} \phi_{\mathbf{j}}(F).$$

Denote $F_n = \phi_{\mathbf{i}_n}^{-1}(E \cap \phi_{\mathbf{i}_n}(F))$. Then the set in the left-hand side of (3.3) can be written as $\lambda F_n + c_n$ for some $c_n \in \mathbb{R}^d$, whilst the set in the right-hand side can be written as $\bigcup_{\mathbf{j} \in \{1, \ldots, N\}^{k_n}} (F + d_{\mathbf{j}})$ with $d_{\mathbf{j}} \in \mathbb{R}^d$. Since Φ satisfies the OSC, these $d_{\mathbf{j}}$'s are uniformly discrete, i.e., there exists a $\delta > 0$ independent of n such that $|d_{\mathbf{j}} - d_{\mathbf{j}'}| > \delta$ for $\mathbf{j} \neq \mathbf{j}'$. Hence we have

$$(3.4) \qquad\qquad \lambda F_n \subseteq \bigcup_{\mathbf{j} \in \{1, \ldots, N\}^{k_n}} (F + t_{\mathbf{j}})$$

where $t_{\mathbf{j}} = d_{\mathbf{j}} - c_n$.

Note that $F_n \subset F$, and $\lim_{n \to \infty} \mathcal{H}^s(F_n) = \mathcal{H}^s(F)$ by (3.2). Due to (3.4) and the fact that $t_{\mathbf{j}}$'s are uniformly discrete, a compactness argument yields that there exists a closed set $\widetilde{F} \subseteq F$ with $\mathcal{H}^s(\widetilde{F}) = \mathcal{H}^s(F)$ and finitely many vectors $a_1, \ldots, a_\ell \in \mathbb{R}^d$ such that

$\lambda \widetilde{F} \subseteq \bigcup_{k=1}^{\ell}(F + a_k)$. Since Φ satisfies the OSC, the above conditions for $\widetilde{F}$ guarantee that $\widetilde{F} = F$. Thus we have

$$(3.5) \qquad \lambda F \subset \bigcup_{k=1}^{\ell}(F + a_k).$$

Without loss of generality, we assume that the above ℓ is the minimal one. Then there exists $x \in F$ such that $\lambda x \in (F + a_1) \backslash (\bigcup_{k=2}^{\ell}(F + a_k))$. Since $\lambda x \notin \bigcup_{k=2}^{\ell}(F + a_k)$, there exists a large integer m and a word $\mathbf{i} \in \{1, \dots N\}^m$ such that $x \in \phi_{\mathbf{i}}(F)$ and $\lambda \phi_{\mathbf{i}}(F) \cap \bigcup_{k=2}^{\ell}(F + a_k) = \emptyset$. According to (3.5), we have $\lambda \phi_{\mathbf{i}}(F) \subseteq F + a_1$, which implies the desired result of the lemma by setting $\xi(x) = \lambda \phi_{\mathbf{i}}(x) - a_1$. ∎

Lemma 3.3. *Let F be the attractor of a homogeneous IFS $\Phi = \{\phi_i = \rho x + t_i\}_{i=1}^{N}$ in $\mathbb{R}^d$ satisfying the OSC with $s = \dim_H F < 1$. Then there exist a closed ball $B_r(x)$ of center x and radius r, and a positive integer k, such that $B_r(x) \cap F \neq \emptyset$ and the following two statements hold:*

(i) *$F \cap \{y \in \mathbb{R}^d : r - |\rho|^k u \le |y - x| \le r + |\rho|^k u\} = \emptyset$, where $u = \mathrm{diam} F$.*

(ii) *Denote $\mathcal{A} = \{\mathbf{i} \in \{1, \cdots, N\}^k : \phi_{\mathbf{i}}(F) \subset B_r(x)\}$ and $M = \#\mathcal{A}$. Then*

$$(3.6) \qquad (M + 1/2) \, |\rho|^{ks} \, \mathcal{H}^s(F) > d_{\max} \, (2r)^s,$$

where $d_{\max}$ is defined by

$$(3.7) \qquad d_{\max} = \sup\{\mathcal{H}^s(F \cap B_{r'}(y))/(2r')^s : y \in \mathbb{R}^d, r' > 0\}.$$

Proof. Since $0 < s < 1$, using L'Hospital's rule we have

$$\lim_{x \to 0} \frac{(1 + hx)^s - 1}{x^s} = 0, \qquad \forall \, h > 0.$$

Therefore there exist $\ell \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$(3.8) \qquad \frac{1}{2} \rho^{\ell s} \mathcal{H}^s(F) - \varepsilon > d_{\max} \left((1 + 8 \, \rho^{\ell-1} \mathrm{diam} F)^s - 1 \right).$$

By the definition of $d_{\max}$ there exists a ball $B_{r'}(x)$ such that $B_{r'}(x) \cap F \neq \emptyset$ and

$$\mathcal{H}^s(F \cap B_{r'}(x)) \ge (d_{\max} - \varepsilon) \, (2r')^s.$$

We may furthermore assume that $\mathcal{H}^s(\partial B_{r'}(x)) = 0$. Let n be the integer so that $\rho^{n+1} < 2r' \le \rho^n$. Then we have

$$
\begin{aligned}
\mathcal{H}^s(F \cap B_{r'}(x)) + \frac{1}{2}\rho^{(n+\ell)s}\mathcal{H}^s(F) \;&>\; (d_{\max} - \varepsilon)\,(2r')^s + \frac{1}{2}\rho^{\ell s}\mathcal{H}^s(F)\,(2r')^s \\
&\ge\; \left(d_{\max} - \varepsilon + \frac{1}{2}\rho^{\ell s}\,\mathcal{H}^s(F)\right)(2r')^s \\
&\ge\; d_{\max}\left(1 + 8\,\rho^{\ell-1}\mathrm{diam}F\right)^s (2r')^s \qquad \text{(by (3.8))} \\
&\ge\; d_{\max}\left(2r' + 8\,\rho^{\ell-1}(2r')\mathrm{diam}F\right)^s \\
&\ge\; d_{\max}\left(2r' + 8\,\rho^{\ell+n}\mathrm{diam}F\right)^s.
\end{aligned}
$$

That is

$$(3.9) \qquad \mathcal{H}^s(F \cap B_{r'}(x)) + \frac{1}{2}|\rho|^{(n+\ell)s}\mathcal{H}^s(F) > d_{\max}\left(2r' + 8\,|\rho|^{\ell+n}\mathrm{diam}F\right)^s.$$

Define $k = \ell + n$ and $r = r' + 2|\rho|^k u$, where $u = \mathrm{diam}\,F$. In the following we show that the statements (i) and (ii) hold for $B_r(x)$ and k.

Assume that (i) is not true, towards a contradiction. Then

$$F \cap \left\{y \in \mathbb{R}^d : r' + |\rho|^k u \le |y - x| \le r' + 3|\rho|^k u\right\} \ne \emptyset.$$

Therefore there exists at least one $\mathbf{i} \in \{1, \ldots, N\}^k$ such that

$$\phi_{\mathbf{i}}(F) \subset \left\{y \in \mathbb{R}^d : r' \le |y - x| \le r' + 4|\rho|^k u\right\}.$$

It together with (3.9) yields

$$
\begin{aligned}
\mathcal{H}^s(F \cap B_{r'+4|\rho|^k u}(x)) \;&\ge\; \mathcal{H}^s(F \cap B_{r'}(x)) + \mathcal{H}^s(\phi_{\mathbf{i}}(F)) \\
&\ge\; \mathcal{H}^s(F \cap B_{r'}(x)) + |\rho|^{ks}\mathcal{H}^s(F) \\
&>\; d_{\max}(2r' + 8\,|\rho|^k u)^s,
\end{aligned}
$$

which contradicts the maximality of $d_{\max}$. This finishes the proof of (i).

To prove (ii), observe that $\bigcup_{\mathbf{i} \in \mathcal{A}} \phi_{\mathbf{i}}(F) \supseteq F \cap B_{r'}(x)$. Thus

$$M|\rho|^{ks}\,\mathcal{H}^s(F) \ge \mathcal{H}^s(F \cap B_{r'}(x)).$$

Combining it with (3.9), we have

$$
\begin{aligned}
(M + 1/2)\,|\rho|^{ks}\,\mathcal{H}^s(F) \;&\ge\; \mathcal{H}^s(F \cap B_{r'}(x)) + 1/2\,|\rho|^{ks}\,\mathcal{H}^s(F) \\
&>\; d_{\max}(2r' + 8\,|\rho|^k u)^s > d_{\max}(2r)^s
\end{aligned}
$$

and we are done. $\blacksquare$

Proof of Theorem 1.1. By Lemma 3.2, it suffices to prove the theorem under the stronger assumption that $\psi(F) \subset F$. Let $B_r(x)$, k, $\mathcal{A}$ and M and $d_{\max}$ be given as in Lemma 3.3. Assume that Theorem 1.1 is false, that is, $\log|\lambda|/\log|\rho| \notin \mathbb{Q}$. We derive a contradiction.

Choose $\varepsilon > 0$ such that $(1-\varepsilon)^s(M+1) \geq (M+1/2)$. Since $\log|\lambda|/\log|\rho| \notin \mathbb{Q}$, there exist $m, n \in \mathbb{N}$ such that $1 - \varepsilon < |\rho|^m/|\lambda|^n < 1$. Define $J = \psi^n(B_r(x))$. We will deduce that

$$(3.10) \qquad \qquad \mathcal{H}^s(J \cap F) > d_{\max}\,|J|^s,$$

which contradicts the maximality of $d_{\max}$.

To show (3.10), set $\tilde{J} = \psi^n(B_{r-|\rho|^k u}(x))$, where $u = \mathrm{diam}F$. By Lemma 3.3,

$$\tilde{J} \cap F \;\supseteq\; \tilde{J} \cap \psi^n(F) = \psi^n(B_{r-|\rho|^k u}(x) \cap F) = \psi^n\left(\bigcup_{\mathbf{i} \in \mathcal{A}} \phi_\mathbf{i}(F)\right).$$

Hence

$$(3.11) \qquad \mathcal{H}^s(\tilde{J} \cap F) \geq \mathcal{H}^s\left(\psi^n\left(\bigcup_{\mathbf{i} \in \mathcal{M}} \phi_\mathbf{i}(F)\right)\right) = M\,|\lambda|^{ns}\,|\rho|^{ks}\,\mathcal{H}^s(F).$$

Define

$$\mathcal{R} := \left\{ \mathbf{i} \in \{1, \cdots, N\}^{m+k} : \; \phi_\mathbf{i}(F) \cap \tilde{J} \neq \emptyset \right\}$$

and $R = \#\mathcal{R}$. It is clear that $\bigcup_{\mathbf{i} \in \mathcal{R}} \phi_\mathbf{i}(F) \supset \tilde{J} \cap F$. Since $|\lambda|^n > |\rho|^m$, we have $\phi_\mathbf{i}(F) \subset J$ for any $\mathbf{i} \in \mathcal{R}$. Thus

$$\mathcal{H}^s(J \cap F) \geq \mathcal{H}^s(\bigcup_{\mathbf{i} \in \mathcal{R}} \phi_\mathbf{i}(F)) = R\,|\rho|^{(m+k)s}\,\mathcal{H}^s(F) \geq \mathcal{H}^s(\tilde{J} \cap F).$$

Combining the second inequality with (3.11) we obtain $R > M$ and thus $R \geq M+1$. Hence we have

$$
\begin{aligned}
\mathcal{H}^s(J \cap F) \;&\geq\; (M+1)\,|\rho|^{(m+k)s}\,\mathcal{H}^s(F) \\
&>\; (M+1)\,(1-\varepsilon)^s\,|\lambda|^{ns}\,|\rho|^{ks}\,\mathcal{H}^s(F) \\
&>\; (M+1/2)\,|\lambda|^{ns}\,|\rho|^{ks}\,\mathcal{H}^s(F) \\
&>\; d_{\max}\,|\lambda|^{ns}\,(2r)^s \qquad (\text{ by } (3.6)) \\
&=\; d_{\max}\,|J|^s.
\end{aligned}
$$

This is a contradiction, finishing the proof of the theorem. ∎

4. THE GENERATING IFSS FOR C

In this section we determine all the generating IFSs for the standard middle-third Cantor set C.

Let $\Phi = \{\phi_0, \phi_1\}$ be the standard generating IFS of C given by (1.1). Denote by $\{0,1\}^*$ (resp. $\{0,1\}^{\mathbb{N}}$) the collection of all finite (resp. infinite) words over the alphabet $\{0,1\}$. A finite subset B of $\{0,1\}^*$ is called *complete* if each infinite word $(i_n)_{n=1}^\infty$ has a prefix in B. For $\mathbf{i} = i_1 i_2 \ldots i_m$, write $\phi_{\mathbf{i}} = \phi_{i_1} \circ \cdots \circ \phi_{i_m}$, and define $\overline{\phi}_{\mathbf{i}}$ by $\overline{\phi}_{\mathbf{i}}(x) = -\phi_{\mathbf{i}}(x) + \phi_{\mathbf{i}}(0) + \phi_{\mathbf{i}}(1)$. Since C is symmetric, we always have $\overline{\phi}_{\mathbf{i}}(C) = \phi_{\mathbf{i}}(C)$. The following result characterizes all the generating IFSs for C.

Proposition 4.1. *An IFS $\Psi = \{\psi_j\}_{j=1}^N$ is a generating IFS of C if and only if there exists a complete subset B of $\{0,1\}^*$ such that*

(i) *For each j, $\psi_j = \phi_{\mathbf{i}}$ or $\psi = \overline{\phi}_{\mathbf{i}}$ for some $\mathbf{i} \in B$.*
(ii) *For each $\mathbf{i} \in B$, at least one of $\phi_{\mathbf{i}}$ and $\overline{\phi}_{\mathbf{i}}$ belongs to Ψ.*

The proof of Proposition 4.1 is based on the following lemma.

Lemma 4.2. *Let ψ be a contracting similitude on $\mathbb{R}$ so that $\psi(C) \subset C$. Then $\psi = \phi_{\mathbf{i}}$ or $\overline{\phi}_{\mathbf{i}}$ for some $\mathbf{i} \in \{0,1\}^*$.*

Proof. Denote by λ the contraction factor of ψ. Let k be the unique integer k so that $3^{-k-1} \leq \lambda < 3^{-k}$. Since the middle hole in $\psi(C)$ has length less than $3^{-(k+1)}$, the interval $\psi([0,1])$ intersects exactly one of the intervals $\phi_{\mathbf{u}}([0,1])$, $\mathbf{u} \in \{0,1\}^{k+1}$. Denote this interval by $\phi_{\mathbf{i}}([0,1])$. Then $\phi_{\mathbf{i}}([0,1])$ contains the two endpoints of $\psi([0,1])$, and its length is not great than that of $\psi([0,1])$. It guarantees that $\psi([0,1]) = \phi_{\mathbf{i}}([0,1])$. Hence $\psi = \phi_{\mathbf{i}}$ or $\overline{\phi}_{\mathbf{i}}$. ∎

Proof of Proposition 4.1. Suppose that B is a complete set in $\{0,1\}^*$ so that the conditions (i) and (ii) hold. Take any $y \in C$. Then there exists an infinite word $\omega = (i_n)_{n=1}^\infty$ such that $y = \lim_{n\to\infty} \phi_{i_1} \circ \cdots \circ \phi_{i_n}(0)$. Since B is complete, there exists $\mathbf{u} \in B$ such that $\mathbf{u}$ is a prefix of ω. Thus $y \in \phi_{\mathbf{u}}(C) = \overline{\phi}_{\mathbf{u}}(C)$, and hence $y \in \psi_j(C)$ for some $1 \leq j \leq N$ by the condition (ii). This means $C \subseteq \bigcup_j \psi_j(C)$. Note that the condition (i) implies $C \supseteq \bigcup_j \psi_j(C)$. Hence $C = \bigcup_j \psi_j(C)$, i.e., Ψ is a generating IFS of C..

Conversely, suppose $C = \bigcup_{j=1}^{N} \psi_j(C)$. By Lemma 4.2, each ψ_j equals $\phi_{\mathbf{u}}$ or $\overline{\phi}_{\mathbf{u}}$ for some $\mathbf{u} \in \{0,1\}^*$. Define $B = \{\mathbf{u} \in \{0,1\}^* : \phi_{\mathbf{u}} = \psi_j \text{ or } \overline{\phi}_{\mathbf{u}} = \psi_j \text{ for some } j\}$. It is clear that B satisfies the condition (i) and (ii). We only need to show that B is complete. To see it, take an arbitrary infinite word $\omega = (i_n)_{n=1}^{\infty}$ and let $y = \lim_{n \to \infty} \phi_{i_1} \circ \cdots \circ \phi_{i_n}(0)$. Then $y \in C$ and thus $y \in \psi_j(C)$ for some j. Therefore $y \in \phi_{\mathbf{u}}(C) = \overline{\phi}_{\mathbf{u}}(C)$ for some $\mathbf{u} \in B$. It implies that u is a prefix of ω. Therefore B is complete, and we are done. ∎

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