SIMULTANEOUS TRANSLATIONAL AND MULTIPLICATIVE TILING AND WAVELET SETS IN $\mathbb{R}^2$

EUGEN IONASCU AND YANG WANG

Abstract. Simultaneous tiling for several different translational sets has been studied rather extensively, particularly in connection with the Steinhaus problem. The study of orthonormal wavelets in recent years, particularly for arbitrary dilation matrices, has led to the study of multiplicative tilings by the powers a matrix. In this paper we consider the following simultaneous tiling problem: Given a lattice in $\mathbb{R}^d$ and a matrix $A \in \mathrm{GL}(d,\mathbb{R})$, does there exist a measurable set T such that both $\{T+\alpha: \alpha \in \mathcal{L}\}$ and $\{A^nT: n \in \mathbb{Z}\}$ are tilings of $\mathbb{R}^d$? This problem comes directly from the study of wavelets and wavelet sets. Such a T is known to exist if A is expanding. When A is not expanding the problem becomes much more subtle. Speegle [22] exhibited examples in which such a T exists for some $\mathcal{L}$ and nonexpanding A in $\mathbb{R}^2$. In this paper we give a complete solutions to this problem in $\mathbb{R}^2$.

1. Introduction

The history of tiling goes as far back as the beginning of civilization. Tiling has been studied in the history of mankind in many different contexts and for differnt purposes. Mathematically we usually study tiling in the context of having a finite set of "shapes" called *prototiles* and using congruent copies of these prototiles to cover the whole Euclidean space without overlapping. Translational tiling is one such example, in which only the translations of the prototiles are used to tile the space.

Recently attentions have been given to *multiplicative tilings*. In a multiplicative tiling there is a finite set of prototiles $\{T_1, T_2, \dots, T_m\}$ and sets of nonsingular $d \times d$ matrices $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_m$ such that

$$\Big\{A_jT_j: \ A_j \in \mathcal{D}_j, 1 \le j \le m\Big\}$$

is a partition of $\mathbb{R}^d$. Here we define a partition in the most general sense, namely the sets are disjoint in Lebesgue measure and their union is $\mathbb{R}^d$ up to a measure zero set, see e.g.

Key words and phrases. wavelet, waveletset, lattice tiling, multiplicative tiling, simultaneous tiling, continued fraction.

Supported in part by the National Science Foundation grant DMS-0456538.

Wang [23] and Speegle [22]. These studies are motivated in large part by the connection with orthonormal wavelets, which we shall discuss later.

We first introduce some notations and terminologies. We say a collection of measurable sets $\{T_j\}$ in $\mathbb{R}^d$ is a *tiling* of $\mathbb{R}^d$ if it is a partition of $\mathbb{R}^d$ in the sense just described above. A measurable set T is said to tile *translationally by* $\mathcal{J}$, where $\mathcal{J}$ is a subset of $\mathbb{R}^d$, if $\{T+\alpha:\ \alpha\in\mathcal{J}\}$ is a tiling of $\mathbb{R}^d$. In this paper, we are primarily concerned with translational tiling by a lattice. Let $A\in \mathrm{GL}(d,\mathbb{R})$ be a $d\times d$ nonsingular matrix. We say a measurable set T *tiles multiplicatively by* A if $\{A^nT:\ n\in\mathbb{Z}\}$ is a tiling of $\mathbb{R}^d$. The main question we ask in this paper is:

Problem. *Given a matrix $A\in \mathrm{GL}(d,\mathbb{R})$ and a lattice $\mathcal{L}$ in $\mathbb{R}^d$, does there exist a measurable set T in $\mathbb{R}^d$ such that T tiles translationally by $\mathcal{L}$ and multiplicatively by A?*

Simultaneous translational tiling using more than one lattice have been studied rather extensively. One of the best known problem in tiling is the classic Steinhaus problem posed by H. Steinhaus sometime in the 1950's, which asks for the existence of a $T\subset\mathbb{R}^2$ that tiles translationally by all lattices of the form $R_\theta\mathbb{Z}^2$ where $\mathbb{R}_\theta$ is a rotation matrix. It was shown by Jackson and Mauldin [16] that such a T exists, but in their construction T is not measurable. The problem remains open for measurable sets. Han and Wang [8] proved that for any two lattices $\mathcal{L}_1,\mathcal{L}_2$ in $\mathbb{R}^d$ with the same co-volume there always exists a measurable $T\subset\mathbb{R}^d$ such that T tiles translationally by both $\mathcal{L}_1$ and $\mathcal{L}_2$. This problem is motivated by the study of Weyl-Heisenberg orthonormal bases for $L^2(\mathbb{R}^d)$. Kolountzakis [18] established conditions on lattices $\mathcal{L}_1,\mathcal{L}_2,\dots,\mathcal{L}_n$ in $\mathbb{R}^d$ with the same co-volume for which a measurable T exists that tiles translationally by each $\mathcal{L}_j$. There are many other related studies on translational simultaneous tilings, see the references in the aforementioned papers.

In contrast, the study of simultaneous translational and multiplicative tiling has only begun very recently. One of the motivations for studying this problem is the connection to orthonormal wavelets. For $A\in\mathrm{GL}(d,\mathbb{R})$ we call a function $f\in L^2(\mathbb{R}^d)$ an *orthogonal wavelet with dilation* A if the set of functions

$$\Big\{|\det(A)|^{\frac{n}{2}}f(A^nx-\alpha):\ n\in\mathbb{Z},\alpha\in\mathbb{Z}^d\Big\}$$

is an orthogonal basis for $L^2(\mathbb{R}^d)$. Again, in general we can substitute $\mathbb{Z}^d$ with any full rank lattice. Since the seminal work of Daubechies [5] there has been an explosion in the study of wavelets and their applications in image compression, digital signal processing and numerical computations. We shall not go into details about wavelets here and shall refer the readers to Daubechies [6]. It should be pointed out that in most studies on wavelets the dilation matrix A is assumed to be *expanding*, i.e. all eigenvalues have $|\lambda| > 1$. Furthermore, a single dilation matrix is involved. In [23], the concept of wavelets is broadened to allow more than one dilation matrix as well as nonexpanding matrices.

The role of tiling has appeared in the study of the following fundamental question in wavelets: Given an expanding matrix $A \in \mathrm{GL}(d, \mathbb{R})$ is it always possible to find an orthogonal wavelet with dilation A? This question was answered by studying functions of the form $f = \widehat{\chi}_T$ where T is a subset of $\mathbb{R}^d$ with finite measure. Dai and Larson [2][1]proved the following theorem:

Theorem 1.1 ([2])**.** *Let $A \in \mathrm{GL}(d, \mathbb{R})$ and let $T \subset \mathbb{R}^d$ have finite measure. Then $f = \widehat{\chi}_T$ is an orthogonal wavelet with dilation A if and only if T tiles $\mathbb{R}^d$ translationally by $\mathbb{Z}^d$ and multiplcatively by A^T.*

A set T that tiles $\mathbb{R}^d$ translationally by $\mathbb{Z}^d$ and multiplcatively by A^T simultaneously is called a *wavelet set with dilation* A. Later, it was proved by Speegle, Dai and Larson [3] that for an expanding or contracting A a wavelet set that is bounded exists. Many other different constructions in the expanding case have been proposed, all of which involve cut and paste, see e.g. [1, 3, 13, 21, 23]. One starts with a set T_0 that tiles by A^T and covers $\mathbb{R}^d$ translationally by $\mathbb{Z}^d$. The goal then is to move pieces of T_0 around using a combination of translations and dilations to get a wavelet set. The fact that A is expanding or contracting plays a crucial role because it can be used multiplicatively to control the size of the pieces, an important part of the constructions.

The existence of wavelet sets for nonexpanding matrices is a much more challenging problem. Wang [23] exhibited an example of a wavelet set whose dilations consist more than the powers of a single matrix, in which some dilations are neither expanding nor contracting. But few people believed, or even thought about, the possibility of an orthogonal wavelet with a single dilation matrix A that is neither expanding or contracting. However, Speegle [22] recently has shown the existence of such wavelet sets (and hence orthogonal wavelets). In particular, he has shown that for $A = \mathrm{diag}\,(a, b)$ where $|b| < 1$ and $|ab| > 1$ and the

[1]In their theorem the matrix A is assumed to be expanding, but it is clearly not needed.

lattice $\mathcal{L} = \mathbb{Z}\mathbf{e}_1 + \mathbb{Z}(\mathbf{e}_2 + \sqrt{5}\mathbf{e}_1)$ there exists a measurable T that tiles $\mathbb{R}^d$ translationally by $\mathbb{Z}^2$ and multiplcatively by A. More generally, the value $\sqrt{5}$ can be replaced with any $\beta \in \mathbb{R}$ if β cannot be approximated by rationals to within certain order $J = J(a,b)$. By a simple linear transformation, one can show the existence of a matrix $A \in \mathrm{GL}(d,\mathbb{R})$ for $d = 2$ which is neither expanding nor expanding, for which there is a wavelet set.

In this paper we give a complete classification in dimension $d = 2$ for the existence of wavelet sets, or equivalently the existence of simultaneous tiling translationally by a lattice and multiplicatively by A. We state our main theorems here.

Theorem 1.2. *Let $A \in \mathrm{GL}(2,\mathbb{R})$ with $|\det(A)| \geq 1$. Let λ_1, λ_2 be the eigenvalues of A with $|\lambda_1| \geq |\lambda_2|$.*

(a) *If $|\lambda_1\lambda_2| = 1$, i.e. $|\det(A)| = 1$, then there exists no wavelet set with dilation A.*
(b) *If $|\lambda_1| > 1$ and $|\lambda_2| \geq 1$, then there exists a wavelet set with dilation A.*
(c) *If $|\lambda_1\lambda_2| > 1$ and $|\lambda_2| < 1$, let $\mathbf{v} = [v_1, v_2]^T$ be an eigenvector of A^T for λ_2. Then there exists a wavelet set with dilation A if and only if $v_1/v_2 \in \mathbb{R} \setminus \mathbb{Q}$.*

Note that the assumption $|\det(A)| \geq 1$ is without any loss of generality. If a wavelet set exists with dilation A then it is also a wavelet set with dilation A^{-1}. We also remark that part (a) of the theorem is due to [19], where it is shown that if $|\det(A)| = 1$ then there exists no T with finite measure such that T tiles multiplicatively by A. Furthermore, they proved that a *bounded* wavelet set exists if and only if A is expanding or contracting. Our next theorem is more general than Theorem 1.2, and it gives a complete classification of the simultaneous tiling problem in $\mathbb{R}^2$.

Theorem 1.3. *Let $A \in \mathrm{GL}(2,\mathbb{R})$ with $|\det(A)| \geq 1$ and eignevalues λ_1, λ_2, $|\lambda_1| \geq |\lambda_2|$. Let $\mathcal{L} = P\mathbb{Z}^2$ be a lattice in $\mathbb{R}^2$ with $P \in \mathrm{GL}(2,\mathbb{R})$.*

(a) *If $|\lambda_1\lambda_2| = 1$, then there exists no measurable $T \subset \mathbb{R}^2$ such that T tiles translationally by $\mathcal{L}$ and multiplicatively by A.*
(b) *If $|\lambda_1| > 1$ and $|\lambda_2| \geq 1$, then there exists a measurable $T \subset \mathbb{R}^2$ such that T tiles translationally by $\mathcal{L}$ and multiplicatively by A.*
(c) *If $|\lambda_1\lambda_2| > 1$ and $|\lambda_2| < 1$, let $\mathbf{v} = [v_1, v_2]^T$ be an eigenvector of PAP^{-1} for λ_2. Then there exists a measurable $T \subset \mathbb{R}^2$ such that T tiles translationally by $\mathcal{L}$ and multiplicatively by A if and only if $v_1/v_2 \in \mathbb{R} \setminus \mathbb{Q}$.*

This work is completed while the first author is visiting the School of Mathematics of Georgia Institute of Technology as part of the Faculty Development Program sponsored by the University System of Georgia. He would like to thank both the Columbus State University and Georgia Tech for the generous support.

2. Some General Results

We focus here on general results concerning simultaneous multiplicative and translational tilings. These results will be used to prove our main theorems. First we introduce some terminologies.

Let $\Omega \subset \mathbb{R}^d$ be a measurable set. We say Ω *packs* $\mathbb{R}^d$ *translationally by* $\mathcal{L}$ or simply Ω *packs by* $\mathcal{L}$, where $\mathcal{L}$ is a lattice in $\mathbb{R}^d$, if $\{\Omega + \alpha : \alpha \in \mathcal{L}\}$ are disjoint in measure. Similarly, we say Ω *packs* $\mathbb{R}^d$ *multiplicatively by* A or simply Ω *packs by* A, where $A \in \mathrm{GL}(d, \mathbb{R})$, if $\{A^n\Omega : n \in \mathbb{Z}\}$ are disjoint in measure. Since the construction of wavelet sets and tiles often involve cut and paste, the concept of packing plays an important role in this paper.

Lemma 2.1. *Let $A \in \mathrm{GL}(d, \mathbb{R})$ and $\mathcal{L} = P\mathbb{Z}^d$ with $P \in \mathrm{GL}(d, \mathbb{R})$. Then a set T tiles multiplicatively by A and translationally by $\mathcal{L}$ if and only if $P^{-1}T$ tiles multiplicatively by $P^{-1}AP$ and translationally by $\mathbb{Z}^d$.*

Proof. If $\{A^nT : n \in \mathbb{Z}\}$ is a tiling then clearly so is $\{P^{-1}A^nT = (P^{-1}AP)^nP^{-1}T : n \in \mathbb{Z}\}$, and conversely. Similarly, if $\{T + P\alpha : \alpha \in \mathbb{Z}^d\}$ is a tiling then so is $\{P^{-1}T + \alpha : \alpha \in \mathbb{Z}^d\}$, and conversely. ∎

In the study of multiplicative tiling it is useful to study the address map. Let Ω be a multiplicative tiles by $A \in \mathrm{GL}(d, \mathbb{R})$. Then the *address map (induced by Ω)* is the map $\tau_\Omega : \mathbb{R}^d \longrightarrow \Omega$ given by

$$\tau_\Omega(x) = y \qquad \text{if} \qquad x = A^n y, y \in \Omega. \tag{2.1}$$

Note that for almost all $x \in \mathbb{R}^d$ there exist unique $n \in \mathbb{Z}$ and $y \in \Omega$ such that $x = A^n y$, so the map is well defined almost everywhere. If $S \subset \mathbb{R}^d$ packs by A then $\tau_\Omega|_S$ is one-to-one almost everywhere on S, and conversely. Furthermore, for such an S we can define the *index map (induced by S and Ω)* $\phi : \Omega \longrightarrow \mathbb{Z}$ given by

$$\phi(y) = \begin{cases} n & \text{if } A^n y \in S \text{ for some } n \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Theorem 2.2. *Let $A \in \mathrm{GL}(d,\mathbb{R})$ and $\mathcal{L}$ be a lattice in $\mathbb{R}^d$. Suppose there exists an $S \subset \mathbb{R}^d$ such that S tiles multiplicatively by A and packs translationally by $\mathcal{L}$. Then there exists a $T \subset \mathbb{R}^d$ such that T tiles multiplicatively by A and translationally by $\mathcal{L}$.*

Proof. Note that we must have $|\det(A)| \neq 1$, for otherwise S would have infinite measure and cannot pack translationally by $\mathcal{L}$, see [19]. It is also proved in [19] that if $|\det(A)| \neq 1$ then we can find an $S_0 \subset \mathbb{R}^d$ such that S_0 tiles multiplicatively by A and furthermore, the construction in the paper clearly shows that we can require S_0 to have nonempty interior. Now, set $S^* = nS_0$ for some n sufficiently large. Again S^* tiles multiplicatively by A. But now S^* contains a set F that tiles by $\mathcal{L}$. Also, since S packs by $\mathcal{L}$, there exists an F^* that tiles by $\mathcal{L}$ with $S \subseteq F^*$.

Now since both F, F^* tiles translationally by $\mathcal{L}$, there exists a bijection (in the sense of almost everywhere) $\rho : F^* \longrightarrow F$ such that $\rho(x) = x + \alpha(x)$ for some unique $\alpha(x) \in \mathcal{L}$. The map $\tau_S : S^* \longrightarrow S$ is also a bijection since both S, S^* tile by A. Let $\phi = \rho|_S$ and $\psi = \tau_S|_F$. Then both $\phi : S \longrightarrow F$ and $\psi : F \longrightarrow S$ are one-to-one.

By the Schröder-Cantor-Bernstein construction there exists a measurable bijection $h : S \longrightarrow F$ having the form

$$h(x) = \begin{cases} \phi(x) & x \in E \\ \psi^{-1}(x) & x \in S \setminus E \end{cases}$$

for some $E \subseteq S$. A more precise form of E can be found in [12]. Clearly for each $x \in S$ there exist a unique $n(x) \in \mathbb{Z}$ and a unique $\alpha(x) \in \mathcal{L}$ such that $h(x) = A^{n(x)}x + \alpha(x)$. Let $T = \{A^{n(x)}x : \ x \in S\}$. It is obvious that T tiles multiplicatively by A because S does. Furthermore, T is congruent to F modulo the lattice $\mathcal{L}$, so T also tiles by $\mathcal{L}$. ∎

The above theorem is established in special forms in [15] and [22]. It shows that to prove the existence of simultaneous multiplicative and translational tilings one only needs to prove the existence of simultaneous multiplicative tiling and translational packing. This is precisely the strategy we follow to prove our main theorems.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^d$ with $\mu(\Omega) < \infty$ tile by $A \in \mathrm{GL}(d,\mathbb{R})$, $|\det(A)| > 1$. Let S_n pack $\mathbb{R}^d$ by A and $\phi_n : \Omega \longrightarrow \mathbb{Z}$ be the index map induced by S_n and Ω. Assume that $\lim_{n\to\infty} \mu(S_n) = 0$ and $\liminf_n \phi_n(x) > -\infty$ for almost all $x \in \Omega$. Then $\lim_{n\to\infty} \mu(\tau_\Omega(S_n)) = 0$.*

Proof. Assume that $\mu(\Omega)$ and $\mu(S_n)$ are all bounded by $c > 0$. Each S_n has a unique decomposition $S_n = \bigcup_{m\in\mathbb{Z}} A^m R_{n,m}$ (up to a null set) where $R_{n,m} \subseteq \Omega$. Furthermore, τ_Ω is injective on S_n so the sets $\{R_{n,m}\}$ are disjoint in measure, and $\tau_\Omega(S_n) = \bigcup_{m\in\mathbb{Z}} R_{n,m}$. We have

$$\sum_{m\in\mathbb{Z}} \mu(R_{n,m}) \le \mu(\Omega) \le c, \qquad \sum_{m\in\mathbb{Z}} \beta^m \mu(R_{n,m}) = \mu(S_n) \le c, \tag{2.3}$$

where $\beta := |\det(A)| > 1$. For any $\varepsilon > 0$, note that $\sum_{m\ge N} \beta^m \mu(R_{n,m}) \le c$ so

$$\sum_{m\ge N} \mu(R_{n,m}) \le c\beta^{-N}.$$

Hence for all $N \ge N_0(\varepsilon)$ we have $\sum_{m\ge N} \mu(R_{n,m}) < \varepsilon$.

Since $\liminf_n \phi_n(x) > -\infty$ for almost all $x \in \Omega$, there exists an $N_1 = N_1(\varepsilon)$ such that for all $N, n \ge N_1$ we have

$$\mu(\{x \in \Omega : \ \phi_n(x) < -N\}) < \varepsilon,$$

which is equivalent to $\sum_{m<-N} \mu(R_{n,m}) < \varepsilon$. Now pick $N \ge N_0, N_1$ and we have

$$\sum_{|m|>N} \mu(R_{n,m}) = \sum_{m>N} \mu(R_{n,m}) + \sum_{m<-N} \mu(R_{n,m}) < 2\varepsilon.$$

In addition,

$$\sum_{|m|\le N} \beta^m \mu(R_{n,m}) \le \mu(S_n) \longrightarrow 0 \quad \text{as} \quad n\to\infty.$$

So there exists an N_2 such that $\sum_{|m|\le N} \mu(R_{n,m}) < \varepsilon$ for all $n \ge N_2$. Thus for $n \ge N_2$ we have

$$\sum_{m\in\mathbb{Z}} \mu(R_{n,m}) < 3\varepsilon.$$

It follows that $\mu(\tau_\Omega(S_n)) = \sum_{m\in\mathbb{Z}} \mu(R_{n,m}) \longrightarrow 0$ as $n\to\infty$. ∎

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^d$ with $\mu(\Omega) < \infty$ tile by $A \in \mathrm{GL}(d,\mathbb{R})$, $|\det(A)| > 1$. Let S_n pack $\mathbb{R}^d$ by A and $\phi_n : \Omega \longrightarrow \mathbb{Z}$ be the index map induced by S_n and Ω. Assume $S^* \subset \mathbb{R}^d$ with $\mu(S^*) < \infty$ such that $\lim_{n\to\infty} \mu(S_n \Delta S^*) = 0$, where Δ denotes the symmetrical difference. Then*

(a) *S^* packs $\mathbb{R}^d$ multiplicatively by A.*
(b) *Assume that $\lim_{n\to\infty} \mu(\tau_\Omega(S_n)\Delta\Omega) = 0$ and $\liminf_n \phi_n(x) > -\infty$ for almost all $x \in \Omega$. Then S^* tiles $\mathbb{R}^d$ multiplicatively by A.*

Proof. (a) is stated in [22] without a proof. It is quite straightforward, but we will furnish a proof here. Assume it is false then $\mu(A^k S^* \cap A^l S^*) = \delta > 0$ for some $k \neq l$. But $\lim_n \mu(S_n \Delta S^*) = 0$. It follows that $\mu(A^k S_n \cap A^l S_n) \geq \delta/2 > 0$ for sufficiently large n, a contradiction.

We now prove (b) by proving $\tau_\Omega(S^*) = \Omega$. Assume it is false. Then $\mu(\Omega \setminus \tau_\Omega(S^*)) > 0$. Hence there exists a $\delta > 0$ such that $\mu(\tau_\Omega(S_n) \setminus \tau_\Omega(S^*)) > \delta$ for all sufficiently large n. Thus $\mu(\tau_\Omega(S_n \setminus S^*) > \delta$ for sufficiently large n. We shall derive a contradiction.

Set $R_n = S_n \setminus S^*$. Then $\lim_n \mu(R_n) = 0$. Note that by assumption $\liminf_n \phi_n(x) > -\infty$ for almost all $x \in \Omega$. Let ψ_n be the index map induced by R_n and Ω. Then either $\psi_n(x) = \phi_n(x)$ or $\psi_n(x) = 0$ because $R_n \subseteq S_n$. Thus $\liminf_n \psi_n(x) > -\infty$ for almost all $x \in \Omega$. It follows from Lemma 2.3 that $\lim_n \mu(\tau_\Omega(R_n)) = 0$, a contradiction. ∎

The next theorem is a stronger and more general version of Theorem 3.2 in [22]. We also give a different proof here, using the above lemmas.

Theorem 2.5. *Let $\mathcal{L}$ be a lattice in $\mathbb{R}^d$ and $A \in \mathrm{GL}(d, \mathbb{R})$ with $|\det(A)| > 1$. Suppose that for any bounded set $S \subset \mathbb{R}^d$ there are infinite many $n \in \mathbb{N}$ such that $A^{-n}S$ packs translationally by $\mathcal{L}$. Then there exists a $T \subset \mathbb{R}^d$ such that T tiles $\mathbb{R}^d$ multiplicatively by A and translationally by $\mathcal{L}$.*

Proof. By [19] there exists an Ω with $\mu(\Omega) = c < \infty$ and Ω tiles multiplicatively by A. If Ω is also bounded then we can find $m_1 > 0$ such that $A^{-m_1}\Omega$ packs by $\mathcal{L}$. Since $A^{-m_1}\Omega$ also tiles by A, the theorem follows from Theorem 2.2. Thus we shall assume Ω is unbounded.

Denote $\Omega_n = \Omega \cap [-M_n, M_n]^d$, where $0 < M_1 < M_2 < \cdots$ with the property that $\mu(\Omega \setminus \Omega_n) \leq 4^{-n}c$. Write $\Omega_0 = \emptyset$ and $R_n = \Omega_n \setminus \Omega_{n-1}$ for $n \geq 1$. Then $\{R_n\}$ are disjoint and

$$\Omega = \bigcup_{k=1}^{\infty} R_k, \qquad \Omega_n = \bigcup_{k=1}^{n} R_k.$$

The idea is to use $\{R_n\}$ to construct a sequence of sets $\{S_n\}$ satisfying the conditions of Lemma 2.4 (b).

Let $S_1 = A^{-m_1}\Omega_1$ such that S_1 packs by $\mathcal{L}$. This m_1 exists since Ω_1 is bounded. We construct recursively S_n for $n \geq 2$ satisfying the following properties:

(a) S_n packs by $\mathcal{L}$ and $\tau_\Omega(S_n) = \Omega_n$.

(b) $\mu(S_n\Delta S_{n-1}) \le 4^{-n}\mu(S_1)$.

(c) Each ϕ_n is bounded and the set $X_n := \{x \in \Omega : \ \phi_n(x) \ne \phi_{n-1}(x)\}$ has $\mu(X_n) \le 2c/4^n$, where $c = \mu(\Omega)$ and ϕ_n is the index map induced by S_n and Ω.

Assume such S_n's exist. Observe that for all $n > m$ we have

$$\mu(S_n\Delta S_m) \le \sum_{k=m}^{n-1} \mu(S_{k+1}\Delta S_k) \le \sum_{k=m}^{n-1} \frac{1}{4^{k+1}}\mu(S_1) < \frac{2}{4^m}\mu(S_1). \tag{2.4}$$

So $\{S_n\}$ is a Cauchy sequence in the sense of symmetrical difference, and there exists an S^* such that $\lim_n \mu(S_n\Delta S^*) = 0$. Furthermore, taking $m = 1$ in (2.4) yields $\mu(S_n\Delta S_1) < \mu(S_1)/2$, so $\mu(S_n) > \mu(S_1)/2$. This means S^* has positive measure. By Lemma 3.1 in [22], S^* packs by A and $\mathcal{L}$. Now let

$$Y_n := \Big\{x \in \Omega : \ \phi_n(x) \ne \phi_k(x) \text{ for all } k > n\Big\}.$$

Then (c) yields

$$\mu(Y_n) \le \sum_{k>n} \mu(X_k) \le \sum_{k>n} \frac{2c}{4^k} < \frac{c}{4^n}.$$

Since each ϕ_n is bounded on Ω, it follows that $\liminf_n \phi_n(x) > -\infty$ for almost all $x \in \Omega$. Hence S^* tiles multiplicatively by A. The theorem follows from Theorem 2.2.

It remains to prove that such S_n's exist. Assume S_{n-1} has been constructed, $n > 1$. We construct S_n. Let $m_n > 0$ such that $A^{-m_n}\Omega_n$ packs by $\mathcal{L}$. Let

$$U_n = S_{n-1} \cap \big(A^{-m_n}\Omega_n + \mathcal{L}\big).$$

Since both S_{n-1} and $A^{-m_n}\Omega_n$ pack by $\mathcal{L}$, it is easy to see that

$$\mu(U_n) \le \mu(A^{-m_n}\Omega_n) \le |\det(A)|^{-m_n}\, 4^{-n}c.$$

Now ϕ_{n-1} is bounded, so $\phi_{n-1}(x) \ge -a_{n-1}$ for some $a_{n-1} > 0$. This means that for each $x \in S_{n-1}$ there exists a $k \le a_{n-1}$ such that $x = A^{-k}\tau_\Omega(x)$. Hence $\mu(\tau_\Omega(U_n)) \le |\det(A)|^{a_{n-1}}\mu(U_n)$. By choosing $m_n \ge a_{n-1}$ sufficiently large we have $\mu(\tau_\Omega(U_n)) \le 4^{-n}c$ and $\mu(A^{-m_n}\Omega_n) \le 4^{-n}\mu(S_1)$. We now define

$$S_n = (S_{n-1} \setminus U_n) \cup A^{-m_n}\tau_\Omega(U_n) \cup A^{-m_n}R_n.$$

It is clear that S_n packs by $\mathcal{L}$, and $\tau_\Omega(S_n) = \tau_\Omega(S_{n-1}) \cup R_n = \Omega_n$. So (a) is satisfied. Also,

$$\mu(S_n\Delta S_{n-1}) \le \mu(U_n) + \mu(A^{-m_n}(\tau_\Omega(U_n) \cup R_n)) \le 2\mu(A^{-m_n}\Omega_n) \le \frac{\mu(S_1)}{4^n}.$$

So (b) is satisfied. Finally, observe that $\phi_n(x) = -m_n$ for $x \in \tau_\Omega(U_n) \cup R_n$ but $\phi_n(x) = \phi_{n-1}(x)$ everywhere else. So ϕ_n is bounded. Furthermore,

$$\mu(X_n) \leq \mu(\tau_\Omega(U_n) \cup R_n) < \frac{2c}{4^n}.$$

This yields (c). The proof of the theorem is now complete. ∎

3. Proof of Main Theorems

We now prove our main theorems. The proofs are divided into several propositions for different cases. A key ingredient is the approximation of irrational numbers by rational numbers.

We first consider the case in which $A \in \mathrm{GL}(2, \mathbb{R})$ has eigenvalues λ_1, λ_2 with $|\lambda_1| > |\lambda_2| = 1$. In this case, both λ_1, λ_2 are necessarily real, and $\lambda_2 = \pm 1$.

Proposition 3.1. *Let $A \in \mathrm{GL}(2, \mathbb{R})$ with eigenvalues λ_1, λ_2, $|\lambda_1| > |\lambda_2| = 1$. Assume A has no rational eigenvectors for λ_2. Then there exists a measurable $T \subset \mathbb{R}^2$ such that T tiles translationally by $\mathbb{Z}^2$ and multiplicatively by A.*

Proof. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors of A for λ_1 and λ_2, respectively. By assumption we may assume $\mathbf{v}_2 = [1, \beta]^T$ where $\beta \in \mathbb{R} \setminus \mathbb{Q}$. We use Theorem 2.5 to complete the proof. Let S be any bounded set in $\mathbb{R}^2$. We prove $A^{-n}S$ packs translationally by $\mathbb{Z}^2$ for all sufficiently large n. Since $S \subseteq \{s\mathbf{v}_1 + t\mathbf{v}_2 : \ |s|, |t| \leq K\}$ for some $K > 0$, we may without loss of generality assume $S = \{s\mathbf{v}_1 + t\mathbf{v}_2 : \ |s|, |t| \leq K\}$. Observe that

$$A^{-n}S = \{s\mathbf{v}_1 + t\mathbf{v}_2 : \ |s| \leq |\lambda_1|^{-n}K, |t| \leq K\},$$

which approaches the straightline segment $L = \{t\mathbf{v}_2 : \ |t| \leq K\}$ in Hausdorff metric.

Now $\mathbf{v}_2 = [1, \beta]^T$ has irrational slope, so the line segments $\{L + \alpha : \ \alpha \in \mathbb{Z}^2\}$ are disjoint. Let $\varepsilon_0 = \mathrm{dist}\,(L, \mathbb{Z}^2 \setminus \{0\}) > 0$. Then the distance between $L + \alpha_1$ and $L + \alpha_2$ is at least ε_0 for any $\alpha_1 \neq \alpha_2$. It follows that for sufficiently large n, by making $|\lambda_1|^{-n}K < \varepsilon_0/3$, the sets $\{A^{-n}S + \alpha : \ \alpha \in \mathbb{Z}^2\}$ are disjoint. Thus $A^{-n}S$ packs by $\mathbb{Z}^2$ for sufficiently large n, proving the proposition. ∎

Proposition 3.2. *Let $A \in \mathrm{GL}(2, \mathbb{R})$ with eigenvalues λ_1, λ_2, $|\lambda_1| > |\lambda_2| = 1$. Assume A has a rational eigenvector for λ_2. Then there exists a measurable $T \subset \mathbb{R}^2$ such that T tiles translationally by $\mathbb{Z}^2$ and multiplicatively by A.*

Proof. Let $\mathbf{v}_2 = [p,q]^T$ be an eigenvector of A for λ_2, with $p,q \in \mathbb{Z}$ and $\gcd(p,q) = 1$. Let $\mathbf{v}_1 \in \mathbb{Z}^2$ with the property that $P = [\mathbf{v}_1, \mathbf{v}_2]$ has $\det(P) = 1$, where the columns of P are $\mathbf{v}_1$ and $\mathbf{v}_2$. For this P we have

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ b & \lambda_2 \end{bmatrix}, \qquad P\mathbb{Z}^2 = \mathbb{Z}^2.$$

Now, take $Q = \left[\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right]$ with $t = \frac{b}{\lambda_1 - \lambda_2}$. Then $Q^{-1}P^{-1}APQ = \mathrm{diag}(\lambda_1, \lambda_2)$. Let $B = \mathrm{diag}(\lambda_1, \lambda_2)$. By Lemma 2.1 it suffices to prove there exists a T such that T tiles translationally by $Q^{-1}P^{-1}\mathbb{Z}^2 = Q^{-1}\mathbb{Z}^2$ and multiplicatively by B.

We prove the existence by an explicit construction. Denote $U_n = [-a_n, a_{n+1}) \cup (a_{n+1}, a_n]$ where $a_n = \frac{1}{2}|\lambda_1|^{-n}$. It is clear that $\lambda_1^{-1}U_n = U_{n+1}$, $\{U_n\}$ is a partition of $\mathbb{R}$, and $\bigcup_{n=0}^{\infty} U_n = [-1/2, 1/2]$ up to a null set. Let $I_n = [-\frac{1}{2} - n, -n] \cup [n, n + \frac{1}{2}]$ and $\Omega_n = U_n \times I_n$, $n \geq 0$. Observe that each I_n tiles $\mathbb{R}$ translationally by $\mathbb{Z}$. So $\Omega_n + \{0\} \times \mathbb{Z} = U_n \times \mathbb{R}$. Set $T = \bigcup_{n=0}^{\infty} \Omega_n$.

Now, $B^{-k}\Omega_n = U_{n+k} \times I_n$. Thus $\bigcup_{k\in\mathbb{Z}} B^{-k}\Omega_n = \mathbb{R} \times I_n$ with the union disjoint. So $\bigcup_{k\in\mathbb{Z}} B^{-k}T = \mathbb{R}^2$ with the union disjoint. Hence T tiles by B. It remains to prove T tiles $\mathbb{R}^2$ by $Q^{-1}\mathbb{Z}^2$. This follows from the observation that $\{T + [0,k]^T : k \in \mathbb{Z}\}$ is a partition of $[-1/2, 1/2] \times \mathbb{R}$. Hence T tiles translationally by $Q^{-1}\mathbb{Z}^2$. ∎

The more difficult part of our theorems concerns the case in which $|\lambda_1| > 1$ and $|\lambda_2| < 1$. In this case again both $\lambda_1, \lambda_2 \in \mathbb{R}$, so there exists a $P \in \mathrm{GL}(2,\mathbb{R})$ such that $P^{-1}AP = \mathrm{diag}(\lambda_1, \lambda_2)$. It thus suffices to consider a diagonal matrix and ask which lattices $\mathcal{L}$ lead to simultaneous translational and multiplicative tilings by $\mathcal{L}$ and A.

Lemma 3.3. *Let $\beta \in \mathbb{R} \setminus \mathbb{Q}$ and let p_n/q_n be the n-th convergent of the continued fraction expansion of β, $q_n > 0$. Denote $M_n = q_n - 1$. Let $c, \varepsilon > 0$. Then there exists an $n_0 = n_0(c, \varepsilon)$ such that if $n \geq n_0$ the for any $p, q \in \mathbb{Z}$ with $\gcd(p,q) = 1$, $1 \leq q \leq M_n$ we have*

$$\left|\beta - \frac{p}{q}\right| \geq \frac{c}{qM_n^{1+\varepsilon}}.$$

Proof. Choose n_0 so that for all $n \geq n_0$ we have $c/M_n^{\varepsilon} < 1/2$, and $q_k \leq q_n - 2$ for all $1 \leq k < n$. Assume the lemma is false, then there exist $p^*, q^* \in \mathbb{Z}$, $\gcd(p^*, q^*) = 1$ and $1 \leq q^* \leq M_n$ such that

$$\left|\beta - \frac{p^*}{q^*}\right| < \frac{c}{q^* M_n^{1+\varepsilon}} < \frac{1}{2q^* M_n} \leq \frac{1}{2q^{*2}}. \tag{3.1}$$

It follows that $p^*/q^* = p_k/q_k$ for some $k < n$, see e.g. [17]. But we know from the properties of continued fractions that

$$\left|\beta - \frac{p^*}{q^*}\right| = \left|\beta - \frac{p_k}{q_k}\right| \geq \frac{1}{q_k(q_{k+1}+q_k)},$$

see also [17]. Now $q_k \leq q_n - 2$, so $q_{k+1} + q_k \leq 2M_n$. Thus $\left|\beta - \frac{p^*}{q^*}\right| \geq \frac{1}{2q^* M_n}$, which contradicts (3.1). ∎

Proposition 3.4. *Let $A = \operatorname{diag}(\lambda_1, \lambda_2)$ with $|\lambda_1| > 1 > |\lambda_2|$ and $|\lambda_1\lambda_2| > 1$. Let $\mathcal{L} = \mathbb{Z}\mathbf{u} + \mathbb{Z}\mathbf{v}$ be a full rank lattice in $\mathbb{R}^2$ with $\mathbf{u} = [u_1, u_2]^T$ and $\mathbf{v} = [v_1, v_2]^T$. Assume that $u_1/v_1 \in \mathbb{R} \setminus \mathbb{Q}$. Then there exists a measurable $T \subset \mathbb{R}^2$ such that T tiles translationally by $\mathcal{L}$ and multiplicatively by A.*

Proof. Write $a = |\lambda_1|$ and $b = |\lambda_2|^{-1}$. Then we have $a > b > 1$. We use Theorem 2.5 and prove that for any bounded $S \subset \mathbb{R}^2$ there exist infinitely many $n > 0$ such that $A^{-n}S$ packs by $\mathcal{L}$. Assume this is false then there exists a bounded $S \subset \mathbb{R}^2$ such that $A^{-n}S$ does not pack by $\mathcal{L}$ for all suffciently large n. We derive a contradiction.

Without loss of generality we assume that $S = [-r, r]^2$ for some r. Hence $A^{-n}S = [-ra^{-n}, ra^{-n}] \times [-rb^n, rb^n]$. Assume that $A^{-n}S$ does not pack translationally by $\mathcal{L}$. Since $\mathcal{L}$ is a laatice, we can find an $\alpha \neq 0$ in $\mathcal{L}$ such that $A^{-n}S \cap (A^{-n}S + \alpha) \neq \emptyset$. Therefore $\alpha \in A^{-n}S - A^{-n}S$, which gives $\alpha \in [-2ra^{-n}, 2ra^{-n}] \times [-2rb^n, 2rb^n]$. Let $\alpha = q\mathbf{u} + p\mathbf{v}$ for $p, q \in \mathbb{Z}$. Since $\mathbf{u}, \mathbf{v}$ are independent, there exists a $C > 0$ such that $|p|, |q| \leq Cb^n$.

Now write $\alpha = [x_0, y_0]^T$. Then $qu_1 + pv_1 = x_0$. Note that $|x_0| \leq 2ra^{-n}$. It follows that

$$\left|\frac{u_1}{v_1} + \frac{p}{q}\right| \leq \frac{2r|v_1|^{-1}}{|q|a^n}. \tag{3.2}$$

Choose $n = m_k$ such that $Cb^{m_k} \leq M_k < Cb^{m_k+1}$ where M_k are defined in Lemma 3.3. Write $a = b^{1+\varepsilon}$, $\varepsilon > 0$. Then

$$a^{m_k} \geq \left(b^{m_k}\right)^{1+\varepsilon} \geq (Cb)^{-1-\varepsilon} M_k^{1+\varepsilon}.$$

Thus from (3.2) we obtain

$$\left|\frac{u_1}{v_1} + \frac{p}{q}\right| = \left|\frac{u_1}{v_1} - \frac{p_0}{|q|}\right| \leq \frac{2r|v_1|^{-1}(bC)^{1+\varepsilon}}{|q|M_k^{1+\varepsilon}}, \tag{3.3}$$

where $p_0/|q| = -p/q$. Note that $|q| \leq M_k$. By Lemma 3.3 we have

$$\left|\frac{u_1}{v_1} - \frac{p_0}{|q|}\right| > \frac{2r|v_1|^{-1}(bC)^{1+\varepsilon}}{|q|M_k^{1+\varepsilon}}$$

for all sufficiently large k. This is a contradiction. ∎

Proposition 3.5. *Let $A = \operatorname{diag}(\lambda_1, \lambda_2)$ with $|\lambda_1| > 1 > |\lambda_2|$ and $|\lambda_1\lambda_2| > 1$. Let $\mathcal{L} = \mathbb{Z}\mathbf{u} + \mathbb{Z}\mathbf{v}$ be a full rank lattice in $\mathbb{R}^2$ with $\mathbf{u} = [u_1, u_2]^T$ and $\mathbf{v} = [v_1, v_2]^T$. Assume that $u_1/v_1 \in \mathbb{Q}$. Then there exists no measurable $T \subset \mathbb{R}^2$ such that T tiles translationally by $\mathcal{L}$ and multiplicatively by A.*

Proof. We first consider the case $\mathbf{u} = [1, c]^T$ and $\mathbf{v} = [0, 1]^T$. Then $F = (-1/2, 1/2]^2$ is a fundamental domain of $\mathcal{L}$, i.e. it tiles by $\mathcal{L}$. We also observe that the set $\Omega = \{[x_1, x_2]^T : \ |\lambda_1|^{-1} < 2|x_1| \leq 1, x_2 \in \mathbb{R}\}$ tiles multiplicatively by A.

Assume that a measurable $T \subset \mathbb{R}^2$ exists such that T tiles translationally by $\mathcal{L}$ and multiplicatively by A. Because T tiles by $\mathcal{L}$ there is a map $p_F : T \longrightarrow F$, which is one-to-one a.e., such that $x = p_F(x) + \alpha(x)$ for some unique $\alpha(x) \in \mathcal{L}$. We have already introduced the address map τ_T induced by T in (2.1). since both Ω and T tile by A, $\tau_T : \Omega \longrightarrow T$ is bijective a.e.. Let $\phi : \Omega \longrightarrow F$ be given by $\phi = p_F \circ \tau_T$.

By definition of τ_T we have $\tau_T(x) = A^{n(x)}x$ for some $n(x) \in \mathbb{Z}$. Let

$$\Omega_n = \{x \in \Omega : \ \tau_T(x) = A^n x\}.$$

Then $\{\Omega_n : \ n \in \mathbb{Z}\}$ is a partition of Ω. Since translations are measure preserving we have $\mu(\phi(\Omega_n)) = |\det(A)|^n \mu(\Omega_n)$. It follows that for $n \geq 0$

$$\sum_{n=0}^{\infty} \mu(S_n) = \sum_{n=0}^{\infty} |\det(A)|^{-n} \mu(\phi(\Omega_n)) \leq \sum_{n=0}^{\infty} \mu(\phi(\Omega_n)) \leq 1. \tag{3.4}$$

For $n < 0$ observe that $A^n[x_1, x_2]^T = [\lambda^n x_1, \lambda_2^n x_2]^T$. Now $|x_1| \leq 1/2$ for any $[x_1, x_2]^T \in \Omega_n$. So $p_F(A^n[x_1, x_2]^T) = [\lambda^n x_1, z_2]^T$ for some $|z_2| \leq 1/2$, and thus

$$\phi(\Omega_n) \subseteq [-|\lambda_1|^n/2, |\lambda_1|^n/2] \times [-1/2, 1/2].$$

Hence $\mu(\phi(\Omega_n)) = |\det(A)|^n \mu(\Omega_n) \leq |\lambda_1|^n$, which yields $\mu(\Omega_n) \leq |\lambda_2|^{-n}$. Thus

$$\sum_{n<0}^{\infty} \mu(\Omega_n) = \sum_{n<0}^{\infty} |\lambda_2|^{-n} < \infty. \tag{3.5}$$

Combining (3.4) and (3.5) yields $\mu(\Omega) < \infty$, a contradiction.

We now complete the proof of the proposition. In the general case, let $u_1/v_1 \in \mathbb{Q}$. Thus we can find a $\beta \in \mathbb{R}$ such that $\beta u_1 = p$, $\beta v_1 = q$ for some $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$. By Lemma 2.1 it suffices to prove there exists no measurable T such that T tiles translationally by $\beta\mathcal{L}$ and multiplicatively by A. Let $r, s \in \mathbb{Z}$ such that $rp + sq = 1$, and let $Q = \left[\begin{smallmatrix} r & -q \\ s & p \end{smallmatrix}\right]$. Then $\det(Q) = 1$,

$$\beta\mathcal{L} = \begin{bmatrix} p & q \\ \beta u_2 & \beta v_2 \end{bmatrix} \mathbb{Z}^2 = \begin{bmatrix} p & q \\ \beta u_2 & \beta v_2 \end{bmatrix} Q\mathbb{Z}^2 = \begin{bmatrix} 1 & 0 \\ b_1 & b_2 \end{bmatrix} \mathbb{Z}^2$$

for some $b_1, b_2 \in \mathbb{R}$. Finally, set $P_1 = \mathrm{diag}(1, b_2^{-1})$. Then $P_1^{-1}AP_1 = A$ and $P_1(\beta\mathcal{L}) = \mathbb{Z}[1, c]^T + \mathbb{Z}[0, 1]^T$ where $c = b_1/b_2$, and it suffices to prove there exists no measurable T such that T tiles translationally by $\mathbb{Z}[1, c]^T + \mathbb{Z}[0, 1]^T$ and multiplicatively by A. But this is precisely what we have proved earlier. ∎

Proof of Theorem 1.3. Let $\mathcal{L} = P\mathbb{Z}^2$ with $P \in \mathrm{GL}(2, \mathbb{R})$. By Lemma 2.1 there exists a measurable set T that tiles by $\mathcal{L}$ and A if and only if there exists a measurable set T_1 that tiles by $\mathbb{Z}^2$ and $P^{-1}AP$.

(a) If $|\det(A)| = 1$ then there exists no T such that $\mu(T) < \infty$ and T tiles multiplicatively by A. Thus any T that tiles by A cannot tile translationally by $\mathcal{L}$.

(b) By Propositions 3.1 and 3.2, there exists a measurable $T_1 \subset \mathbb{R}^2$ such that T_1 tiles by $\mathbb{Z}^2$ and $P^{-1}AP$. So there exists a measurable set $T \subset \mathbb{R}^2$ that tiles by $\mathcal{L}$ and A.

(c) Let $B = P^{-1}AP$. In this case both λ_1 and λ_2 are real. Hence there exists a $Q \in \mathrm{GL}(2, \mathbb{R})$ such that $Q^{-1}BQ = \mathrm{diag}(\lambda_1, \lambda_2)$. By Lemma 2.1 the existence of a measurable T_1 that tiles $\mathbb{R}^2$ translationally by $\mathbb{Z}^2$ and multiplicatively by B is equivalent to the existence of a measuable T_2 such that T_2 tiles $\mathbb{R}^2$ translationally by $Q^{-1}\mathbb{Z}^2$ and multiplicatively by $Q^{-1}BQ = \mathrm{diag}(\lambda_1, \lambda_2)$. Let $Q = [\mathbf{u}, \mathbf{v}]$ where $\mathbf{u}, \mathbf{v}$ are the columns of Q, which are the eigenvectors of $B = P^{-1}AP$ for the eigenvalues λ_1 and λ_2, respectively. Hence

$$Q = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}, \qquad Q^{-1} = \frac{1}{\det(Q)} \begin{bmatrix} v_2 & -v_1 \\ -u_2 & u_1 \end{bmatrix}.$$

By Propositions 3.4 and 3.5, a measurable T_2 that tiles translationally by $Q^{-1}\mathbb{Z}^2$ and multiplicatively by $\mathrm{diag}(\lambda_1, \lambda_2)$ if and only if $v_1/v_2 \in \mathbb{R} \setminus \mathbb{Q}$. This proves the theorem. ∎

Proof of Theorem 1.2. Since a wavelet set in $\mathbb{R}^2$ is a measurable set T that tiles translationally by $\mathbb{Z}^2$ and multiplicatively by A^T, this theorem is a clearly corollary of Theorem 1.3. ∎

References

[1] J. J. Benedetto and M. T. Leon, The construction of multiple dyadic minimally supported frequency wavelets on $\mathbb{R}^d$, in *The Functional And Harmonic Analysis Of Wavelets And Frames*, Amer. Math. Soc., Providence, RI, 1999.

[2] X. Dai and D. R. Larson, Wandering vectors for unitary systems and orthogonal wavelets, *Mem. Amer. Math. Soc.* **134**, no. 640 (1998).

[3] X. Dai, D. R. Larson, and D. Speegle, Wavelet sets in $\mathbb{R}^n$, *J. Fourier Anal. Appl.* **3** (1997), 451–456

[4] X. Dai, D. Larson and D. Speegle, Wavelet in $\mathbb{R}^n$. II. , *Wavelets, multiwavelets, and their applications* (*San Diego, CA, 1997*), 15–40, Contemp. Math., 216, Amer. Math. Soc., Providence, RI, 1998.

[5] I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **41** (1988), 906–996.

[6] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia 1991.

[7] X. Fang and X.-H. Wang, Construction of minimally supported frequency wavelets, *J. Fourier Anal. Appl.* **2** (1996), 315–327.

[8] D.-G. Han and Y. Wang, Lattice Tiling and the Weyl-Heisenberg families, *Geometric and Funct. Anal.* **11** (2001), no. 4, 742–758.

[9] E. Hernández, X.-H. Wang and G. Weiss, Smoothing minimally supported frequency wavelets. I, *J. Fourier. Anal. Appl.* **2** (1996), 329–340.

[10] E. Hernández, X.-H. Wang and G. Weiss, Smoothing minimally supported frequency wavelets. II, *J. Fourier. Anal. Appl.* **3** (1997), 23–41.

[11] E. Hernández and G. Weiss, *A first course on wavelets*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1996.

[12] E. Ionascu, A new construction of wavelet sets, *Real Anal. Exchange*,**28** (2002/03), no. 2, 593–609.

[13] E. J. Ionascu, D. R. Larson and C. M. Pearcy, On wavelet sets, *J. Fourier Anal. Appl.* **4** (1998), 711–721.

[14] E. Ionascu, D. Larson, and C. Pearcy, On the unitary systems affiliated with orthonormal wavelet theory in n-dimensions, *J. Funct. Anal.*, **157** (1998), 413–431.

[15] E. J. Ionascu and C. M. Pearcy, On subwavelet sets, *Proc. Amer. Math. Soc.* **126** (1998), 3549–3552.

[16] S. Jackson and R. D. Mauldin, On a lattice problem of H. Steinhaus, *J. of AMS* **15** (2002), 817–856.

[17] A. Y. Khinchin, *Continued fractions*, Dover Publications Inc., Mineola, New York, 1997.

[18] M. Kolountzakis, Multi-lattice tiles, *Internat. Math. Res. Notices* **19** (1997), 937–952.

[19] D. Larson, E. Schulz, D. Speegle and K. Taylor, Explicit cross sections of singly generated group actions, to appear.

[20] G. Olafsson and D. Speegle, Wavelets, wavelet sets, and linear actions on $\mathbb{R}^n$, in *Wavelets, Frames and Operator Theory*, 253-281, Contemp. Math. 345, Amer. Math. Soc. Providence, RI, 2004.

[21] P. M. Soardi and D. Weiland, Single wavelets in n-dimensions, *J. Fourier Anal. Appl.* **4** (1998), 299–315.

[22] D. Speegle, Dilation and translation tillings of $\mathbb{R}^n$ for non-expansive dilations, *Collect. Math.* **54** (2003), no. 2, 163–179.

[23] Y. Wang, Wavelets, tilings and spectral sets, Duke Math J. **114** (2004), no. 1, 43-57.

Columbus State University, 4225 University Ave., Columbus, GA 31907, USA.

E-mail address: `ionascu@math.gatech.edu`

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA.

E-mail address: `wang@math.gatech.edu`