Lattice Tiling and the Weyl-Heisenberg Frames

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Abstract

Let $\mathcal{L}$ and $\mathcal{K}$ be two full rank lattices in $\mathbb{R}^d$. We prove that if $\mathrm{v}(\mathcal{L}) = \mathrm{v}(\mathcal{K})$, i.e. they have the same volume, then there exists a measurable set Ω such that it tiles $\mathbb{R}^d$ by both $\mathcal{L}$ and $\mathcal{K}$. A counterexample shows that the above tiling result is false for three or more lattices. Furthermore, we prove that if $\mathrm{v}(\mathcal{L}) \leq \mathrm{v}(\mathcal{K})$ then there exists a measurable set Ω such that it tiles by $\mathcal{L}$ and packs by $\mathcal{K}$. Using these tiling results we answer a well known question on the density property of Weyl-Heisenberg frames.

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1 Introduction

Let $\mathcal{L}$ and $\mathcal{K}$ be two full-rank lattices in $\mathbb{R}^d$, and let $g(x) \in L^2(\mathbb{R}^d)$. The *Weyl-Heisenberg family*, also known as the *Gabor family*, is the following family of functions in $L^2(\mathbb{R}^d)$:

$$\mathbf{G}(\mathcal{L}, \mathcal{K}, g) := \left\{ e^{2\pi i \langle \ell, x\rangle} g(x-\kappa) \;\middle|\; \ell \in \mathcal{L}, \kappa \in \mathcal{K} \right\}. \tag{1.1}$$

Such a family was first introduced by Gabor [Ga] in 1946 for signal processing, and is still widely used today. For recent developments on Weyl-Heisenberg (Gabor) analysis, we refer to the book [FS] by Feichtinger and Strohmer, and a survey paper [Ca] by Casazza.

In signal processing we often require the Weyl-Heisenberg family be either an orthonormal basis (windowed Fourier transform) or a frame of $L^2(\mathbb{R}^d)$. Recall that a family of functions $\{f_j\}$ in $L^2(\mathbb{R}^d)$ is a frame if there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|f\|_2^2 \leq \sum_j |\langle f, f_j\rangle|^2 \leq C_2 \|f\|_2^2 \tag{1.2}$$

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for all $f \in L^2(\mathbb{R}^d)$. If $C_1 = C_2 = 1$ we say $\{f_j\}$ is a *normalized tight frame.* One of the well known questions is the so-called density problem for Weyl-Heisenberg families:

Question 1: Let $\mathcal{L}$ and $\mathcal{K}$ be two full-rank lattices in $\mathbb{R}^d$. Under what conditions can we find a function $g \in L^2(\mathbb{R}^d)$ such that the Weyl-Heisenberg family $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is an orthonormal basis (frame) of $L^2(\mathbb{R}^d)$?

Question 1 has been answered completely in the one dimension case. Let $\mathcal{L} = a\mathbb{Z}$ and $\mathcal{K} = b\mathbb{Z}$. Suppose that $|ab| \leq 1$. Then it is trivial to show that $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is a tight frame when $g = \frac{1}{\sqrt{|b|}}\chi_{[0,|b|]}$, which is an orthonormal basis if $|ab| = 1$. Conversely, Rieffel [Rie] proves the following density theorem, which asserts that it is necessary that $|ab| \leq 1$ for $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ to be complete in $L^2(\mathbb{R})$. In higher dimensions, analogous necessary conditions have been established, see [RSh], [RSt] and [CDH]. Let $\mathcal{L} = A\mathbb{Z}^d$ and $\mathcal{K} = B\mathbb{Z}^d$ where A and B are real $d \times d$ nonsingular matrices. The density result states that one necessarily has $|\det(AB)| = 1$ if $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is an orthonormal basis, and $|\det(AB)| \leq 1$ if $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is a frame. Interestingly the converse, which is trivial in the one dimension, remained unsolved. In this paper we prove the converse by studying a seemingly unrelated problem concerning lattice tiling in $\mathbb{R}^d$.

We now consider lattice tiling in $\mathbb{R}^d$. Let Ω be a measurable set in $\mathbb{R}^d$ (not necessarily bounded), and let $\mathcal{L}$ be a full rank lattice in $\mathbb{R}^d$. We say Ω *tiles* $\mathbb{R}^d$ by $\mathcal{L}$, or Ω is a *fundamental domain* of $\mathcal{L}$, if

(i) $\bigcup_{\ell \in \mathcal{L}} (\Omega + \ell) = \mathbb{R}^d$ a.e.;

(ii) $(\Omega + \ell) \cap (\Omega + \ell')$ has Lebesgue measure 0 for any $\ell \neq \ell'$ in $\mathcal{L}$.

We say that Ω *packs* $\mathbb{R}^d$ by $\mathcal{L}$ if only (ii) holds. Equivalently, Ω tiles $\mathbb{R}^d$ by $\mathcal{L}$ if and only if

$$\sum_{\ell \in \mathcal{L}} \chi_\Omega(x - \ell) = 1 \quad \text{for a.e. } x \in \mathbb{R}^d, \tag{1.3}$$

and Ω packs $\mathbb{R}^d$ by $\mathcal{L}$ if and only if

$$\sum_{\ell \in \mathcal{L}} \chi_\Omega(x - \ell) \leq 1 \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{1.4}$$

Let $\mathrm{v}(\mathcal{L})$ denote the volume of $\mathcal{L}$, i.e. $\mathrm{v}(\mathcal{L}) = |\det(A)|$ for $\mathcal{L} = A\mathbb{Z}^d$. Clearly, $\mu(\Omega) = \mathrm{v}(\mathcal{L})$ if Ω tiles by $\mathcal{L}$, and $\mu(\Omega) \leq \mathrm{v}(\mathcal{L})$ if Ω packs by $\mathcal{L}$. Furthermore, if Ω packs $\mathbb{R}^d$ by $\mathcal{L}$ and $\mu(\Omega) = \mathrm{v}(\mathcal{L})$, then Ω necessarily tiles $\mathbb{R}^d$ by $\mathcal{L}$. One of the questions we study here is:

Question 2: Let $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m$ be full rank lattices in $\mathbb{R}^d$ such that $\mathrm{v}(\mathcal{L}_1) = \mathrm{v}(\mathcal{L}_2) = \cdots = \mathrm{v}(\mathcal{L}_m)$. Does there exist a measurable set Ω in $\mathbb{R}^d$ such that Ω tiles $\mathbb{R}^d$ by $\mathcal{L}_j$ for each $1 \leq j \leq m$?

Question 2 is closely related to a well known open problem of Steinhaus', which asks whether there exists a set Ω that tiles $\mathbb{R}^2$ by every lattice of the form $R_\theta\mathbb{Z}^2$ where R_θ is the rotation matrix by the angle θ. Kolountzakis [Ko] shows that Question 2 has an affirmative answer if the sum $\mathcal{L}_1^* + \cdots + \mathcal{L}_m^*$ is direct, where $\mathcal{L}_i^*$ denotes the dual lattice of $\mathcal{L}_i$. A summary on the problem of Steinhaus' can also be found in [Ko]. It should be pointed out that the requirement that the sum of the lattices be direct is rather strong. In particular it is not satisfied if two of the matrices A_j contain rational columns, where $\mathcal{L}_j = A_j\mathbb{Z}^d$. We prove:

Theorem 1.1 *Let $\mathcal{L}, \mathcal{K}$ be two full rank lattices in $\mathbb{R}^d$ such that $\mathrm{v}(\mathcal{L}) = \mathrm{v}(\mathcal{K})$. Then there exists a measurable set Ω in $\mathbb{R}^d$ such that Ω tiles $\mathbb{R}^d$ by both $\mathcal{L}$ and $\mathcal{K}$.*

The answer to Question 2 is negative for $m \geq 3$ in general in dimensions $d \geq 2$, as first pointed out in [Ko]. The following is a counterexample example:

Example 1.1. Consider the following three lattices in $\mathbb{R}^2$,

$$\mathcal{L}_1 = \mathbb{Z}^2, \quad \mathcal{L}_2 = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\mathbb{Z}^2, \quad \mathcal{L}_3 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}\mathbb{Z}^2.$$

Then $\mathrm{v}(\mathcal{L}_i) = 1$, and there exists no measurable set Ω that tiles by each $\mathcal{L}_i$. The product lattices $\mathcal{L}_i \times \mathbb{Z}^{d-2}$ also yield a counterexample to Question 2 in dimensions $d > 2$ for $m \geq 3$. ∎

Theorem 1.1 is in fact a corollary of the following more general theorem:

Theorem 1.2 *Let $\mathcal{L}, \mathcal{K}$ be two full rank lattices in $\mathbb{R}^d$ such that $\mathrm{v}(\mathcal{L}) \geq \mathrm{v}(\mathcal{K})$. Then there exists a measurable set Ω in $\mathbb{R}^d$ such that Ω tiles $\mathbb{R}^d$ by $\mathcal{K}$ and packs $\mathbb{R}^d$ by $\mathcal{L}$.*

We apply Theorems 1.1 and 1.2 to prove the following density theorem for Weyl-Heisenberg families, answering Question 1:

Theorem 1.3 *Let $\mathcal{L}$, $\mathcal{K}$ be two full rank lattices in $\mathbb{R}^d$. Then*

(i) *There exists a $g(x) \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ if and only if $\mathrm{v}(\mathcal{L})\mathrm{v}(\mathcal{K}) = 1$.*

(ii) *There exists a $g(x) \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is a frame of $L^2(\mathbb{R}^d)$ if and only if $\mathrm{v}(\mathcal{L})\mathrm{v}(\mathcal{K}) \leq 1$.*

In §2 we prove our results on lattice tiling and packing. In §3 we prove several results on Weyl-Heisenberg families, of which Theorem 1.3 is a corollary.

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2 Lattice Tiling

In this section we prove Theorem 1.2, which also implies Theorem 1.1. We first introduce some notations. The torus $\mathbb{R}^d/\mathbb{Z}^d$ is denoted by $\mathbb{T}^d$, and $\pi_d : \mathbb{R}^d \to \mathbb{T}^d$ denotes the canonical map. The Haar measure of $\mathbb{T}^d$ will be denoted by $\nu(\cdot)$, with $\nu(\mathbb{T}^d) = 1$.

Before proceeding with our proofs we examine the structure of subgroups of $\mathbb{T}^d$. A subset $S \subseteq \mathbb{T}^d$ is called a *subspace* if $S = \pi_d(V)$ where $V \subset \mathbb{R}^d$ is a linear subspace. The subspace S is called *rational* if V is rational, i.e. it has a basis consisting of vectors in $\mathbb{Q}^d$. It is known that any closed subspace of $\mathbb{T}^d$ must be rational, and the closure of any subspace $S = \pi_d(V)$ of $\mathbb{T}^d$ is $\pi_d(V')$ where V' is the smallest rational subspace in $\mathbb{R}^d$ containing V (see e.g. Lagarias and Wang [LW1]).

Lemma 2.1 *Let G be a closed subgroup of $\mathbb{T}^d$. Then*

$$G = S \oplus F \tag{2.1}$$

where S is a rational subspace of $\mathbb{T}^d$ and F is a finite group.

Proof. This result is Proposition 11 in Bourbaki [Bou], §1.5. It states that $G = S \oplus F$ where S is isomorphic to $\mathbb{T}^h$ for some $0 \leq h \leq n$ and F is a finite subgroup of $\mathbb{T}^d$. It is clear in the proof that $S = \pi_d(V)$ for some vector subspace of $\mathbb{R}^d$. The rationality of V also follows from the proof. Another proof of the rationality of V can be found in [LW1]. ∎

For any $s \in \mathbb{T}^d$ let τ_s denote the translation $\tau_s(x) = x + s$ in $\mathbb{T}^d$. Suppose that $\Omega \subseteq \mathbb{T}^d$ and S is a countable subset of $\mathbb{T}^d$. We say that $\tilde{\Omega}$ is *S-shifted* from Ω, or $\tilde{\Omega}$ is an *S-shift* of Ω, if Ω has a measure disjoint partition $\Omega = \bigcup_{s\in S} \Omega^{(s)}$ such that

$$\tilde{\Omega} = \bigcup_{s\in S} \tau_s(\Omega^{(s)}),$$

where the above union is measure disjoint. We say a subset Ω of $\mathbb{T}^d$ is a *polytope* (respectively, cube, parallelopiped, etc.) if it is the projection of a polytope (respectively, cube, parallelopiped, etc.) in $\mathbb{R}^d$.

An essential lemma for proving Theorem 1.2 is:

Lemma 2.2 *Let S be a dense countable subset of $\mathbb{T}^d$. Let Ω and R be finite unions of polytopes in $\mathbb{T}^d$ such that $\nu(\Omega) \leq \nu(R)$. Then there exists an S-shift $\tilde{\Omega}$ of Ω such that $\tilde{\Omega} \subseteq R$.*

Proof. The idea of the proof is to cut R into small cubes and Ω into slightly smaller cubes and translate the smaller cubes of Ω into the cubes of R using τ_s, applying the fact that S is dense in $\mathbb{T}^d$.

Since Ω and R are finite unions of polytopes we may find a finite set of measure disjoint cubes $\mathcal{C} = \{C_1, C_2, \ldots, C_N\}$ in Ω and a finite set of measure disjoint cubes $\mathcal{E} = \{E_1, E_2, \ldots, E_M\}$ in R with the following properties:

(a) All cubes C_i have the same size with length $\varepsilon > 0$, and all cubes E_i have the same size with length $\delta > 0$.

(b) $\sum_{i=1}^{N} \nu(C_i) \geq \frac{1}{2}\nu(\Omega)$ and $\sum_{i=1}^{M} \nu(E_i) \geq \frac{1}{2}\nu(R)$.

(c) $\delta > \varepsilon \geq \frac{1}{2}\delta$.

Observe that properties (a) and (b) are clearly possible if we take ε and δ sufficiently small. Given $\mathcal{C}$ and $\mathcal{E}$ with properties (a) and (b) we can then always subdivide the cubes so that property (c) is met.

Let $L = \max\{N, M\}$. Since C_i is strictly smaller in size than E_i, and since S is dense in $\mathbb{T}^d$, we may find $s_i \in S$ such that $\tau_{s_i}(C_i) \subseteq E_i$ for $1 \leq i \leq L$. Denote

$$\tilde{\Omega}_1 = \bigcup_{i=1}^{L} \tau_{s_i}(C_i) \quad \text{and} \quad \Omega_1 = \Omega \setminus \Big(\bigcup_{i=1}^{L} C_i\Big).$$

Claim: $\nu(\tilde{\Omega}_1) \geq 2^{-(d+1)}\nu(\Omega)$.

To see this, if $L = N$ then $\nu(\tilde{\Omega}_1) = \sum_{i=1}^{N} \nu(C_i) \geq 2^{-1}\nu(\Omega)$, and the claim holds. On the other hand, if $L = M$ then

$$\begin{aligned} \nu(\tilde{\Omega}_1) &= \sum_{i=1}^{M} \nu(C_i) = M\varepsilon^d \geq M\Big(\frac{\delta}{2}\Big)^d \\ &= \frac{1}{2^d}\sum_{i=1}^{M} \nu(E_i) \geq 2^{-(d+1)}\nu(R) \geq 2^{-(d+1)}\nu(\Omega). \end{aligned}$$

This proves the claim.

To summarize, we have shown that the following procedure can be completed:

For any finite unions of polytopes Ω and R in $\mathbb{T}^d$ with $\nu(\Omega) \leq \nu(R)$ there exists a finite collection of disjoint cubes $\{C_i : 1 \leq i \leq L\}$ in Ω, such that an S-shift $\tilde{\Omega}_1$ of these cubes satisfies $\nu(\tilde{\Omega}_1) \geq c_0\nu(\Omega)$ for $c_0 = 2^{-(d+1)}$ and $\tilde{\Omega}_1 \subseteq R$.

We perform the above procedure inductively for Ω_k and R_k in place of Ω and R, $k = 0, 1, 2, \ldots$, starting with $\Omega_0 = \Omega$ and $R_0 = R$. From Ω_k and R_k we obtain some disjoint cubes $\{C_i^k : 1 \leq i \leq N_k\}$ in Ω_k, such that an S-shift $\tilde{\Omega}_{k+1}$ of the cubes satisfies $\nu(\tilde{\Omega}_{k+1}) \geq c_0\nu(\Omega_k)$ and $\tilde{\Omega}_{k+1} \subseteq R_k$. Set $\Omega_{k+1} = \Omega_k \setminus (\bigcup_{i=1}^{L_k} C_i^k)$ and $R_{k+1} = R_k \setminus \tilde{\Omega}_{k+1}$. Observe that the procedure discribed in this proof guarantees that both Ω_{k+1} and R_{k+1} are still finite unions of polytopes in $\mathbb{T}^d$, since they are obtained by removing a finitely many cubes from finite unions of polytopes.

Now we have obtained measure disjoint sets $\tilde{\Omega}_k$ for $k \geq 1$. Let $\tilde{\Omega} = \bigcup_{k\geq 1} \tilde{\Omega}_k$. Then $\tilde{\Omega}$ is S-shifted from a subset of Ω, and $\tilde{\Omega} \subseteq R$. But note that

$$\nu(\Omega_{k+1}) = \nu(\Omega_k) - \nu(\tilde{\Omega}_{k+1}) \leq (1 - c_0)\nu(\Omega_k).$$

Hence $\nu(\Omega_k) \leq (1-c_0)^k \nu(\Omega) \to 0$ as $k \to \infty$. It follows that $\tilde{\Omega}$ is in fact S-shifted from the entire Ω, not just a subset of it. This proves the lemma. ∎

The notion of S-shift of a set obviously applies to $\mathbb{R}^d$. For any $s \in \mathbb{R}^d$ we denote $\tau_s(x) := x + s$ (a slight abuse of notation). Let S be a countable subset of $\mathbb{R}^d$ and $\Omega \subseteq \mathbb{R}^d$. We say that $\tilde{\Omega}$ is an *S-shift* of Ω if Ω has a measure disjoint partition $\Omega = \bigcup_{s \in S} \Omega^{(s)}$ such that

$$\tilde{\Omega} = \bigcup_{s \in S} \tau_s(\Omega^{(s)}),$$

where the above union is measure disjoint.

Corollary 2.3 *Let S be a countable subset of $\mathbb{R}^d$ such that $\pi_d(S)$ is dense in $\mathbb{T}^d$. Let Ω and R be finite unions of polytopes in $\mathbb{R}^d$ and $\mathbb{T}^d$, respectively, with $\mu(\Omega) \leq \nu(R)$. Then there exists an S-shift $\tilde{\Omega}$ of Ω such that $\pi_d : \tilde{\Omega} \longrightarrow R$ is one-to-one.*

Proof. Since Ω is a finite union of polytopes we may partition Ω into Ω_k for $k = 1, 2, \ldots, m$, each Ω_k a finite union of polytopes, such that $\pi_d : \Omega_k \longrightarrow \mathbb{T}^d$ is one-to-one. Now, partition R into R_k for $1 \leq k \leq m$ with the properties that each R_k is a finite union of polytopes and $\nu(R_k) \geq \mu(\Omega_k)$. Let $S^* = \pi_d(S)$. S^* is dense in $\mathbb{T}^d$, so by Lemma 2.2 there exist S^*-shifts $\tilde{\Omega}_k^*$ of $\pi_d(\Omega_k)$ such that $\tilde{\Omega}_k^* \subseteq R_k$. Since $\pi_d : \Omega_k \longrightarrow \mathbb{T}^d$ is one-to-one, we may find a $\tilde{\Omega}_k$ in $\mathbb{R}^d$ such that $\tilde{\Omega}_k$ is an S-shift of Ω_k and $\pi_d(\tilde{\Omega}_k) = \tilde{\Omega}_k^*$. Set $\tilde{\Omega} = \bigcup_{k=1}^m \tilde{\Omega}_k$. Then $\tilde{\Omega}$ is an S-shift of Ω and $\pi_d : \tilde{\Omega} \longrightarrow R$ is one-to-one. ∎

Proof of Theorem 1.2. Without loss of generality we may assume that $\mathcal{L} = \mathbb{Z}^d$ and $\mathcal{K} = A\mathbb{Z}^d$ where $A \in M_d(\mathbb{R})$ with $|\det A| \leq 1$. We will call A *good* if there exists an Ω that tiles $\mathbb{R}^d$ by $A\mathbb{Z}^d$ and packs $\mathbb{R}^d$ by $\mathbb{Z}^d$.

Let $\mathcal{J}$ be any full-rank lattice in $\mathbb{R}^d$. Two measurable sets Ω_1 and Ω_2 are said to be *$\mathcal{J}$-congruent* if Ω_1 is a $\mathcal{J}$-shift of Ω_2. The lattice property assures that $\mathcal{J}$-congruence is an equivalent relation. Furthermore, suppose that Ω_1 and Ω_2 are $\mathcal{J}$-congruent. Then Ω_1 tiles (packs) by $\mathcal{J}$ if and only if Ω_2 does. Our goal is to find a fundamental domain Ω_2 of $\mathcal{K}$ and construct a $\mathcal{K}$-congruent set Ω_1 that packs by $\mathcal{L}$.

Now note that $\overline{\pi_d(\mathcal{K})}$ is a closed subgroup of $\mathbb{T}^d$. So $\overline{\pi_d(\mathcal{K})} = S \oplus F$ for some rational subspace S and finite set F. We divide our proof into three cases: $S = \mathbb{T}^d$, $S = \{0\}$ and neither of the above. The last case is the most difficult case, and we hope the proof of the first two cases will make the general idea more clear.

Case I: $S = \mathbb{T}^d$

Under this condition $\pi_d(\mathcal{K}) = \pi_d(A\mathbb{Z}^d)$ is dense in $\mathbb{T}^d$. This case includes the condition in [Ko] but not equivalent to it.

We will construct an Ω that tiles $\mathbb{R}^d$ by $\mathcal{K}$ and packs by $\mathbb{Z}^d$. Start with Ω_1 being the parallelopiped spanned by the columns of A. Since $\mu(\Omega_1) \leq 1$, it follows from Corollary 2.3 that there exists a $\mathcal{K}$-shift Ω of Ω_1 such that $\pi : \Omega \longrightarrow \mathbb{T}^d$ is one-to-one. Hence Ω packs by $\mathbb{Z}^d$. It is $\mathcal{K}$-congruent to Ω_1 so it tiles by $\mathcal{K}$. This proves the theorem in Case I.

Case II. $S = \{0\}$

In this case $\mathcal{K} = A\mathbb{Z}^d$ and $\mathbb{Z}^d$ are commensurable. Equivalently, $A \in M_d(\mathbf{Q})$ is a rational matrix. To prove A is good we make use of the Smith canonical form.

Sub Lemma 1 *Let $P, Q \in M_n(\mathbb{Z})$ be unimodular matrices (i.e. $|\det P| = |\det Q| = 1$). Then A is good if and only if PAQ is good.*

Proof. Suppose that A is good. Then there exists an Ω such that $\Omega + A\mathbb{Z}^d$ is a tiling of $\mathbb{R}^d$ and $\Omega + \mathbb{Z}^d$ is a packing of $\mathbb{R}^d$. So

$$\begin{aligned} P(\Omega) + PA\mathbb{Z}^d &= P(\Omega) + PAQQ^{-1}\mathbb{Z}^d \\ &= P(\Omega) + PAQ\mathbb{Z}^d \end{aligned}$$

is a tiling of $\mathbb{R}^d$. Similarly,

$$P(\Omega) + P\mathbb{Z}^d = P(\Omega) + \mathbb{Z}^d$$

is a packing of $\mathbb{R}^d$. Hence PAQ is good. Conversely, if PAQ is good then it follows immediately that $A = P^{-1}(PAQ)Q^{-1}$ is good since P^{-1}, Q^{-1} are unimodular matrices in $M_d(\mathbb{Z})$. □

Since $A \in M_d(\mathbf{Q})$, $A = \frac{1}{q}\tilde{A}$ with $\tilde{A} \in M_d(\mathbb{Z})$ for some $q \in \mathbb{Z}$. The Smith canonical form (see Newman [New]) for $\hat{A}$ implies that there exist unimodular integral matrices P, Q such that

$$P\tilde{A}Q = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_d \end{bmatrix}$$

where each $r_i \in \mathbb{Z}$ and $r_i | r_{i+1}$. By Sub Lemma 1 we may without loss of generality assume that

$$A = \frac{1}{q}\text{diag } (r_1, r_2, \ldots, r_d),$$

where each $r_i \in \mathbb{Z}$ and $r_i | r_{i+1}$.

We prove A is good. Write

$$A = \text{diag } \Big(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_d}{q_d}\Big), \quad \text{with} \quad (p_i, q_i) = 1.$$

The rectangular parallelopiped spanned by the columns of A is

$$\Omega_1 = [0, \frac{p_1}{q_1}) \times \cdots \times [0, \frac{p_d}{q_d}),$$

which is a fundamental domain of $\mathcal{K}$. Let T be the smaller rectangular parallelopiped $T = [0, \frac{1}{q_1}) \times \cdots \times [0, \frac{1}{q_d})$. Then

$$\Omega_1 = T + \Big\{[\frac{k_1}{q_1}, \ldots, \frac{k_d}{q_d}]^T : \ 0 \leq k_i < p_i\Big\} := T + \mathcal{F}.$$

Our goal is to construct a $\mathcal{K}$-shift Ω of Ω_1 by translating the smaller rectangular parallelopiped so that Ω packs by $\mathbb{Z}^d$. To do so, observe that the unit cube $[0,1)^d$ satisfies

$$[0,1)^d = T + \Big\{[\frac{k_1}{q_1}, \ldots, \frac{k_d}{q_d}]^T : \ 0 \le k_i < q_i\Big\} := T + \mathcal{G}.$$

Now order the elements of $\mathcal{F}$ and $\mathcal{G}$ (say lexicographically),

$$\mathcal{F} = \{\alpha_1, \alpha_2, \ldots, \alpha_M\}, \quad \mathcal{G} = \{\beta_1, \beta_2, \ldots, \beta_N\}.$$

It follows from $|\det(A)| \le 1$ that $M \le N$. We prove that there exists a $\gamma_i \in \mathcal{K}$ for each $1 \le i \le M$ such that

$$\alpha_i + \gamma_i \equiv \beta_i \ (\text{mod } 1). \tag{2.2}$$

To do so, note that

$$\mathcal{K} = \Big\{[\frac{p_1}{q_1}m_1, \ldots, \frac{p_d}{q_d}m_d]^T : \ m_i \in \mathbb{Z}\Big\}$$

Assume that

$$\alpha_i = [\frac{k_1}{q_1}, \ldots, \frac{k_d}{q_d}]^T, \quad \beta_i = [\frac{n_1}{q_1}, \ldots, \frac{n_d}{q_d}]^T.$$

Since $(p_j, q_j) = 1$, there exists an m_j such that $k_j + p_j m_j \equiv n_j$ (mod q_j) for each j. Taking $\gamma_i = [\frac{p_1}{q_1}m_1, \ldots, \frac{p_d}{q_d}m_d]^T$ yields $\alpha_i + \gamma_i \equiv \beta_i$ (mod 1).

Finally, set $\Omega = T + \{\alpha_i + \gamma_i : \ 1 \le i \le M\}$. The fact that $\gamma_i \in \mathcal{K}$ implies that Ω is $\mathcal{K}$-congruent to Ω_1 and so it tiles by $\mathcal{K}$. It also follows from (2.2) that Ω is $\mathbb{Z}^d$-congruent to $T + \{\beta_i : \ 1 \le i \le M\}$, a subset of $[0,1)^d$. Hence Ω packs by $\mathbb{Z}^d$.

A corollary of the proof is that we may choose our Ω to be bounded — in fact, a finite union of congruent parallelopiped.

Case III. None of the Above

Here we have $\overline{\mathcal{K}(\text{mod } 1)} = S \oplus F$ where S is a rational subspace of dimension e with $0 < e < d$.

Sub Lemma 2 *There exist unimodular matrices P and Q such that*

$$PAQ = \begin{bmatrix} A_1 & B \\ 0 & D \end{bmatrix},$$

where $D = \text{diag }(r_1, \ldots, r_{d-e})$ for $r_i \in \mathbf{Q}$, and $[A_1 \ B]\mathbb{Z}^d$ (mod 1) is dense in $[0,1]^e$.

Proof. Let $S = \pi_d(V)$ where V is a e-dimensional rational subspace of $\mathbb{R}^d$. It is known (see e.g. [Sch]) that there exists a unimodular $P_1 \in M_d(\mathbb{Z})$ such that

$$P_1 V = \mathbb{R}^e \times \{0\} \subset \mathbb{R}^d,$$

namely P_1 maps V to the first e coordinates of $\mathbb{R}^d$. Hence

$$P_1 A = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix},$$

where E_1 is $e \times d$ and $E_1(\mathbb{Z}^d) \pmod 1$ is dense in $[0,1]^e$. Clearly, E_2 is rational, or $E_2(\mathbb{Z}^d) \pmod 1$ would be infinite. Now the Smith canonical form applied to E_2 yields

$$P_2 E_2 Q = [0\ D], \text{ where } D = \text{diag } (r_1, \ldots, r_{d-e}) \text{ for } r_i \in \mathbf{Q}.$$

By denoting $E_1 = [A_1\ B]$ we obtain

$$P_2 P_1 A Q = \begin{bmatrix} A_1 & B \\ 0 & D \end{bmatrix},$$

proving Sub Lemma 2. □

So by Sub Lemma 2 we may without loss of the generality assume that

$$A = \begin{bmatrix} A_1 & B \\ 0 & D \end{bmatrix}$$

with $D = \text{diag } (\frac{p_1}{q_1}, \ldots, \frac{p_{e-d}}{q_{e-d}})$ for $(p_i, q_i) = 1$, $q_i > 0$, and $[A_1\ B]\mathbb{Z}^d \pmod 1$ dense in $[0,1]^e$. For simplicity denote $r = d - e$. An element α of $\mathcal{K}$ has the form $\alpha = [\alpha_e, \alpha_r]^T$, in which $\alpha_e \in [A_1\ B]\mathbb{Z}^d$ and $\alpha_r = [\frac{p_1 m_1}{q_1}, \ldots, \frac{p_r m_r}{q_r}]^T \in D\mathbb{Z}^r$ for $m_1, \ldots, m_r \in \mathbb{Z}$.

Sub Lemma 3 *Let $\beta_1,\ \beta_2 \in \mathbb{R}^r$ such that*

$$\beta_1 = [\frac{k_1}{q_1}, \ldots, \frac{k_r}{q_r}]^T,\ \beta_1 = [\frac{l_1}{q_1}, \ldots, \frac{l_r}{q_r}]^T, \quad \textit{where all}\ \ k_i,\ l_i \in \mathbb{Z}.$$

Let $\mathcal{J} = \{[\alpha_e, \alpha_r]^T \in \mathcal{K} :\ \beta_1 + \alpha_r \equiv \beta_2 \pmod 1\}$. Then there exists a $\gamma \in \mathcal{K}$ such that

$$\mathcal{J} \supseteq \gamma + N\mathcal{K}, \quad \textit{where} \quad N = q_1 q_2 \cdots q_r.$$

Proof. Since $(p_i, q_i) = 1$ it is well known that the solutions to the linear Diophantine equation $k_i + p_i x \equiv l_i \pmod 1$, which is equivalent to

$$\frac{k_i}{q_i} + \frac{p_i x}{q_i} \equiv \frac{l_i}{q_i} \pmod 1,$$

are $x \in q_i\mathbb{Z} + a_i$ for some $a_i \in \mathbb{Z}$. Therefore the set

$$\mathcal{I}_r := \{\alpha_r \in D\mathbb{Z}^r :\ \beta_1 + \alpha_r \equiv \beta_2 \pmod 1\}$$

satisfies $\mathcal{I}_r \supseteq D(N\mathbb{Z}^r + \gamma_r)$ for $\gamma_r = [a_1, \ldots, a_r]^T$. Hence

$$\begin{aligned} \mathcal{J} &= \left\{[\alpha_e, \alpha_r]^T \in \mathcal{K} :\ \alpha_r \in \mathcal{I}_r\right\} \\ &\supseteq \left\{\begin{bmatrix} A_1 & B \\ 0 & D \end{bmatrix} \begin{bmatrix} z_e \\ N z_r + \gamma_r \end{bmatrix} :\ z_e \in \mathbb{Z}^e, z_r \in \mathbb{Z}^r\right\} \\ &\supseteq \left\{\begin{bmatrix} A_1 & B \\ 0 & D \end{bmatrix} \begin{bmatrix} z_e \\ N z_r \end{bmatrix} + \gamma :\ z_e \in \mathbb{Z}^e, z_r \in \mathbb{Z}^r\right\} \\ &\supseteq N\mathcal{K} + \gamma, \quad \text{where} \quad \gamma := \begin{bmatrix} A_1 & B \\ 0 & D \end{bmatrix} \begin{bmatrix} 0 \\ \gamma_r \end{bmatrix}. \end{aligned}$$

This proves the sub lemma. $\square$

Sub Lemma 4 *Let $\mathcal{J}$ be as in Sub Lemma 3. Then the set*

$$\{\alpha_e \ (mod\ 1): \ [\alpha_e, \alpha_r]^T \in \mathcal{J} \ for\ some\ \alpha_r \in \mathbb{R}^r\}$$

is dense in $[0,1]^e$.

Proof. By Sub Lemma 3 the set $\{\alpha_e\}$ contains the set $N[A_1\ B]\mathbb{Z}^d + \gamma_e$ for some $\gamma_e \in \mathbb{R}^e$. since $[A_1\ B]\mathbb{Z}^d$ (mod 1) is dense in $[0,1]^e$, $N[A_1\ B]\mathbb{Z}^d + \gamma_e$ (mod 1) is also dense in $[0,1]^e$. $\square$

Now a fundamental domain of $\mathcal{K} = A\mathbb{Z}^d$ is

$$\tilde{\Omega} = \Omega^e \times \Omega^r, \quad \text{where} \ \ \Omega^e = A_1([0,1)^e), \ \Omega^r = D([0,1)^r).$$

Let $T^r = [0, \frac{1}{q_1}) \times \cdots \times [0, \frac{1}{q_r})$. Then

$$\Omega^r = T^r \oplus \mathcal{F}_r, \quad \text{where} \ \ \mathcal{F}_r = \Big\{[\frac{k_1}{q_1}, \ldots, \frac{k_r}{q_r}]^T : \ 0 \le k_i < p_i\Big\},$$

which yields

$$\tilde{\Omega} = \Omega^e \times (T^r \oplus \mathcal{F}_r) = \bigcup_{\alpha \in \mathcal{F}_r} \Omega^e \times (T^r + \alpha). \tag{2.3}$$

Meanwhile, the $\mathbb{Z}^d$-tile $[0,1)^d$ has a decomposition

$$[0,1)^d = [0,1)^e \times [0,1)^r = \bigcup_{\alpha \in \mathcal{F}_r} R_\alpha \times [0,1)^r, \tag{2.4}$$

in which $[0,1)^e$ is partitioned into $|\mathcal{F}_r|$ disjoint rectangular parallelopiped R_α of equal volume in $\mathbb{R}^e$ indexed by the elements of $\mathcal{F}_r$. Since $\mu(\tilde{\Omega}) \le 1$ we have $\mu(\Omega^e \times (T^r + \alpha)) \le \mu(R_\alpha \times [0,1)^r)$.

Sub Lemma 5 *For each $\alpha \in \mathcal{F}_r$ there exists a $\mathcal{K}$-shift Ω_α of $\Omega^e \times (T^r + \alpha)$ such that $\pi_d : \Omega_\alpha \longrightarrow R_\alpha \times [0,1)^r$ is one-to-one, where we view $R_\alpha \times [0,1)^r$ as a subset of $\mathbb{T}^d$.*

Proof. Observe that

$$[0,1)^r = T^r \oplus \mathcal{G}_r, \quad \text{where} \ \ \mathcal{G}_r = \Big\{[\frac{k_1}{q_1}, \ldots, \frac{k_r}{q_r}]^T : \ 0 \le k_i < q_i\Big\}.$$

Therefore

$$R_\alpha \times [0,1)^r = R_\alpha \times (T^r \oplus \mathcal{G}_r) = \bigcup_{\beta \in \mathcal{G}_r} R_\alpha \times (T^r + \beta). \tag{2.5}$$

Since $\Omega^e = A_1([0,1)^e)$ is a parallelopiped, we may partition it into $\Omega^e = \bigcup_{\beta \in \mathcal{G}_r} \Omega^e_\beta$ in which all Ω^e_β are parallelopiped with the same volume. Hence

$$\Omega^e \times (T^r + \alpha) = \bigcup_{\beta \in \mathcal{G}_r} \Omega^e_\beta \times (T^r + \alpha).$$

We only need to prove that there exists a $\mathcal{K}$-shift $\Omega_{\alpha,\beta}$ of $\Omega^e_\beta \times (T^r + \alpha)$ such that

$$\pi_d : \Omega_{\alpha,\beta} \longrightarrow R_\alpha \times (T^r + \beta) \;\text{ is one-to-one.} \tag{2.6}$$

To prove (2.6), let

$$\mathcal{J} = \left\{[\alpha_e, \alpha_r]^T \in \mathcal{K} : \; \alpha + \alpha_r \equiv \beta \pmod 1\right\}.$$

Then by Sub Lemma 4 the set

$$\mathcal{J}_e := \left\{\alpha_e \in \mathbb{R}^e : \; [\alpha_e, \alpha_r]^T \in \mathcal{K} \text{ for some } \alpha_r \in \mathbb{R}^r\right\}$$

has the property that $\mathcal{J}_e \pmod 1$ is dense in $[0,1]^e$. It follows from Corollary 2.3 that there exists a $\mathcal{J}_e$-shift $\tilde{\Omega}^e_\beta$ of Ω^e_β such that

$$\pi_e : \; \tilde{\Omega}^e_\beta \longrightarrow R_\alpha \;\text{ is one-to-one,} \tag{2.7}$$

where we view R_α as a subset of $\mathbb{T}^e$. The $\mathcal{J}_e$-shift $\tilde{\Omega}^e_\beta$ of Ω^e_β has the form

$$\tilde{\Omega}^e_\beta = \bigcup_{\gamma \in \mathcal{J}_e} (\Omega^e_{\beta,\gamma} + \gamma)$$

where $\{\Omega^e_{\beta,\gamma}\}$ is a partition of Ω^e_β. Set

$$\Omega_{\alpha,\beta} = \bigcup_{\gamma \in \mathcal{J}_e} \left(\Omega^e_{\beta,\gamma} \times (T^r + \alpha) + \gamma'\right)$$

where for each $\gamma \in \mathcal{J}_e$ the element γ' is any element in $\mathcal{J}$ whose first e coordinates is γ. Clearly $\Omega_{\alpha,\beta}$ is a $\mathcal{J}$-shift of $\Omega^e \times (T^r + \alpha)$. Furthermore,

$$\pi_d : \Omega_{\alpha,\beta} \longrightarrow R_\alpha \times (T^r + \beta) \;\text{ is one-to-one}$$

as a result of (2.7). The sub lemma is proved by letting $\Omega_\alpha = \bigcup_{\beta \in \mathcal{G}_r} \Omega_{\alpha,\beta}$. □

To conclude the proof of Theorem 1.2 in Case III we let

$$\Omega = \bigcup_{\alpha \in \mathcal{F}_r} \Omega_\alpha.$$

Then Ω is a $\mathcal{K}$-shift of the fundamental domain $\tilde{\Omega}$ of $\mathcal{K}$. Hence Ω tiles $\mathbb{R}^d$ by $\mathcal{K}$. Furthermore by Sub Lemma 5,

$$\pi_d : \; \Omega \longrightarrow \bigcup_{\alpha \in \mathcal{F}_r} R_\alpha \times (T^r + \beta) = [0,1)^d \;\text{ is one-to-one.}$$

This proves the theorem in Case III, which completes the overall proof of the theorem. ■

Corollary 2.4 *Let $\mathcal{L}, \mathcal{K}$ be two full rank lattices in $\mathbb{R}^d$ such that $\mathrm{v}(\mathcal{L}) \geq \mathrm{v}(\mathcal{K})$. Suppose that $\mathcal{L}$ and $\mathcal{K}$ are commensurable. Then there exists an Ω that is a finite union of congruent rectangular parallelopipeds in $\mathbb{R}^d$ such that Ω tiles $\mathbb{R}^d$ by $\mathcal{K}$ and packs $\mathbb{R}^d$ by $\mathcal{L}$.*

In general it is not known whether we can always make the set Ω a bounded set.

Proof of Theorem 1.1. By Theorem 1.2 we may find a measurable set Ω such that Ω tiles $\mathbb{R}^d$ by $\mathcal{K}$ and packs by $\mathcal{L}$. But $\mathrm{v}(\mathcal{L}) = \mathrm{v}(\mathcal{K})$. So if Ω packs by $\mathcal{L}$ then it must tile by $\mathcal{L}$. This proves the theorem. ∎

Proof of Example 1. Here we give a Fourier analysis proof of the nonexistence, which differs for that of [Ko]. Assume that there exists an Ω that tiles $\mathbb{R}^d$ by each of the three lattices $\mathcal{L}_i$. By a standard result in Fourier analysis, the zero set of $\hat{\chi}_\Omega(\xi)$ must contain $\mathcal{L}_i^* \setminus \{0\}$ for each i, where $\mathcal{L}_i^*$ is the dual lattice of $\mathcal{L}_i$. Now,

$$\mathcal{L}_1^* = \mathbb{Z}^2, \quad \mathcal{L}_2^* = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix} \mathbb{Z}^2, \quad \mathcal{L}_3^* = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \mathbb{Z}^2.$$

Observe that $\mathcal{L}_1^* \cup \mathcal{L}_2^* \cup \mathcal{L}_3^*$ contains the lattice

$$\mathcal{J} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \mathbb{Z}^2.$$

Hence

$$\{\xi : \hat{\chi_\Omega}(\xi) = 0\} \supseteq \mathcal{J} \setminus \{0\}.$$

It follows that $\{e^{2\pi i \langle \alpha, x\rangle} : \alpha \in \mathcal{J}\}$ is an orthogonal family of exponentials in $L^2(\Omega)$. But this would imply that Ω is the union of fundamental domains of $\mathcal{J}^*$, the dual lattice of $\mathcal{J}$ (see [JoPe] or [LW2]), which yields the contradiction $\mu(\Omega) \geq 2$. ∎

3 Weyl-Heisenberg Frames

Let $\mathcal{L} = A\mathbb{Z}^d$ and $\mathcal{K} = B\mathbb{Z}^d$ where A and B are real $d \times d$ nonsingular matrices. We prove density results for Weyl-Heisenberg families $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ in higher dimensions.

We first give a more detailed survey of existing results. As mentioned earlier, in the one dimension $d = 1$ where $\mathcal{L} = a\mathbb{Z}$ and $\mathcal{K} = b\mathbb{Z}$, the condition $|ab| = 1$ ($|b| \leq 1$) is obviously sufficient for the existence of a Weyl-Heisenberg orthonormal basis (frame) $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ for $L^2(\mathbb{R})$, by simply taking $g = \sqrt{|a|}\chi_{[0,|b|)}$. However, the sufficiency is much more complicated in higher dimensions, as the geometry of lattices can be quite complex. It is the main objective of this section to prove the sufficiency.

In the other direction, it is known that in the one dimension $|ab| \leq 1$ is also the necessary condition for the existence of a function $g \in L^2(\mathbb{R})$ such that $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is *complete* (not necessarily a frame) in $L^2(\mathbb{R})$. Rieffel [Rie] proves this as a corollary of results on von Neumann algebras associated with two lattices of Lie groups. For the case that $|ab| > 1$ is rational Daubechies [Dau] provides a constructive proof of the incompleteness of $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ through the use of Zak transform. In higher dimensions density results similar to Rieffel's have been established in various contexts. Ramanathan and Steger [RSt] introduces a technique that applies to Weyl-Heisenberg frames in $\mathbb{R}^d$ in which the lattices are replaced by countable, non-lattice sets that are uniformly separated. They are also able to recapture the density result of Rieffel in $\mathbb{R}^d$ ([RSt], Corollary 1). Ron and Shen ([RSh], Corollary 2.7) prove that if there exists a $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is a frame for $L^2(\mathbb{R}^d)$ then $|\det(AB)| \leq 1$. Christensen, Deng and Heil [CDH] extend results of Ramanathan and Steger to multiple generating functions, from which the density result of Ron and Shen also follows. In [GH1] and [GH2] Gabardo and the first author introduce a simple and general approach to the incompleteness property for arbitrary group-like unitary systems, and prove in particular that if there is a function $g \in L^2(\mathbb{R})$ such that $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is complete for $L^2(\mathbb{R}^d)$ then $|\det AB| \leq 1$. For the purpose of self-containment, we will provide here a very elementary and short proof for $|\det AB| \leq 1$ by assuming that there exists a function $g \in L^2(\mathbb{R})$ such that $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is a frame for $L^2(\mathbb{R}^d)$.

A function $g \in L^2(\mathbb{R}^d)$ is called a *pre-frame function* (with respect to $\mathcal{L}$ and $\mathcal{K}$) if $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is a *Bessel sequence*, i.e. there exists a constant $C > 0$ such that

$$\sum_{\ell \in \mathcal{L}, \kappa \in \mathcal{K}} |\langle f, e^{2\pi i \langle \ell, x \rangle} g(x - \kappa) \rangle|^2 \leq C \|f\|_2^2$$

for all $f \in L^2(\mathbb{R}^d)$. For a pre-frame function g we define an *analysis operator* $\mathfrak{S}_g : L^2(\mathbb{R}^d) \to l^2(\mathcal{L} \times \mathcal{K})$ by

$$\mathfrak{S}_g f = \sum_{\ell \in \mathcal{L}, \kappa \in \mathcal{K}} \langle f, e^{2\pi i \langle \ell, x \rangle} g(x - \kappa) \rangle e_{\ell, \kappa}$$

where $\{e_{\ell,\kappa}\}$ is the standard orthonormal basis for $l^2(\mathcal{L} \times \mathcal{K})$. Clearly $\mathfrak{S}_g$ is a bounded linear operator, and hence $\mathfrak{S}_g^* \mathfrak{S}_g$ is a bounded linear operator on $L^2(\mathbb{R}^d)$. It easy to check that $\mathfrak{S}_g^* \mathfrak{S}_g$ commutes with the modulation operator $\mathfrak{M}_\ell$ and the translation operator $\mathfrak{T}_\kappa$ defined by

$$\mathfrak{M}_\ell f = e^{2\pi i \langle \ell, x \rangle} f(x), \qquad \mathfrak{T}_\kappa f = f(x - \kappa).$$

Lemma 3.1 *There exist pre-frame functions $\{f_\alpha : \alpha \in \mathbb{Z}^d\}$ (with respect to the lattices $\mathcal{L} = A\mathbb{Z}^d$ and $\mathcal{K} = B\mathbb{Z}^d$ such that*

$$\sum_\alpha \mathfrak{S}_{f_\alpha}^* \mathfrak{S}_{f_\alpha} = I$$

and $\sum_\alpha \|f_\alpha\|_2^2 = |\det AB|$.

Proof. Without loss of generality we consider the case $B = I$. Let $\Omega = (0, 1]^d$ and $G_\alpha = A^T\Omega \cap (\Omega + \alpha)$ for $\alpha \in \mathbb{Z}^d$. Note that $\{\Omega + \alpha : \alpha \in \mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$. Thus $\bigcup_{\alpha \in \mathbb{Z}^d} G_\alpha = A^T\Omega$, and $\bigcup_{\alpha \in \mathbb{Z}^d} (A^T)^{-1} G_\alpha = \Omega$.

Write $E_\alpha = (A^T)^{-1} G_\alpha$. Then $\{E_\alpha : \alpha \in \mathbb{Z}^d\}$ is a partition of Ω. Let $f_\alpha = \sqrt{|\det A|}\chi_{E_\alpha}$. Since $A^T E_\alpha - \alpha = G_\alpha - \alpha \subseteq \Omega$, it is easy to check that $\{e^{2i\pi \langle A\beta, x \rangle} f_\alpha(x) : \beta \in \mathbb{Z}^d\}$ is a normalized tight frame for $L^2(E_\alpha)$. Therefore $\mathbf{G}(\mathcal{L}, \mathcal{K}, f_\alpha)$ is a normalized tight frame for $L^2\Big(\bigcup_{\beta \in \mathbb{Z}^d} (E_\alpha + \beta)\Big)$.

Let $F_\alpha = \bigcup_{\beta \in \mathbb{Z}^d} (E_\alpha + \beta)$. Then $\{F_\alpha\}$ is a partition of $\mathbb{R}^d$. Thus $I = \sum_{\alpha \in \mathbb{Z}^d} \mathfrak{S}_{f_\alpha}^* \mathfrak{S}_{f_\alpha}$, and

$$\sum_{\alpha \in \mathbb{Z}^d} \|f_\alpha\|_2^2 = |\det A| \sum_{\alpha \in \mathbb{Z}^d} \mu(E_\alpha) = |\det A| \; \mu(\Omega) = |\det A|.$$

∎

Lemma 3.2 *Assume that there exists a function $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is a frame for $L^2(\mathbb{R}^d)$. Then $|\det AB| \le 1$.*

Proof. Let $\mathfrak{S}_g$ be the analysis operator associated with $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$. Observe that $\mathfrak{S}_g^*\mathfrak{S}_g$ commutes with both the translation and the modulation operators, so $(\mathfrak{S}_g^*\mathfrak{S}_g)^{-1/2}g$ generates a normalized tight Weyl-Heisenberg frame for $L^2(\mathbb{R}^d)$. This means without loss the generality we may assume that $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is already a normalized tight frame for $L^2(\mathbb{R}^d)$. Denote $(f)_{\ell,\kappa} = e^{2\pi i\langle \ell, x\rangle} f(x-\kappa)$ for any function f and let f_α be as in Lemma 3.1. Since

$$|\langle g, (f_\alpha)_{\ell,\kappa}\rangle| = |\langle (g)_{-\ell,-\kappa}, f_\alpha\rangle|,$$

it follows that

$$\begin{aligned}
1 \ \ge\ \|g\|_2^2 \ &=\ \langle g, g\rangle \\
&= \sum_{\alpha\in\mathbb{Z}^d} \langle \mathfrak{S}_{f_\alpha}^* \mathfrak{S}_{f_\alpha} g, g\rangle \\
&= \sum_{\alpha\in\mathbb{Z}^d} \langle \mathfrak{S}_{f_\alpha} g, \mathfrak{S}_{f_\alpha} g\rangle \\
&= \sum_{\alpha\in\mathbb{Z}^d} \Big\langle \sum_{\ell\in\mathcal{L},\kappa\in\mathcal{K}} \langle g, (f_\alpha)_{\ell,\kappa}\rangle e_{\ell,\kappa}, \sum_{\ell\in\mathcal{L},\kappa\in\mathcal{K}} \langle g, (f_\alpha)_{\ell,\kappa}\rangle e_{\ell,\kappa} \Big\rangle \\
&= \sum_{\alpha\in\mathbb{Z}^d} \sum_{\ell\in\mathcal{L},\kappa\in\mathcal{K}} |\langle g, (f_\alpha)_{\ell,\kappa}\rangle|^2 \\
&= \sum_{\alpha\in\mathbb{Z}^d} \sum_{\ell\in\mathcal{L},\kappa\in\mathcal{K}} |\langle (g)_{-\ell,-\kappa}, f_\alpha\rangle|^2 \\
&= \sum_{\alpha\in\mathbb{Z}^d} \|f_\alpha\|_2^2 = |\det AB|.
\end{aligned}$$

∎

Note that the above argument also implies that $\|g\|_2^2 = |\det AB|$ for any normalized tight frame $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ for $L^2(\mathbb{R}^d)$.

Theorem 3.3 *Let $\mathcal{L} = A\mathbb{Z}^d$ and $\mathcal{K} = B\mathbb{Z}^d$ be two full rank lattices in $\mathbb{R}^d$. Then the following statements are equivalent:*

(i) *There exists $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is a normalized tight frame for $L^2(\mathbb{R})$.*

(ii) *There exists $g \in L^2(\mathbb{R}^d)$ such that $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is complete in $L^2(\mathbb{R}^d)$.*

(iii) $\mathrm{v}(\mathcal{L})\mathrm{v}(\mathcal{K}) = |\det(AB)| \le 1$.

Proof. (i) ⇒ (ii) is obvious. Lemma 3.2 gives the implication (i) ⇒ (iii). Now, (ii) ⇒ (i) follows from Theorem 2.1 of [GH2].

Finally, we prove (iii) $\Rightarrow$ (i). Since $|\det(AB)| \leq 1$, we have $|\det B| \leq |\det(A^T)^{-1}|$. By Theorem 1.2 there exists a measurable set Ω in $\mathbb{R}^d$ such that Ω tiles $\mathbb{R}^d$ by BZ^d and packs $\mathbb{R}^d$ by $(A^T)^{-1}\mathbb{Z}^d$. An elementary argument will imply that $g = \frac{1}{\sqrt{|\det A|}}\chi_\Omega$ generates a normalized tight Weyl-Heisenberg frame for $L^2(\mathbb{R}^d)$. ∎

We remark that if the matrices B and $(A^T)^{-1}$ commensurate and $|\det(AB)| \leq 1$ then by Corollary 2.4 we may find a compactly supported function $g(x)$ such that $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is a normalized tight frame for $L^2(\mathbb{R}^d)$.

Proof of Theorem 1.3. It suffices to note that if $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is a normalized tight frame for $L^2(\mathbb{R}^d)$, then $\|g\|_2^2 = |\det AB|$ (see the remark following the proof of Lemma 3.2), and that a normalized tight frame $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ if and only if $\|g\|_2 = 1$. ∎

Since every Weyl-Heisenberg frame $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is similar to a normalized tight Weyl-Heisenberg frame $\mathbf{G}(\mathcal{L}, \mathcal{K}, h)$ in the sense that there exists a bounded invertible operator $\mathfrak{P}$ on $L^2(\mathbb{R}^d)$ such that $(\mathfrak{P}g)_{\ell,\kappa} = h_{\ell,\kappa}$ for all $\ell \in \mathcal{L}$ and $\kappa \in \mathcal{K}$, the following corollary follows immediately:

Corollary 3.4 *Let $\mathcal{L} = A\mathbb{Z}^d$ and $\mathcal{K} = B\mathbb{Z}^d$ be two full rank lattices in $\mathbb{R}^d$. Then the following statements are equivalent:*

(i) $\mathrm{v}(\mathcal{L})\mathrm{v}(\mathcal{K}) = |\det AB| = 1$.

(ii) *Every Weyl-Heisenberg frame $\mathbf{G}(\mathcal{L}, \mathcal{K}, g)$ is a Riesz basis for $L^2(\mathbb{R}^d)$.*

References

[BHW] J. J. Benedetto, C. Heil and D. F. Walnut, Gabor systems and the Balian-Low theorem, in *Gabor analysis and algorithms*, 85-122, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 1998.

[Bou] N. Bourbaki, *General Topology, Part 2*, Addison-Wesley, 1966.

[Ca] P. Casazza, Modern tools for Weyl-Heisenberg (Gabor) frame theory, preprint.

[CDH] O. Christensen, B. Deng and C. Heil, Density of Gabor frames, *Appl. Comput. Harm. Anal* **7**(1999), 292-304.

[Dau] I. Daubechies, The wavelet transformation, time-frequence localization and signal analysis, *IEEE Trans. Inform. Theory* **36** (1990), 961-1005.

[Dau2] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF regional conference series in Applied Math., Philadelphia (1992).

[DLL] I. Daubechies, H. Landau and Z. Landau, Gabor time frequency lattices and the Wexler-Raz identity *J. Fourier. Anal. Appl.* **1** (1995), 437-478.

[FJ] H. G. Feichtinger and A. J. Janssen, Validity of WH-frame bound conditions depends on lattice parameters, *Appl. Comput. Harmon. Anal.* **8** (2000), 104–112.

[FS] H. G. Feichtinger and T. Strohmer (eds), *Gabor Analysis and Algorithms: Theory and Applications, Applied and Numerical Harmonic Analysis,* Birkhuser, 1998.

[GH1] J. P. Gabardo and D. Han, Frame parametrizations and density property for group-like unitary systems with applications to the Weyl-Heisenberg systems, preprint.

[GH2] J. P. Gabardo and D. Han, Subspace Weyl-Heisenberg frames, preprint.

[Ga] D. Gabor, Theory of Communication, *J. Inst. Elec. Eng.* (London) **93** (1946), 429-457.

[JoPe] P. Jorgensen and S. Pedersen, Spectral theory for Borel sets in $\mathbb{R}^n$ of finite measure, *J. Funct. Anal.* **107** (1992), 72-104.

[Ko] M. Kolountzakis, Multi-lattice tiles, *Internat. Math. Res. Notices* **19** (1997), 937-952

[LW1] J. C. Lagarias and Y. Wang, Integral self–affine tiles in R^n part II: lattice tilings, *J. Fourier Analysis and Appl.* **3** (1997), 83-101.

[LW2] J. C. Lagarias and Y. Wang, The spectral set conjecture and factorization of finite abelian groups, *J. Functional Analysis* **145** (1997), 73-98.

[New] M. Newman, *Integral Matrices*, Academic Press, New York 1972.

[Rie] M. A. Rieffel, Von Neumann algebras associated with pairs of lattices in Lie groups, *Math. Ann.* **257**(1981), 403-413.

[RSh] A. Ron and Z. Shen, Weyl-Heisenberg frames and Riesz bases in $L^2(\mathbb{R}^d)$, *Duke Mathematical Journal* **89** (1997), 237-282.

[RSt] J. Ramanathan and T. Steger, Incompleteness of sparse coherent states, *Appl. Comp. Harm. Anal.* **2** (1995), 148-153.

[Sch] L. Schrijver, *Theory of Linear and Integer Programming*, John Wiley and Sons, Inc., New York (1965).